

Real Analysis
Lecture notes ICTP 2025

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Preliminaries

This course assumes familiarity with basic notions from

- functions, such as injectivity, bijectivity, images, and preimages,
- topology on $\mathbb{R}^d$, such as closed and open sets,
- analysis on the real line, such as sequences, series, limits, $\liminf$

and $\limsup$,

- calculus on the real line, such as the chain and product rule for derivatives, and the Riemann integral.

We recall the following notions and notations that are particularly important for the course.

Sets and set operations We denote by $\mathbb{N}$ the **natural numbers** $\mathbb{N} := \{1, 2, 3, \dots\}$ and by $\mathbb{R}$ the **real numbers**. For two sets A and B their **union** $A \cup B$ consists of all points x that belong to A or to B . Their **intersection** $A \cap B$ consists of all points that belong to both A and B . For sets A_n that are indexed by for example by natural numbers $n \in \mathbb{N}$ in the case of a sequence $A_1, A_2, \dots$, we denote by

$$\bigcup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup \dots$$

their union, i.e. the set of all points that for any n belong to A_n . More generally, if $\mathcal{A}$ is a collection of sets A , we denote by

$$\bigcup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A$$

the set of all points x for which there exists an $A \in \mathcal{A}$ with $x \in A$. The **set difference** $A \setminus B$ consists of all points that belong to A and not to B . Two sets A, B are **disjoint** if $A \cap B = \emptyset$. We say that a collection of sets $\mathcal{A}$ is disjoint, if any two $A, B \in \mathcal{A}$ with $A \neq B$ are disjoint.

For two sets A, B we define their product $A \times B$ as the set of all pairs with the first element from A and the second element from B , i.e.

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

More generally,

$$A_1 \times \dots \times A_n = \{(a_1, \dots, a_n) : \forall k = 1, \dots, n \ a_k \in A_k\}.$$

A set A is **countable** if there exists a surjection $f : \mathbb{N} \rightarrow A$, i.e. for each $a \in A$ exists an $n \in \mathbb{N}$ with $f(n) = a$. We also write $a_n := f(n)$.

Euclidean space The basic space where our study takes place is Euclidean space, that is, for any natural number $d \in \mathbb{N}$, the space $\mathbb{R}^d$, which consist of all d -tuples $x = (x_1, \dots, x_d)$ of real numbers $x_n \in \mathbb{R}$ with $n = 1, \dots, d$.

We assign to each point $x \in \mathbb{R}^d$ its (Euclidean) norm

$$|x| = (x_1^2 + \dots + x_d^2)^{\frac{1}{2}}.$$

We can add and substract points $x, y \in \mathbb{R}^d$ componentwise,

$$\begin{aligned}x + y &= (x_1 + y_1, \dots, x_d + y_d) \\x - y &= (x_1 - y_1, \dots, x_d - y_d).\end{aligned}$$

Euclidean distance The (Euclidean) distance between them is $|x - y|$. The most elementary subsets of $\mathbb{R}$ we consider are open and

closed intervals. For $a, b \in \mathbb{R}$ denote by $[a, b]$ the set of all $x \in \mathbb{R}$ with $a \leq x \leq b$ and by (a, b) the set of all $x \in \mathbb{R}$ with $a < x < b$. In Euclidean space with larger dimensions d those sets generalize to rectangles and balls.

Definition 0.0.1. For $a, b \in \mathbb{R}$ such that for $n = 1, \dots, d$ we have $a_n \leq b_n$, the closed and open rectangles that have a and b as opposite corners are

$$(a_1, b_1) \times \dots \times (a_d, b_d), \quad [a_1, b_1] \times \dots \times [a_d, b_d].$$

By **rectangle** we always mean a closed rectangle unless specified otherwise.

The (open) **ball** with center $x \in \mathbb{R}^d$ and radius $r > 0$ consist of those $y \in \mathbb{R}^d$ with $|x - y| < r$ and is denoted by $B(x, r)$. The corresponding closed ball $\overline{B}(x, r)$ consist of those $y \in \mathbb{R}^d$ with $|x - y| \leq r$.

Let $A \subset \mathbb{R}^d$. A point $x \in \mathbb{R}^d$ is an **interior point** of A if there exists an $r > 0$ with $B(x, r) \subset A$. A point $x \in \mathbb{R}^d$ is a **limit point** of A if for every $r > 0$ exists a $y \in A$ with $|x - y| < r$. We denote by $\overset{\circ}{A}$ the **interior** of A , the set of all interior points of A . We denote by $\overline{A}$ the **closure** of A , the set of all limit points of A . We denote by

$$\partial A := \overline{A} \setminus \overset{\circ}{A}$$

the **boundary** of A . By this definition, the interior of an open or closed ball is the corresponding open ball, and its closure is the corresponding closed ball. The same is true for rectangles.

