

Overview Real Analysis

Julian Weigt

1 Measure theory

1.1 Lebesgue outer measure

Definition Lebesgue outer measure:

$$\mathcal{L}(E) = \inf\left\{\sum_{Q \in \mathcal{Q}} |Q| : E \subset \bigcup \mathcal{Q}\right\}$$

Definition outer measure $\mu : 2^\Omega \rightarrow [0, \infty]$:

1. empty set
2. (monotonicity)
3. countable subadditivity

1.2 Measurable sets

1.2.1 Carathéodory's theorem

Definition σ -algebra M :

1. empty set
2. complement
3. countable union

Definition measure $\mu : M \rightarrow [0, \infty]$:

1. empty set
2. countable subadditivity

Definition $A \subset \Omega$ Carathéodory measurable:

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A)$$

Theorem The set of all Carathéodory measurable sets is a σ -algebra and an outer measure restricted to all Carathéodory measurable sets is a measure.

Lemma measure continuity (upwards and downwards)

1.2.2 A non-measurable set

Theorem Using the axiom of choice and equivalence classes, one can decompose a set $[0, 1] \subset E \subset [-1, 2]$ into countably many pieces which are translates of one another. The translation invariance of Lebesgue outer measure implies that this set cannot be Lebesgue measurable.

1.2.3 Metric measures

Definition $d : \Omega \times \Omega \rightarrow [0, \infty)$ metric:

1. $d(x, x) = 0$
2. symmetry
3. triangle inequality

Definition Borel σ -algebra: smallest σ -algebra generated by all open sets.

Definition metric outer measure μ : If $d(A, B) > 0$ then $\mu(A \cup B) = \mu(A) + \mu(B)$.

Theorem If μ is a metric outer measure then all Borel sets are μ -measurable.

Definition

- Borel (outer) measure: All Borel sets are measurable.
- Borel regular (outer) measure: Every $A \subset \Omega$ has an $A \subset B$ with $\mu(A) = \mu(B)$.
- Radon (outer) measure: Borel regular and for all compact K $\mu(K) < \infty$.

Proposition Inner and outer regularity of Radon (outer) measures and Lebesgue (outer) measure.

1.3 Measurable functions

1.3.1 Definition and extent of the class

Definition $f : \Omega \rightarrow [-\infty, \infty]$ is μ -measurable if all superlevelsets are μ -measurable, or the preimages of Borel sets etc.

Lemma Examples:

- continuous functions (if Borel measure)
- limits
- composition of measurable and continuous function
- sums, products

1.3.2 Approximation

Definition characteristic function, simple function, step function

Theorem Measurable functions are pointwise limits of simple functions (approximate negative and positive part from below separately).

Definition σ -finite

Theorem Measurable function on $\mathbb{R}^d$ are Lebesgue-almost everywhere pointwise limits of step functions.

Theorem Egorov: For every $\varepsilon > 0$ any pointwise convergent sequence of functions on Ω converges uniformly outside a set of measure at most ε .

Theorem Lusin: For every $\varepsilon > 0$ and any measurable function f on $\mathbb{R}^d$ exists a closed C with $L(C) < \varepsilon$ such that f restricted to $\mathbb{R}^d \setminus C$ is continuous.

2. Integration

2.1 The Lebesgue integral

We define the Lebesgue integral of functions in the following order

1. characteristic functions
2. canonical form of simple functions
3. simple functions (have to show uniqueness)
4. nonnegative functions via approximation from below by simple functions
5. measurable functions via decomposition into positive and negative part

Proposition The Lebesgue integral satisfies

- linearity
- additivity
- monotonicity
- zero if and only if function zero a.e.
- triangle inequality

Main work is proof of linearity for nonnegative functions. For that need to prove bounded version of Monotone Convergence Theorem first

Definition A measure function is integrable if the negative or positive part have finite integral.

Theorem Fatou's lemma: For $f_n \geq 0$

$$\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n$$

Proof uses measure continuity.

Theorem Monotone convergence theorem For $f_n \geq 0$ monotonously increasing

$$\int \lim_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} \int f_n$$

Follows from Fatou's lemma.

Theorem Dominated Convergence: Limiting and integration can be swapped if all functions are dominated by a single function that has a finite integral.

Follows from Fatou's lemma.

