Quiz 1
Real Analysis ICTP 2025

Julian Weigt

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1. For $\mathcal{Q} = \{Q_n : n \in \mathbb{N}\}$ is
$$\bigcup \mathcal{Q} = \bigcup_{n=1}^{\infty} Q_n?$$

Solution: Yes, this is another way of writing the definition.

2. For $x, y \in \mathbb{R}^d$ and $i \in \{1, ..., d\}$ is it true, that $x_i + y_i = (x + y)_i$?

Solution: Yes, this is another way of writing the definition.

3. Is it true for all $x \in \mathbb{R}$ that $(x + \infty) - \infty = x + (\infty - \infty)$?

Solution: For every $x \in \mathbb{R}$ we have $x+\infty = \infty$, and $\infty-\infty$ is undefined. That means both sides of the equality are undefined, which makes the whole equality statement malformed. It is not a formal logical statement, which means it does not make sense to ask if it is true or false.

4. In $\mathbb{R}$, what is the value of $\mathcal{L}_*(\{2\})$?

Solution: We have $\mathcal{L}_*(\{2\}) = 0$ because for each $\varepsilon > 0$ the set consisting of the single cube $[2-\varepsilon, 2+\varepsilon]$ is a cover of $\{2\}$, and $\lvert[2-\varepsilon, 2+\varepsilon]\rvert = 2\varepsilon$.

5. Is it true for any set of cubes $Q_1, ..., Q_n \subset \mathbb{R}^d$ that
$$\mathcal{L}_*(Q_1 \cup ... \cup Q_n) = \mathcal{L}_*(Q_1) + ...\mathcal{L}_*(Q_n)?$$

Solution: No, for example if all cubes are the same $Q_1 = ... = Q_n$.

6. Let μ_* be an outer measure on Ω. Is it true for all sets $A, B \subset \Omega$ that $\mu_*(A \cup B) \leq \mu_*(A) + \mu_*(B)$?

Solution: Yes, by definition.

7. Let μ_* be an outer measure on Ω. Is it true for all sets $A, B \subset \Omega$ with $A \cap B = \emptyset$ that $\mu_*(A \cup B) = \mu_*(A) + \mu_*(B)$?

Solution: No. See exercise sheet 1 for a counterexample.

8. Let μ_* be an outer measure on Ω. Is it true for all measurable sets $A, B \subset \Omega$ with $A \cap B = \emptyset$ that $\mu_*(A \cup B) = \mu_*(A) + \mu_*(B)$?

Solution: Yes, since by definition $\mu(A \cup B) = \mu((A \cup B) \cap A) + \mu((A \cup B) \setminus A) = \mu(A) + \mu(B)$.