Real Analysis
Lecture notes ICTP 2025

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Contents

Preliminaries

This course assumes familiarity with basic notions from

- functions, such as injectivity, bijectivity, images, and preimages,
- topology on $\mathbb{R}^d$, such as closed and open sets,
- analysis on the real line, such as sequences, series, limits, $\liminf$ and $\limsup$,
- calculus on the real line, such as the chain and product rule for derivatives, and the Riemann integral.

We recall the following notions and notations that are important for the course.

Sets and set operations We denote by $\mathbb{N}$ the **natural numbers** $\mathbb{N} := \{1, 2, 3, ...\}$ and by $\mathbb{R}$ the **real numbers**. For two sets A and B their **union** $A \cup B$ consists of all points x that belong to A or to B. Their **intersection** $A \cap B$ consists of all points that belong to both A and B. For sets A_n that are indexed by for example by natural numbers $n \in \mathbb{N}$ in the case of a sequence $A_1, A_2, ...$, we denote by

$$\bigcup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup ...$$

their union, i.e. the set of all points that for any n belong to A_n. More generally, if $\mathcal{A}$ is a collection of sets A, we denote by

$$\bigcup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A$$

the set of all points x for which there exists an $A \in \mathcal{A}$ with $x \in \mathcal{A}$. The **set difference** $A \setminus B$

consists of all points that belong to A and not to B. Two sets A, B are **disjoint** if $A \cap B = \emptyset$. We say that a collection of sets $\mathcal{A}$ is disjoint, if any two $A, B \in \mathcal{A}$ with $A \neq B$ are disjoint.

For two sets A, B we define their product $A \times B$ as the set of all pairs with the first element from A and the second element from B, i.e.

$$A \times B = \{(a, b) : a \in A,\ b \in B\}.$$

More generally,

$$A_1 \times ... \times A_n = \{(a_1, ..., a_n) : \forall k = 1, ..., n\ a_k \in A_k\}.$$

A set A is **countable** if there exists a surjection $f : \mathbb{N} \to A$, i.e. for each $a \in A$ exists an $n \in \mathbb{N}$ with $f(n) = a$. We also write $a_n := f(n)$.

Euclidean space The basic space where our study takes place is Euclidean space, that is, for any natural number $d \in \mathbb{N}$, the space $\mathbb{R}^d$, which consist of all d-tuples $x = (x_1, ..., x_d)$ of real numbers $x_n \in \mathbb{R}$ with $n = 1, ..., d$.

We assign to each point $x \in \mathbb{R}^d$ its (Euclidean) norm

$$|x| = (x_1^2 + ... + x_d^2)^{\frac{1}{2}}.$$

We can add and substract points $x, y \in \mathbb{R}^d$ componentwise,

$$x + y = (x_1 + y_1, ..., x_d + y_d)$$
$$x - y = (x_1 - y_1, ..., x_d - y_d).$$

Euclidean distance The (Euclidean) distance between them is $|x - y|$. The most elementary subsets of $\mathbb{R}$ we consider are open and closed intervals. For $a, b \in \mathbb{R}$ denote by $[a, b]$ the set of all $x \in \mathbb{R}$ with $a \leq x \leq b$ and by (a, b) the set of all $x \in \mathbb{R}$ with $a < x < b$. In Euclidean space with larger dimensions d those sets generalize to rectangles and balls. For $a, b \in \mathbb{R}$ such that for $n = 1, ..., d$ we have $a_n \leq b_n$, the open and closed rectangles that have a and b as opposite corners are

$$(a_1, b_1) \times ... \times (a_d, b_d), \quad [a_1, b_1] \times ... \times [a_d, b_d].$$

The (open) ball with center $x \in \mathbb{R}^d$ and radius $r > 0$ consist of those $y \in \mathbb{R}^d$ with $|x - y| < r$ and is denoted by $B(x, r)$. The corresponding closed ball $\overline{B}(x, r)$ consist of those $y \in \mathbb{R}^d$ with $|x - y| \leq r$.

Let $A \subset \mathbb{R}^d$. A point $x \in \mathbb{R}^d$ is an **interior point** of A if there exists an $r > 0$ with $B(x,r) \subset A$. A point $x \in \mathbb{R}^d$ is a **limit point** of A if for every $r > 0$ exists a $y \in A$ with $|x - y| < r$. We denote by $\mathring{A}$ the **interior** of A, the set of all interior points of A. We denote by $\overline{A}$ the **closure** of A, the set of all limit points of A. We denote by

$$\partial A := \overline{A} \setminus \mathring{A}$$

the **boundary** of A. By this definiton, the interior of an open or closed ball is the corresponding open ball, and its closure is the corresponding closed ball. The same is true for rectangles.

The extendend real line is the set $\mathbb{R} \cup \{-\infty, \infty\}$. We partially extend addition and multiplication from $\mathbb{R}$ to the extended real line by defining

$$\forall x \in \mathbb{R} \cup \{\infty\} : x + \infty := \infty \quad \forall x > 0 : x \cdot \infty := \infty.$$

We further extend this by prescribing commutativity and associativity and multiplying both definitions with -1. This only leaves $\infty - \infty$

and $0 \cdot \infty$ undefined. In this sense we can treat a statement like

$$\lim_{n \to \infty} a_n = \infty$$

as an equality on the extended real line. We also extend the relations $<, \leq, >, \geq$ to the extended real line via

$$\forall x \in \mathbb{R} \cup \{-\infty\} : x < \infty, \qquad \infty = \infty,$$

with the corresponding definitions for $-\infty$.