Proposition Let μ be σ -finite. Then

1. For any f

$$\int_{\Omega \setminus B} f < \varepsilon$$

for B large enough.

2. For any f and any $\varepsilon > 0$ exist $\delta > 0$ such that

$$\int_E f < \varepsilon$$

for any E with $\mu(E) < \delta$.

2.2 L^p spaces

Definition of L^p

- Triangle inequality
- Hölder's inequality

Proposition L^p functions can be approximated by

- simple functions
- step functions (if $\Omega = \mathbb{R}^d$ and $p < \infty$)
- compactly supported continuous functions (if $\Omega = \mathbb{R}^d$ and $p < \infty$)

2.3 Fubini's theorem

Definition product (outer) measure

Theorem Fubini: The order of double integrals can be swapped. More precisely: The product measure can be written as iterated measures.

Proof strategy: Show that product measure and iterated integration measure agree by going incrementally from rectangles to general sets.

3 Differentiation and integration on $\mathbb{R}$

3.1 The Lebesgue differentiatio theorem

Theorem Lebesgue differentiation theorem: For Lebesgue almost every $x \in \mathbb{R}^d$ we have

$$\lim_{r \rightarrow 0} \frac{1}{L(B(x, r))} \int_{B(x, r)} f = f(x).$$

Can be strengthened to

$$\lim_{r \rightarrow 0} \frac{1}{L(B(x, r))} \int_{B(x, r)} |f - f(x)| = 0.$$

Such x are called Lebesgue points.

Definition Hardy-Littlewood maximal operator

Definition $L^{1,\infty}$ (weak L^1)

Theorem Hardy-Littlewood maximal function theorem: The Hardy-Littlewood maximal operator is bounded as a map $L^1(\mathbb{R}^d) \rightarrow L^{1,\infty}(\mathbb{R}^d)$

It is proved using the Vitali covering lemma.

Since the Hardy-Littlewood maximal operator is also bounded on L^∞ , by Marcinkiewicz interpolation it is bounded on L^p for all $1 < p \leq \infty$.

The Lebesgue differentiation theorem is proved by using that it holds for continuous functions and by approximation measurable functions by continuous and estimating the difference using the Hardy-Littlewood maximal function theorem.

3.2 Radon measures

Definition absolute continuity of measures

Definition upper semicontinuity of functions

Theorem There is a 1-1 correspondence between increasing functions and Radon measures on the real line.

Definition absolute continuity of functions on the real line

Lemma Both notions of absolute continuity are the same in the sense of the 1-1 correspondence.

3.3 The Cantor set

Definition Cantor set C : Starting with $[0, 1]$ iteratively take away middle thirds of intervals. Or: The set of all real numbers whose ternary expansions do not have 1s.

C has the following properties

- uncountable
- zero Lebesgue measure
- totally disconnected
- no isolated points

Definition Cantor measure μ_C

Lemma The Cantor measure has the following properties

- $\mu_C(C) = 1$
- $\mu_C(\mathbb{R} \setminus C) = 0$
- $\mu_C(\{x\}) = 0$

3.4 Functions of bounded variation

Definition Variation of a function, $\text{var}_{[a,b]}(f)$

Lemma A function with bounded variation is the difference of two increasing functions (their positive and negative variations).

4 Further topics

4.1 Signed measures

Definition signed measure: countably additive set function that maps into $(-\infty, \infty]$.

Examples difference of measures, integrals of integrable functions, Radon measure defined via functions of bounded variation

Proposition A signed measure is the difference of two measures (can be written in terms of total variation measure).

4.2 Convolution and approximation of the identity

Definition Convolution $f * g$.

Proposition Convolutions smoothen: If $g \in C_c^k$ then $f * g \in C^k$.

Theorem Young's convolution inequality

For g with $\int g = 1$ define scaling g_r .

Theorem For every Lebesgue point of f we have

$$\lim_{r \rightarrow 0} (f * g_r)(x) = f(x)$$

Theorem For $1 \leq p < \infty$

$$\lim_{r \rightarrow 0} \|f - f * g_r\|_p = 0.$$

Is proven using the following:

Theorem Minkowski's integral inequality: Generalization of triangle inequality for L^p -spaces. (proof skipped)

Proposition

$$\lim_{x \rightarrow 0} \|f(\cdot - x) - f\|_p = 0$$