Quiz 4
Real Analysis ICTP 2025

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1. Let $A_1, A_2, ... \subset \Omega$ be measurable and $a_1, a_2, ... \geq 0$. Is the function $f = \sum_{n=1}^{\infty} a_n 1_{A_n}$ measurable?

> **Solution:** Yes, because for each $k \in \mathbb{N}$ the function $f_k = \sum_{n=1}^{k} a_k 1_{A_k}$ is measurable and $f = \lim_{k\to\infty} f_k$.

2. Let $a_1, ..., a_n, b_1, ..., b_n \in \mathbb{R}$ and $A_1, ..., A_n, B_1, ..., B_n \subset \Omega$ such that

$$\sum_{k=1}^{n} a_n 1_{A_n} = \sum_{k=1}^{n} b_n 1_{B_n}.$$

Is it true, that $\{a_1, ..., a_n\} = \{b_1, ..., b_n\}$ and $\{A_1, ..., A_n\} = \{B_1, ..., B_n\}$?

> **Solution:** No, it may happen that $A_1 = A_2 = B_1 = B_2$ and $a_1 + a_2 = b_1 + b_2$ but $\{a_1, a_2\} \neq \{b_1, b_2\}$, for example for $a_1 = 0, a_2 = 2,\ b_1 = b_2 = 1$.
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> It similarly happen that $a_1 = a_2 = b_1 = b_2$ and $A_1 \cup A_2 = B_1 \cup B_2$ but $\{A_1, A_2\} \neq \{B_1, B_2\}$.

3. Let $f_1, f_2 : \Omega \to [0, \infty]$ and $f = f_1 - f_2$. Does that imply $f_1 = f^+$ and $f_2 = f^-$?

What, if for all $x \in \Omega$ we have $f_1(x) = 0$ or $f_2(x) = 0$?

> **Solution:** No, for example if $f = 1$ is constant, then $f^+ = 1$ and $f^- = 0$ but $f_1 = 2$ and $f_2 = 1$ satisfy $f = f_1 - f_2$.
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> The statement is true however under the assumption that for all $x \in \Omega$ we have $f_1(x) = 0$ or $f_2(x) = 0$.

4. Let μ be a measure on $\mathbb{R}^d$. Is $\mathbb{R}^d$ σ-finite with respect to μ?

What, if μ is a Radon measure?

> **Solution:** No, for example for the counting measure.
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> If μ is a Radon measure the answer is yes, because $\mathbb{R}^d = B(0,1) \cup B(0,2) \cup ...$ and $\mu(B(0,n)) \leq \mu(\overline{B(0,1)}) < \infty$.

5. Given $f, f_1, f_2, ... : \Omega \to \mathbb{R}$, what does it mean that $f_n \to f$ uniformly?

> **Solution:** It means that for every $\varepsilon > 0$ exists an $n \in \mathbb{N}$ such that for all $k \geq n$ and $x \in \Omega$ we have $|f_k(x) - f(x)| < \varepsilon$.

6. Find an example of functions $f, f_1, f_2, ... : (0,1) \to \mathbb{R}$ such that $f_n \to f$ pointwise, but not uniformly.

> **Solution:** Set $f_n(x) = x^{1/n}$. Then $f_n \to 0$ pointwise but not uniformly: Let $n \in \mathbb{N}$. Then for $x_n = 1/2^{1/n} \in (0,1)$ we have $|f_n(x_n) - 0| = 1/2$.

7. Find $f : \mathbb{R} \to \mathbb{R}$ and a closed set $C \subset \mathbb{R}$ such that $f : C \to \mathbb{R}$ is continuous but $f : \mathbb{R} \to \mathbb{R}$ is not continuous in every $x \in C$.

   Can you also find an open set with that property?

> **Solution:** Set $C = [0,1]$ and $f(x) = 1$ if $x \in C$ and $f(x) = 0$ if $x \notin C$.
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> Such an example cannot be found with an open set U instead. The reason is, that if $x \in U$ and $x_n \to x$, then there exists a $k \in \mathbb{N}$ such that for all $n \geq k$ we have $x_n \in U$. As a consequence, $f : U \to \mathbb{R}$ is continuous in x if and only if $f : \mathbb{R}^d \to \mathbb{R}$ is continuous in x.

8. What is the Lebesgue integral $\int f \, \mathrm{d}\mathcal{L}$ for

   (i) $f = 1_{\mathbb{R}} - 1_{\mathbb{R}}$

   (ii) $f = 1_{[0,\infty)} - 1_{(-\infty,0)}$

> **Solution:**
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> (i) 0, since the canonical form of f is $f = 0$.
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> (ii) The integral is not defined because both $\int f^+ \, \mathrm{d}\mathcal{L} = \infty$ and $\int f^- \, \mathrm{d}\mathcal{L} = \infty$.