Recall also

$$\inf \emptyset = \infty, \qquad \sup \emptyset = -\infty.$$

Convergent sums Let $a_1, a_2, ... \geq 0$. Then their sum does not depend on the order of summation, i.e. for any bijection $\sigma : \mathbb{N} \to \mathbb{N}$ we have

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\sigma(n)}.$$

Here, both sides of the equality may be infinite. The same conclusion is true if $a_1, a_2, ... \in \mathbb{R}$ and

$$\sum_{n=1}^{\infty} |a_n| < \infty.$$

For a countable set $A = \{a_1, a_2, ...\} \subset [0, \infty]$ this allows for the notation

$$\sum_{a \in A} a = \sum_{n=1}^{\infty} a_n.$$

Chapter 1

Measure Theory

The main textbook sources are [SS05] and [EG15]. Other inspirational material are the lecture notes in real analysis by Emanuel Carneiro and the lecture notes in measure theory [Kin24] and real analysis [Kin25] by Juha Kinnunen.

1.1 Lebesgue outer measure

[1]

Our first goal is to rigorously assign a volume to subsets of $\mathbb{R}^d$. A set whose volume we already know is the rectangle: For $a, b \in \mathbb{R}^d$ the volume of the rectangle $R = (a_1, b_1) \times ... \times (a_d, b_d)$ is the product of its side lengths,

$$|R| = (b_1 - a_1) \cdot ... \cdot (b_d - a_d). \tag{1.1.1}$$

The corresponding closed rectangle has the same volume.

We say that a collection $\mathcal{R}$ of rectangles is **almost disjoint** if for any two $R_0, R_1 \in \mathcal{R}$ with $R_0 \neq R_1$ their interiors $\mathring{R}_0$ and $\mathring{R}_1$ are disjoint.

Lemma 1.1.1. Let $n \in \mathbb{N}$ and let $R_1, ..., R_n$ be almost disjoint rectangles such that

$$R = R_1 \cup ... \cup R_n$$

is a rectangle, too. Then

$$|R| = \sum_{k=1}^{n} |R_k|.$$

[1] This section follows Sections 1.1.1 and 1.1.2 from [SS05] for the construction of the Lebesgue measure.

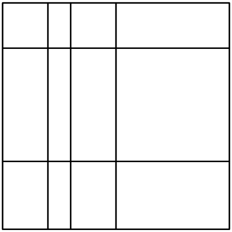

Figure 1.1: The grid case.

By our intuition about volumes this is clearly true. But it requires a proof because all we know so far about our mathematical notion of volume is the abstract formula (1.1.1).

Proof. We first consider the case that the rectangles form a grid, that is, for each $l = 1, ..., d$ there are $a_l = a_l^0 < ... < a_l^{N_d} = b_l$ such that each rectangle R_k is of the form $[a_1^i, a_1^{i+1}] \times ... \times [a_d^j, a_d^{j+1}]$. Then

$$|R| = \prod_{l=1}^{d}(b_l - a_l) = \prod_{l=1}^{d}\sum_{i=1}^{N_d}(a_l^i - a_l^{i-1})$$

$$= \sum_{i_1=1}^{N_1} ... \sum_{i_d=1}^{N_d} \prod_{l=1}^{d} (a_l^{i_l} - a_l^{i_l-1}) = \sum_{k=1}^{n} |R_k|. \tag{1.1.2}$$

For the general case, we subdecompose each rectangle R_i via the extended faces of the rectangles $R_1, ..., R_n$ into rectangles $R_i = R_i^1 \cup ... \cup R_i^{N_i}$. This subdecomposition is a grid which means that by the previous case

$$|R_i| = \sum_{j=1}^{N_i} |R_i^j|.$$

Moreover, the rectangles $\{R_i^j : j = 1, ..., N_i,\ i = 1, ..., n\}$ form a grid for R, so that

$$|R| = \sum_{i=1}^{n} \sum_{j=1}^{N_i} |R_i^j| = \sum_{i=1}^{n} |R_i|. \tag{1.1.3}$$

□

Lemma 1.1.2. Let $n \in \mathbb{N}$ and let $R, R_1, ..., R_n$ be rectangles with

$$R \subset R_1 \cup ... \cup R_n$$

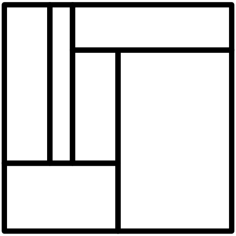

Figure 1.2: The general case.

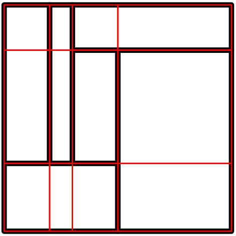

Figure 1.3: Subdecomposition into grid.

