Quiz 6
Real Analysis ICTP 2025

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1. Is it true that for $a, b \geq 0$ and $f, g \geq 0$ we have

$$\|af + bg\|_{L^p(\mathbb{R}^d)} = a\|f\|_{L^p(\mathbb{R}^d)} + b\|g\|_{L^p(\mathbb{R}^d)}?$$

> **Solution:** No, only for $p = 1$ or for $g = 0$. But it remains true as an inequality for all $1 \leq p \leq \infty$.

2. Is it true that for F, E disjoint we have

$$\|f\|_{L^p(E \cup F)} = \|f\|_{L^p(E)} + \|f\|_{L^p(F)}?$$

> **Solution:** No, only for $p = 1$.

3. Is it true for $0 \leq f \leq g$ that $\|f\|_{L^p(\mathbb{R}^d)} \leq \|g\|_{L^p(\mathbb{R}^d)}$?

> **Solution:** Yes.

4. Does $\|f\|_{L^p(\mathbb{R}^d)} = 0$ imply $f = 0$ everywhere?

> **Solution:** No, only almost everywhere.

5. What is another way to write $\|f\|_{L^\infty(\mathbb{R}^d)}$ if $f : \mathbb{R}^d \to \mathbb{R}$ is continuous?

> **Solution:** If f is continuous then $\|f\|_{L^\infty(\mathbb{R}^d)} = \sup_x |f(x)$.

6. If $E \subset \mathbb{R}^2$ is $\mathcal{L}^2$-measurable, is it true that for each $x \in \mathbb{R}$ the set $E_x = \{y \in \mathbb{R} : (x, y) \in E\}$ is $\mathcal{L}^1$-measurable?

> **Solution:** No (assuming AC). For example take a nonmeasurable set $S \subset [0, 1]$ and set $E = \{(0, y) : y \in S\}$. Then $\mathcal{L}^2(E) = 0$ and thus E is $\mathcal{L}^2$-measurable, but $E_0 = S$ is not $\mathcal{L}^1$-measurable.

7. If $f : \mathbb{R}^2 \to [-\infty, \infty]$ is $\mathcal{L}^2$-integrable, is it true that for every $x \in \mathbb{R}$ the function $y \mapsto f(x, y)$ is $\mathcal{L}^1$-integrable?

Solution: No, only for $\mathcal{L}^1$-almost every $x \in \mathbb{R}$. For example $f = \infty \cdot (1_{\{0\}\times[1,2]} - 1_{\{0\}\times[-1,0]}$ has $\int f \, \mathrm{d}\mathcal{L}^2 = 0$ but $y \mapsto f(0, y)$ is not $\mathcal{L}^1$-integrable.