Quiz 2
Real Analysis ICTP 2025

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1. Does Lebesgue outer measure assign a number to every subset $E \subset \mathbb{R}^d$? Does this number always represent a reasonable notion of volume of the set?

> **Solution:** Yes, Lebesgue outer measure assigns a number to every subset $E \subset \mathbb{R}^d$. However, for it to be a reasonable notion of volume, it should be countably additive, or at least additive. At this point we do not know if this is the case. In fact, we will later see in **??** that Lebesgue outer measuer is not countably additive if we assume the axiom of choice.

2. Let $(\Omega, \mathcal{M}, \mu)$ be a measure space. Is it true for all $A, B \in \mathcal{M}$ with $A \cap B = \emptyset$ that $\mu_*(A \cup B) = \mu_*(A) + \mu_*(B)$?

> **Solution:** Yes, by definition.

3. What is the relation between outer measures and measures?

> **Solution:** An outer measure restricted to its Caratheodory measurable sets is a measure. There might be very few of those sets though, and an outer measure might not be a measure on 2^Ω.

4. Does $d(x, y) = \sqrt{|x - y|}$ define a metric on $\mathbb{R}$?

> **Solution:** Yes, since for $a, b \geq 0$ we have $(\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{a}\sqrt{b} \geq (\sqrt{a + b})^2$.

5. Let μ_* be a metric outer measure. Is it true for all Borel sets B that $\mu_*(B) < \infty$?

> **Solution:** No, Borel sets B are measurable but might have infinite measure.

6. Let Q, P be dyadic cubes. What are the options of sets that $P \cap Q$ can be?

> **Solution:** Q, P and $\emptyset$.

7. Does for any Borel set B exist an open set $U \supset B$ with $\mathcal{L}(U) = \mathcal{L}(B)$?

Solution: No, for example take $B = \{0\}$.