Then
$$|R| \leq \sum_{k=1}^{n} |R_k|.$$

Proof. This follows by the same proof as Lemma 1.1.1, except that some rectangles of the grid that decomposes R may belong to more than one of the rectangles $R_1, ..., R_n$. More precisely, the last equality in (1.1.2) and the first equality in (1.1.3) become inequalities. $\square$

We want to use a similar idea of writing a set in terms of sets whose volume we know in order to define its volume. A **cube** $Q \subset \mathbb{R}^d$ is a rectangle whose sidelengths are all identical, i.e. for $a_1, ..., a_d \in \mathbb{R}^d$ and $r > 0$ it is of the form

$$Q = (a_1, a_1 + r) \times ... \times (a_d, a_d + r).$$

Its volume thus is $|Q| = r^d$. For any set $E \subset \mathbb{R}^d$ we define its **outer Lebesgue measure** by

$$\mathcal{L}_*(E) = \inf\Big\{ \sum_{Q \in \mathcal{Q}} |Q| :$$
$$\mathcal{Q} \text{ is a countable set of closed cubes}$$

$$\text{with } E \subset \bigcup \mathcal{Q} \Big\}.$$

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Lemma 1.1.3. For each closed cube Q we have

$$\mathcal{L}_*(Q) = |Q|.$$

Proof. Since $\{Q\}$ is a cover of Q we have $\mathcal{L}_*(Q) \leq |Q|$. For the reverse inequality let $\varepsilon > 0$. Then there exists a countable cover $\mathcal{Q}$ of Q such that

$$\mathcal{L}_*(Q) \leq \sum_{P \in \mathcal{Q}} |P| + \varepsilon.$$

Let $\delta > 0$ and for each $P \in \mathcal{Q}$ denote by $\tilde{P}$ the open cube with the same center as P and volume $1+\delta$ times the volume of P. Then $\tilde{P} \supset P$ which means that $\tilde{\mathcal{P}} := \{\tilde{P} : P \in \mathcal{Q}\}$ is an open cover of the compact set Q and thus has a finite subcover $\mathcal{P}$. By Lemma 1.1.2 we can conclude

$$\begin{aligned} |Q| &\leq \sum_{\tilde{P} \in \mathcal{P}} |\tilde{P}| = (1+\delta) \sum_{\tilde{P} \in \mathcal{P}} |P| \\ &\leq (1+\delta) \sum_{P \in \mathcal{Q}} |P| \leq (1+\delta)(\mathcal{L}_*(Q) + \varepsilon). \end{aligned}$$

Since $\varepsilon, \delta > 0$ were arbitrarily small we can conclude $|Q| \leq \mathcal{L}_*(Q)$ and finish the proof. $\square$

Remark 1.1.4. We need to allow countable sequences of cubes in the definition of the outer measure. If we only allowed finite sequences then unbounded sets would always have infinite outer Lebesgue measure. However, sets like $\mathbb{N} \subset \mathbb{R}$ should have zero volume.

For a set Ω denote by 2^Ω the set of all subsets of Ω. We say that a set function $\mu_* : 2^\Omega \to [0, \infty]$ is an **outer measure** if it has the following two properties:

(i) (empty set) $\mu_*(\emptyset) = 0$.

(ii) (countable subadditivity) For each $E, E_1, E_2, ... \subset \Omega$ with $E \subset E_1 \cup E_2 \cup ...$ we have

$$\mu_*(E) \leq \sum_{n=1}^{\infty} \mu_*(E_n).$$

Observe that countable subadditivity implies the **monotonicity** property, that for each $E_0 \subset E_1 \subset \Omega$ we have $\mu_*(E_0) \leq \mu_*(E_1)$.

Proposition 1.1.5. Lebesgue outer measure is an outer measure or $\mathbb{R}^d$.

Proof. In order to show the empty set property it suffices to observe that the empty cover $\emptyset$ is a cover of $\emptyset$, and that an empty sum equals zero.

In order to prove the countable subadditivity let $\varepsilon > 0$. Then for each $n = 1, 2, ...$ exists a cover $\mathcal{Q}_n$ of E_n such that

$$\sum_{Q \in \mathcal{Q}_n} |Q| \leq \mathcal{L}_*(E)_n + 2^{-n}\varepsilon.$$

Then $\mathcal{Q} := \mathcal{Q}_1 \cup \mathcal{Q}_2 \cup ...$ is a cover of $E_1 \cup E_2 \cup ...$ and thus

$$\begin{aligned} &\mathcal{L}_*\Big(\bigcup_{n=1}^{\infty} E_n\Big) \\ &\leq \sum_{Q \in \mathcal{Q}_1 \cup \mathcal{Q}_2 \cup ...} |Q| = \sum_{n=1}^{\infty} \sum_{Q \in \mathcal{Q}_n} |Q| \\ &\leq \sum_{n=1}^{\infty} (\mathcal{L}_*(E_n) + 2^{-n}\varepsilon) = \varepsilon + \sum_{n=1}^{\infty} \mathcal{L}_*(E_n). \end{aligned}$$

Since $\varepsilon > 0$ was arbitrarily small this finishes the proof. $\square$

For Lebesgue outer measure to represent a reasonable notion of volume, it should be true that if we divide a set into parts, the volumes of the parts should sum up to the volume of the original set. This property is called **additivity** if we ask it to hold for a division into finitely many parts, and **countable additivity** for countably many. In Proposition 1.1.5 we have only proven countable subadditivity for Lebesgue outer measure, i.e. that the volumes of the parts sum up to *at least* the volume of the original set. Unfortunately, we cannot strengthen this to countable additivity. More precisely, we cannot prove that for any sequence $E_1, E_2, \ldots \subset \mathbb{R}^d$ of disjoint sets we have

$$\mathcal{L}_*\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} \mathcal{L}_*(E_n). \qquad (1.1.4)$$

In fact, we will see in Section 1.2.2 that this property can indeed fail if we assume the **axiom of choice**.

