Quiz 3
Real Analysis ICTP 2025

Julian Weigt

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1. Is Lebesgue measure a Radon measure? Is Lebesgue outer measure a Radon outer measure?

> **Solution:** Yes.

2. Do the annuli $A_n = \overline{B(x,n)} \setminus B(x, n-1)$ form a decomposition, i.e. is it true that $\Omega = A_1 \cup A_2 \cup ...$ and for any $n \neq k$ we have $A_n \cap A_k = \emptyset$?

> **Solution:** Almost, but no. It is true that $\Omega = A_1 \cup A_2 \cup ...$, but $A_n \cap A_{n+1} = \partial B(x,n)$, and $\partial B(x,n) \neq \emptyset$ for certain metric spaces such as $\mathbb{R}^d$ with the Euclidean metric.

3. Is it true, that for any $E \subset \Omega \ni x$ and any Radon measure μ that $\mu(E \setminus B(x,n)) \to 0$ as $n \to \infty$?

> **Solution:** No, take $E = \mathbb{R}^d$ and $\mu = \mathcal{L}$. It is true however if $\mu(E) < \infty$ as a consequence of the "measure continuity" Lemma 1.2.7.

4. Let μ be the counting measure on $\mathbb{R}$, i.e. $\mu(E)$ equals the number of elements in E if E is finite, and $\mu(E) = \infty$ otherwise. Is μ a Radon measure? Is μ a Radon outer measure?

> **Solution:** All sets $E \subset \mathbb{R}$ are measurable, in particular Borel sets, but μ is not a Radon (outer) measure because compact sets do not have finite μ-measure.

5. Does Proposition 1.2.18 hold for the counting measure?

> **Solution:** No. If E is closed and nonempty then for any open $U \supset E$ the set $U \setminus E$ is open and nonempty and thus infinite. If E is open and nonenmpty and $C \subset E$ is closed then the same is true for $E \setminus C$.

6. Is it true that for any measurable $A \subset \mathbb{R}^d$ with $\mathcal{L}(A) < \infty$ and $\varepsilon > 0$ exists a compact set $K \subset A$ with $\mathcal{L}(A \setminus K) < \varepsilon$? What, if $\mathcal{L}(A) = \infty$?

Solution: No, if $\mathcal{L}(A) = \infty$, take $A = \mathbb{R}^d$.

If $\mathcal{L}(A) < \infty$, then yes. By Proposition 1.2.18 there exist a closed set $C \subset A$ with $\mathcal{L}(A \setminus C) < \varepsilon/2$ and $\mathcal{L}(C) \leq \mathcal{L}(A) < \infty$. Thus by the "measure continuity" lemma there exists an $n \in \mathbb{N}$ such that $K := C \cap \overline{B(0, n)}$, which is compact, satisfies $\mathcal{L}(A \setminus K) = \mathcal{L}(A \setminus C) + \mathcal{L}(C \setminus K) < \varepsilon$.

7. Let $E \subset \Omega$ be measurable. Is the characteristic function given by

$$1_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

measurable?

Solution: Yes, because for any $a \in \mathbb{R}$ we have $\{f < a\} \in \{\emptyset, \mathbb{R}^d \setminus A, \mathbb{R}^d\}$, all of which are measurable.

8. Are polynomials Lebesgue measurable functions?

Solution: Yes, because they are continuous.