Quiz 5
Real Analysis ICTP 2025

Julian Weigt

October 6, 2025

1. What happens if we relax the assumption $a_1, ..., a_n \geq 0$ in Proposition 2.1.3?

> **Solution:** The formula $\sum_{k=1}^{n} a_k \mu(E_k)$ might not be well defined if there are sets with $\mu(E_k) = \infty$. But, Proposition 2.1.3 continues to hold for all a_k, A_k for which $\sum_{k=1}^{n} a_k \mu(A_k)$ is well defined.

2. Let $f, f_1, f_2, ... : \mathbb{R}^d \to [0, \infty]$ such that $f_n \to f$ $\mathcal{L}$-almost everywhere. Does $\int f_n \,\mathrm{d}\mathcal{L}$ converge?

> **Solution:** No.

3. With f, f_n from Question 2, if $\int f_n \,\mathrm{d}\mathcal{L}$ converges, what are the possible values for its limit in terms of $\int f \,\mathrm{d}\mathcal{L}$?

> **Solution:** It can can converge to all values larger or equal to $\int f \,\mathrm{d}\mathcal{L}$.

4. With f, f_n from Question 2, under which conditions does $\int f_n \,\mathrm{d}\mathcal{L}$ necessarily converge to $\int f \,\mathrm{d}\mathcal{L}$?

> **Solution:**
>
> (i) If f_n converges pointwise to f from below (monotone convergence theorem). Or, more generally, if $f_n \leq f$ pointwise almost everywhere.
>
> (ii) If there exists an $E \subset \mathbb{R}^d$ with $\mathcal{L}(E) < \infty$ and an $N \in \mathbb{N}$ with $f_n \leq N$ (Lemma 2.1.7).

5. Is Proposition 2.1.3 also true for ∞ instead of n?

> **Solution:** Yes, as a consequence of the monotone convergence theorem.