1.2 Measurable sets

1.2.1 Carathéodory's theorem

[2]

As we will see, (1.1.4) actually does hold for a vast amount of sets. To determine a sufficient class of those sets we elevate to a more abstract setting.

We say that a collection $\mathcal{M} \subset 2^{\Omega}$ of sets is a **σ-algebra** if

(i) (empty set) $\emptyset \in \mathcal{M}$,

(ii) (complement) for each $E \in \mathcal{M}$ we have $\Omega \setminus E \in \mathcal{M}$, and

(iii) (countable union) for each $E_1, E_0, ... \in \mathcal{M}$ we have
$$\bigcup_{n=1}^{\infty} E_n \in \mathcal{M}.$$

Let $\mathcal{M}$ be a σ-algebra. A set function $\mu : \mathcal{M} \to [0, \infty]$ is called a **measure** if it has the following two properties.

[2]This section follows the more abstract Section 6.1.1 from [SS05].

(i) (empty set) $\mu(\emptyset) = \emptyset$.

(ii) (countable additivity) For all disjoint $E_0, E_1, \ldots \in \mathcal{M}$ we have

$$\mu\Big(\bigcup_{n=0}^{\infty} E_n\Big) = \sum_{n=0}^{\infty} \mu(E_n).$$

A triple $(\Omega, \mathcal{M}, \mu)$ of a set Ω, a σ-algebra $\mathcal{M}$ and a measure μ is called a **measure space**.

Note, that countable additivity is our missing property (1.1.4). That means we call an outer measure μ_* on a selected collection of sets a measure, if on those sets it is not only countable subadditive but countable additive. But how do we find that selected collection? As we will see soon, the following criterion will do.

Given an outer measure μ_*, we say that a set $A \subset \Omega$ is **Carathéodory measurable** if for all $B \subset \Omega$ we have

$$\mu_*(B) = \mu_*(B \cap A) + \mu_*(B \setminus A). \tag{1.2.1}$$

For brevity we will just say **measurable** instead of Carathéodory measurable. As we will see, for Lebesgue outer measure, essentially all the sets that we care about in analysis satisfy this Carathéodory criterion.

Note, that

$$\mu_*(B) \leq \mu_*(B \cap A) + \mu_*(B \setminus A)$$

always holds by subadditivity of an outer measure. That means (1.2.1) is equivalent to

$$\mu_*(B) \geq \mu_*(B \cap A) + \mu_*(B \setminus A).$$

Theorem 1.2.1. *Given an outer measure μ_* on Ω, the set $\mathcal{M}$ of all Carathéodory measurable subsets of Ω forms a σ-algebra.*

Theorem 1.2.2. *Given an outer measure μ_* on Ω, the map μ_* restricted to the set $\mathcal{M}$ of all Carathéodory measurable subsets of Ω is a measure.*

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Proof of Theorems 1.2.1 and 1.2.2. We have to show the following properties: Let $E_1, E_2, \ldots \subset \Omega$ be measurable. Then

(i) $\emptyset$ is measurable,

(ii) $\Omega \setminus E_1$ is measurable,

(iii) $G = E_1 \cup E_2 \cup ...$ is measurable, and

(iv) if $E_1, E_2, ...$ are disjoint then

$$\mu_*(G) = \sum_{n=1}^{\infty} \mu_*(E_n).$$

$E = \emptyset$ satisfies (1.2.1), and since E_1 satisfies (1.2.1) also $\Omega \setminus E$ does. That proves (i) and (ii).

Let $A \subset \Omega$. Then

$$\begin{aligned} \mu(A) &= \mu(A \cap E_1) + \mu(A \setminus E_1) \\ &= \mu(A \cap E_1 \cap E_2) + \mu(A \cap E_1 \setminus E_2) \\ &\quad + \mu(A \setminus E_1 \cap E_2) + \mu(A \setminus (E_1 \cup E_2)) \\ &\geq \mu(A \cap (E_1 \cup E_2)) + \mu(A \setminus (E_1 \cup E_2)). \end{aligned}$$

That means $E_1 \cup E_2$ is measurable, and by induction we can conclude that for any $n \in \mathbb{N}$ the set $G_n := E_1 \cup ... \cup E_n$ is measurable.

Set $\tilde{E}_1 = E_1$ and for each $n \geq 2$ set $\tilde{E}_n = E_n \setminus G_{n-1}$. Then

$$\tilde{E}_n = \Omega \setminus [(\Omega \setminus E_n) \cup G_{n-1}]$$

is measurable and $G_n = \tilde{E}_1 \cup ... \cup \tilde{E}_n$, $G = \tilde{E}_1 \cup \tilde{E}_2 \cup ....$ Then

$$\mu(A \cap G_n) = \mu(A \cap G_n \cap \tilde{E}_n) + \mu(A \cap G_n \setminus \tilde{E}_n)$$
$$= \mu(A \cap \tilde{E}_n) + \mu(A \cap G_{n-1})$$

and by induction we can conclude

$$\mu(A \cap G_n) = \sum_{k=1}^{n} \mu(A \cap \tilde{E}_k).$$

Therefore

$$\mu(A) = \mu(A \cap G_n) + \mu(A \setminus G_n) \geq \sum_{k=1}^{n} \mu(A \cap \tilde{E}_k) + \mu(A \setminus G)$$

and letting $n \to \infty$ we obtain

$$\mu(A) \geq \sum_{k=1}^{\infty} \mu(A \cap \tilde{E}_k) + \mu(A \setminus G)$$
$$\geq \mu(A \cap G) + \mu(A \setminus G) \geq \mu(A).$$

That means the previous inequality is an equality, which implies (iii). If the $E_1, E_2, ...$ are disjoint then $E_n = \tilde{E}_n$. Setting $A = G$ in the previous equality thus implies (iv). □

It follows from the definition that any set $E \subset \Omega$ with $\mu_*(E) = 0$ is measurable as

$$\mu_*(A \cap E) + \mu_*(A \setminus E) = \mu_*(A \setminus E) \leq \mu(A).$$

This property is called that the σ-algebra $\mathcal{M}$ of measurable sets is **complete**.

Lemma 1.2.3. Let $(\Omega, \mathcal{M}, \mu)$ be a measure space and let $E_1, E_2, ... \in \mathcal{M}$. If $E_1 \subset E_2 \subset ...$ then

$$\lim_{n\to\infty} \mu(E_n) = \mu\Big(\bigcup_{n=1}^{\infty} E_n\Big),$$

where both sides may be infinite. If $\mu(E_1) < \infty$ and $E_1 \supset E_2 \supset ...$ then

$$\lim_{n\to\infty} \mu(E_n) = \mu\Big(\bigcap_{n=1}^{\infty} E_n\Big).$$

Proof. Exercise. □

Note, that we ask (1.2.1) to hold for all sets $B \subset \Omega$, which in the end may include sets that do not belong to the collection $\mathcal{M}$ of measurable sets.

By Theorem 1.2.5, Lebesgue outer measure is not a measure on the σ-algebra $2^{\mathbb{R}}$ if we assume the axiom of choice. In the next section we will see that still essentially all the sets that we care about in analysis are Lebesgue measurable.

1.2.2 A non-measurable set

Lemma 1.2.4 (Translation invariance of Lebesgue measure)**.** Let $E \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$ and denote

$$E + x = \{y + x : y \in E\}.$$

Then

$$\mathcal{L}_*(E + x) = \mathcal{L}_*(E).$$

Proof. It suffices to show $\mathcal{L}_*(E + x) \leq |E|$ because from that we also obtain $\mathcal{L}_*(E) = \mathcal{L}_*(E + x - x) \leq \mathcal{L}_*(E + x)$.

For any $\varepsilon > 0$ exists a cover $\mathcal{Q}$ of E with

$$\mathcal{L}_*(E) \leq \varepsilon + \sum_{Q \in \mathcal{Q}} |Q|.$$

Then $\tilde{\mathcal{Q}} = \{Q + x : Q \in \mathcal{Q}\}$ is a cover of $E + x$

and thus since $|Q - x| = |Q|$ we obtain

$$\mathcal{L}_*(E + x) \leq \sum_{\tilde{Q} \in \tilde{\mathcal{Q}}} |\tilde{Q}| = \sum_{\tilde{Q} \in \tilde{\mathcal{Q}}} |\tilde{Q} - x| = \sum_{Q \in \mathcal{Q}} |Q|$$
$$\leq \mathcal{L}_*(E) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitraly small we obtain $\mathcal{L}_*(E + x) \leq |E|$ and finish the proof. (Exercise?) □

Theorem 1.2.5. *Assume the axiom of choice holds. Then there exist disjoint sets $E_0, E_1, ... \subset \mathbb{R}$ for which (1.1.4) fails.*

Proof. For $x, y \in [0, 1]$ denote $x \sim y$ if $x - y$ is a rational number. Then $\sim$ is an equivalence relation. (Exercise here?) That means there is a decomposition $\mathcal{A}$ of $[0, 1]$, i.e. the union

$$[0, 1] = \bigcup \mathcal{A}$$

is disjoint, such that for any $x, y \in [0, 1]$ we have $x \sim y$ if and only if x and y belong to the same set $A \in \mathcal{A}$. By the axiom of choice there exists a set E that contains exactly one element from each set $A \in \mathcal{A}$. That means for each $x \in [0, 1]$ exists a $y \in E$ and a rational $q \in \mathbb{Q} \cap [-1, 1]$

such that $x = y + q$. Moreover, for each $x \in E$ and $q \in \mathbb{Q}$ we have $x + q \notin E$. We can conclude that the sets in $\{E + q : q \in \mathbb{Q} \cap [-1, 1]\}$ form a countable disjoint cover of $[0, 1]$ and belong to $[-1, 2]$.

By Lemma 1.2.4 we have $\mathcal{L}_*(E+q) = \mathcal{L}_*(E)$. Then by

$$1 = \mathcal{L}_*([0,1]) \leq \sum_{q\in\mathbb{Q}\cap[0,1]} \mathcal{L}_*(E+q) = \sum_{q\in\mathbb{Q}\cap[0,1]} \mathcal{L}_*(E).$$

we must have $\mathcal{L}_*(E) > 0$. This however means

$$\begin{aligned}\sum_{q\in\mathbb{Q}\cap[0,1]} \mathcal{L}_*(E+q) &= \sum_{q\in\mathbb{Q}\cap[0,1]} \mathcal{L}_*(E) = \infty \\ &> 3 \geq \mathcal{L}_*([-1,2]) \\ &\geq \mathcal{L}_*\Big(\bigcup_{q\in\mathbb{Q}\cap[0,1]} E+q\Big),\end{aligned}$$

i.e. (1.1.4) fails. □

1.2.3 Metric measures

Our definition of measurability works in a very general setting of a mere set Ω. Since the fundamental domain in this course is $\mathbb{R}^d$ we allow ourselves to a ssume a bit more structure. Given a set Ω a map $\mathrm{d} : \Omega \times \Omega \to [0, \infty)$ is called a **metric** if

(i) for all $x \in \Omega$ we have $\mathrm{d}(x, x) = 0$,

(ii) (symmetry) for all $x, y \in \Omega$ we have $\mathrm{d}(x, y) = \mathrm{d}(y, x)$, and

(iii) (triangle inequality) for all $x, y, z \in \Omega$ we have $\mathrm{d}(x, z) \leq \mathrm{d}(x, y) + \mathrm{d}(y, z)$.

The pair (Ω, d) is called a **metric space**.

Given a metric d, we can define an **(open) ball** centered in $x \in \Omega$ with radius $r > 0$,

$$B(x, r) = \{y \in \Omega : \mathrm{d}(x, y) < r\}.$$

We say that a set $A \subset \Omega$ is **open** if for every $x \in A$ exists an $r > 0$ such that $B(x, r) \subset A$. We say that $A \subset \Omega$ is **closed** if $\Omega \setminus A$ is open. We define the **Borel σ-algebra** $\mathscr{B}_\Omega$ to be the smallest σ algebra that contains all open sets $A \subset \Omega$ (Exercise). Its members $E \in \mathscr{B}_\Omega$ are called **Borel** sets. We want to show that all Borel subsets of $\mathbb{R}^d$ are Lebesgue measurable. This will be a consequence of the fact that Lebesgue outer measure is a metric outer measure.

We extend the metric to $\mathrm{d} : 2^\Omega \times 2^\Omega \to [0, \infty)$ by defining

$$\mathrm{d}(A, B) = \inf\{\mathrm{d}(x, y) : x \in A,\ y \in B\}.$$

An outer measure μ_* is called a **metric outer measure** if for all $A, B \subset \Omega$ with $\mathrm{d}(A, B) > 0$ we have

$$\mu_*(A \cup B) = \mu_*(A) + \mu_*(B).$$

Lemma 1.2.6. Lebesgue outer measure is a metric outer measure.

Proof. Let $A_0, A_1 \subset \mathbb{R}^d$ with $\delta := \mathrm{d}(A_0, B_0) > 0$. It suffices to prove

$$\mathcal{L}(A_0 \cup A_1) \geq \mathcal{L}(A_0) + \mathcal{L}(A_1).$$

Let $\varepsilon > 0$. Then there exists a countable cover $\mathcal{Q}$ of $A_0 \cup A_1$ with closed cubes such that

$$\sum_{Q \in \mathcal{Q}} |Q| \leq \mathcal{L}(A_0 \cup A_1) + \varepsilon.$$

There is a subdivision $\tilde{\mathcal{Q}}$ of the cubes in $\mathcal{Q}$ in cubes with diameter less than δ and. That means the cover $\bigcup \mathcal{Q}$ and by Lemma 1.1.1 we have

$$\sum_{Q \in \mathcal{Q}} |Q| = \sum_{Q \in \tilde{\mathcal{Q}}} |Q|.$$

Since $\mathrm{d}(A_0, A_1) = \delta$, any $Q \in \tilde{\mathcal{Q}}$ cannot intersect both A_0 and A_1. That means the sets $\tilde{\mathcal{Q}}_i = \{Q \in \tilde{\mathcal{Q}} : Q \cap A_i \neq \emptyset\}$ are disjoint for $i = 0, 1$. Moreover, $\tilde{\mathcal{Q}}_i$ is a cover of A_i. We can conclude

$$\begin{aligned}\mathcal{L}(A_0 \cup A_1) + \varepsilon \geq \sum_{Q \in \tilde{\mathcal{Q}}} |Q| &\geq \sum_{Q \in \tilde{\mathcal{Q}}_0} |Q| + \sum_{Q \in \tilde{\mathcal{Q}}_1} |Q| \\ &\geq \mathcal{L}(A_0) + \mathcal{L}(A_1).\end{aligned}$$

Since $\varepsilon > 0$ was arbitrarily small this finishes the proof. (Exercise with subdivision hint?) □

Theorem 1.2.7. *Let μ_* be a metric outer measure. Then all Borel sets are μ_*-measurable.*

Proof. By the definition of Borel sets and Theorem 1.2.1 it suffices to show that all closed sets are μ_*-measurable. To that end it suffices to show that for all $A \subset \Omega$ and all closed $B \subset \Omega$ we have

$$\mu_*(A) \geq \mu_*(A \cap B) + \mu_*(A \setminus B). \qquad (1.2.2)$$

For $n \in \mathbb{N}$ denote

$$A_n = \{x \in A \setminus B : \mathrm{d}(B, \{x\}) \geq 1/n\}.$$

Since B is closed we have $A = A_1 \cup A_2 \cup \ldots$. Moreover, for each n we have $\mathrm{d}(B, A_n) \geq 1/n$ and since μ_* is a metric outer measure this means

$$\mu_*(A) \geq \mu_*(A \cap B \cup A_n) = \mu_*(A \cap B) + \mu_*(A_n). \tag{1.2.3}$$

Set $C_n = A_{n+1} \setminus A_n$. Then

$$\mathrm{d}(C_{n+1}, A_n) \geq \frac{1}{n(n+1)}.$$

Since μ_* is a metric outer measure, by induction this implies

$$\mu_*(A) \geq \mu_*\Big(\bigcup_{k=1}^{n} C_{2k}\Big) = \sum_{k=1}^{n} \mu_*(C_{2k})$$

and similarly

$$\mu_*(A) \geq \mu_*\Big(\bigcup_{k=1}^{n} C_{2k-1}\Big) = \sum_{k=1}^{n} \mu_*(C_{2k-1}).$$

Since $\mu_*(A) < \infty$ (why?) we can conclude

$$\sum_{k=1}^{\infty} \mu_*(C_k) < \infty$$

Therefore,

$$\mu_*(A_n) \leq \mu_*(A \setminus B) \leq \mu_*(A_n) + \sum_{k=n+1}^{\infty} \mu_*(C_k).$$

Letting $n \to \infty$ this means $\mu_*(A_n) \to \mu_*(A \setminus B)$ and thus from (1.2.3) we can conclude (1.2.2) and finish the proof. □

An measure for which all open, or, equivalently, all Borel sets, are measurable is also called a **Borel measure**.

Proposition 1.2.8. Let μ be a Borel measure which is finite for all balls with finite radius. Then for any Borel set E and any $\varepsilon > 0$ exists an open set $U \supset E$ and a closed set $C \subset E$ such that

$$\mu(U \setminus E) < \varepsilon, \qquad \mu(E \setminus C) < \varepsilon.$$

Proof. First consider the case that E is a countable union of closed sets $E = C_1 \cup C_2 \cup ....$ Since finite unions of closed sets are closed, it suffices to consider the case that the sequence is increasing, i.e. $C_1 \subset C_2 \subset ....$ Take any $x \in \Omega$. Let $\varepsilon > 0$ and $n \in \mathbb{N}$. Then since $\mu(\overline{B(x,n)} \setminus B(x,n-1))$ is finite by assumption, by Lemma 1.2.3 there exists an $N(n)$ such that

$$\mu\big([E \setminus C_{N(n)}] \cap [\overline{B(x,n)} \setminus B(x,n-1)]\big) \leq 2^{-n}\varepsilon.$$

Since $C_{N(n)}$ and $\overline{B(x,n)} \setminus B(x,n-1))$ are closed in $\overline{B(x,n)} \setminus B(x,n-1))$, also the set

$$C = \bigcup_{n=1}^{\infty} C_{N(n)} \cap \overline{B(x,n)} \setminus B(x,n-1)$$

is closed and satisfies

$$\mu(E \setminus C) \leq \sum_{n=1}^{\infty} 2^{-n}\varepsilon = \varepsilon.$$

We now proceed essentially by induction. Denote by $\mathcal{M}$ the set of sets E that satisfy the conclusion of the proposition. It suffices to show that $\mathcal{M}$ is a σ-algebra which contains all open sets. So, let U be open. Then we can write U as a countable union of closed sets,

$$U = \bigcup_{n=1}^{\infty} \{x \in U : \mathrm{d}(\{x\}, \Omega \setminus U, \geq) 1/n\}.$$

By the previous argument we can conclude $U \in \mathcal{M}$.

Now we show that $\mathcal{M}$ is a σ-algebra. Since open and closed sets are complements it follows that $\mathcal{M}$ is closed under complement. It remains to show that $\mathcal{M}$ is closed under countable union. So, let $E_1, E_2, ... \in \mathcal{M}$ and set $E = E_1 \cup E_2 \cup ...$ and let $\varepsilon > 0$. By inductive assumption for each $n \in \mathbb{N}$ exists an open set U_n and a closed set C_n with $\mu(U_n \setminus E_n) < 2^{-n}\varepsilon$ and $\mu(E_n \setminus C_n) < 2^{-n}\varepsilon$. That means $U = U_1 \cup U_2 \cup ...$ and $K = C_1 \cup C_2 \cup ...$ satisfy

$$\mu(U \setminus E) \leq \sum_{n=1}^{\infty} \mu(U_n \setminus E_n) \leq \sum_{n=1}^{\infty} 2^{-n}\varepsilon = \varepsilon,$$

$$\mu(E \setminus K) \leq \sum_{n=1}^{\infty} \mu(E_n \setminus C_n) \leq \sum_{n=1}^{\infty} 2^{-n}\varepsilon = \varepsilon.$$

The set U is open, but K might not be. However, K is a countable union of open sets and thus by the first argument there exists a closed set $C \subset K$ with $\mu(K \setminus C) < \varepsilon$. We can conclude

$$\mu(E \setminus C) \leq \mu(E \setminus K) + \mu(K \setminus C) \leq 2\varepsilon$$

and finish the proof. □

Such a measure for which all measurable sets can be approximated from within by closed sets is called **inner regular**. If all measurable sets can be approximated from the outside by open sets then the measure is called **outer regular**. An inner and outer regular Borel measure is called a **Radon measure**.

Lemma 1.2.9. Let μ be Borel measure for which all balls have finite measure and such that for each measurable set E we have

$$\mu(E) = \inf\{\mu(U) : U \text{ open}, \ E \subset U\}.$$

Then for every measurable $E \subset \mathbb{R}^d$ we have

$$\mu(E) = \sup\{\mu(C) : C \text{ closed}, \ C \subset E\}.$$

Proof. Similarly to the previous proof it suffices to consider subsets E of a closed ball B. By assumption exists a sequence $U_1, U_2, \dots$ of open sets with $U_n \supset B \setminus E$ and $\mu(B \cap U_n) \to \mu(B \setminus E)$. Thus, we have

$$\mu(B) = \mu(B \cap U_n) + \mu(B \setminus U_n) \to \mu(B \setminus E) + \mu(E).$$

Thus, the closed sets $B \setminus U_n$ approximate E as desired. ☐

Corollary 1.2.10. For each Lebesgue measurable set E exists a Borel set B and sets N, M with Lebesgue outer measure zero such that $E = (B \setminus N) \cup M$.

Conversely, each such set $(B \setminus N) \cup M$ is Lebesgue measurable.

Proof. (Exercise) ☐

In this sense the set of Lebesgue measurable sets is the completion of the Borel σ-algebra with respect to sets with Lebesgue outer measure zero.

1.3 Measurable functions

1.3.1 Definition and extent of the class

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space. We consider functions with values in the extended real line, $f : \Omega \to [-\infty, \infty]$, i.e. real valued functions that can also attain the values $\pm\infty$. Such a function f is called **(μ-)measurable** if for every $a \in \mathbb{R}$ the set

$$\{f < a\} := f^{-1}([-\infty, a)) := \{x \in \Omega : f(x) \in [-\infty, a)\}$$

is (μ-)measurable.

Lemma 1.3.1. Let $f : \Omega \to [-\infty, \infty]$. Then following are equivalent to f being measurable

(i) For every $a \in \mathbb{R}$ the set $\{f \geq a\}$ is measurable.

(ii) For every $a \in \mathbb{R}$ the set $\{f < a\}$ is measurable.

(iii) For every $a \in \mathbb{R}$ the set $\{f \leq a\}$ is measurable.

(iv) The function $-f$ is measurable.

If $f : \Omega \to (-\infty, \infty)$ then measurability is also equivalent to each of the following:

(v) For every $a, b \in \mathbb{R}$ the set $\{a < f < b\}$ is measurable. Equivalently we can replace $<$ by $\leq$ in either instance.

(vi) For every open $U \in \mathbb{R}$ the set $f^{-1}(U)$ is measurable.

(vii) For every closed $C \in \mathbb{R}$ the set $f^{-1}(C)$ is measurable.

The latter also apply to functions $f : \Omega \to [-\infty, \infty]$ if in addition we require $f^{-1}(\{-\infty\})$ and $f^{-1}(\{\infty\})$ to be measurable.

Proof. Exercise □

Lemma 1.3.2. Let $f : \Omega \to (-\infty, \infty)$ be continuous. Then f is measurable.

Lemma 1.3.3. Let $f_1, f_2, ... : \Omega \to [-\infty, \infty]$ be measurable. Then the following functions are continuous

(i) $x \mapsto \sup_n f_n(x)$

(ii) $x \mapsto \inf_n f_n(x)$

(iii) $x \mapsto \limsup_n f_n(x)$

(iv) $x \mapsto \liminf_n f_n(x)$

(v) If $(f_n)_n$ converges pointwise, then $x \mapsto \lim_n f_n(x)$ is measurable.

Proof. Exercise. □

Lemma 1.3.4. Let $f : \Omega \to (-\infty, \infty)$ be measurable and $g : \mathbb{R} \to \mathbb{R}$ be continuous. Then $g \circ f$ is measurable

Proof. Exercise. □

Lemma 1.3.5. Let $f, g : \Omega \to (-\infty, \infty)$ be measurable. Then $f + g$ and fg are measurable.

We say that a statement that involves $x \in \Omega$ holds **(μ-)almost everywhere** if the set of all $x \in \Omega$ for which the statement fails is μ-measurable and has zero μ-measure. For example, given $f, g : \Omega \to [-\infty, \infty]$, we say that $f(x) = g(x)$ for μ-almost everywhere x if $\mu(\{x \in \Omega : f(x) \neq g(x)\}) = 0$.

Lemma 1.3.6. Let $f, g : \Omega \to [-\infty, \infty]$ such that f is measurable and $f(x) = g(x)$ for μ-almost every x. Then g is μ-measurable.

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