The extended real line is the set $\mathbb{R} \cup \{-\infty, \infty\}$. We partially extend addition and multiplication from $\mathbb{R}$ to the extended real line by defining

$$\forall x \in \mathbb{R} \cup \{\infty\} : x + \infty := \infty \qquad \forall x > 0 : x \cdot \infty := \infty.$$

We further extend this by prescribing commutativity and associativity and multiplying both definitions with -1 . This only leaves $\infty - \infty$ and $0 \cdot \infty$ undefined. In this sense we can treat a statement like

$$\lim_{n \rightarrow \infty} a_n = \infty$$

as an equality on the extended real line. We also extend the relations $<, \leq, >, \geq$ to the extended real line via

$$\forall x \in \mathbb{R} \cup \{-\infty\} : x < \infty, \quad \infty = \infty,$$

with the corresponding definitions for $-\infty$.

Recall also

$$\inf \emptyset = \infty, \quad \sup \emptyset = -\infty.$$

Convergent sums Let $a_1, a_2, \dots \geq 0$. Then their sum does not depend on the order of summation, i.e. for any bijection $\sigma : \mathbb{N} \rightarrow \mathbb{N}$

we have

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\sigma(n)}.$$

Here, both sides of the equality may be infinite. The same conclusion is true if $a_1, a_2, \dots \in \mathbb{R}$ and

$$\sum_{n=1}^{\infty} |a_n| < \infty.$$

For a countable set $A = \{a_1, a_2, \dots\} \subset [0, \infty]$ this allows for the notation

$$\sum_{a \in A} a = \sum_{n=1}^{\infty} a_n.$$

Chapter 1

Measure Theory

The main textbook sources are [SS05] and [EG15]. Other inspirational material are the lecture notes in real analysis by Emanuel

Carneiro and the lecture notes in measure theory [Kin24] and real analysis [Kin25] by Juha Kinnunen.

1.1 Lebesgue outer measure

1

Our first goal is to rigorously assign a volume to subsets of $\mathbb{R}^d$. A set whose volume we already know is the rectangle: For $a, b \in \mathbb{R}^d$ the volume of the rectangle $R = (a_1, b_1) \times \dots \times (a_d, b_d)$ is the product of its side lengths,

$$|R| = (b_1 - a_1) \cdot \dots \cdot (b_d - a_d). \quad (1.1.1)$$

The corresponding closed rectangle has the same volume.

¹This section follows Sections 1.1.1 and 1.1.2 from [SS05] for the construction of the Lebesgue measure.

Definition 1.1.1. We say that a collection $\mathcal{R}$ of rectangles is **almost disjoint** if for any two $R_0, R_1 \in \mathcal{R}$ with $R_0 \neq R_1$ their interiors $\overset{\circ}{R}_0$ and $\overset{\circ}{R}_1$ are disjoint.

Lemma 1.1.2. Let $n \in \mathbb{N}$ and let $R_1, \dots, R_n$ be almost disjoint rectangles such that

$$R = R_1 \cup \dots \cup R_n$$

is a rectangle, too. Then

$$|R| = \sum_{k=1}^n |R_k|.$$

By our intuition about volumes this is clearly true. But it requires a proof because all we know so far about our mathematical notion of volume is the abstract formula (1.1.1).

Proof. We first consider the case that the rectangles form a grid, that is, for each $l = 1, \dots, d$ there are $a_l = a_l^0 < \dots < a_l^{N_d} = b_l$ such

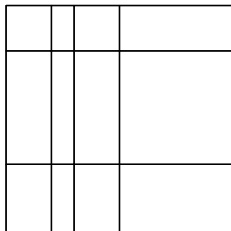


Figure 1.1: The grid case.

that each rectangle R_k is of the form $[a_1^i, a_1^{i+1}] \times \dots \times [a_d^j, a_d^{j+1}]$. Then

$$\begin{aligned}
|R| &= \prod_{l=1}^d (b_l - a_l) = \prod_{l=1}^d \sum_{i=1}^{N_d} (a_l^i - a_l^{i-1}) \\
&= \sum_{i_1=1}^{N_1} \dots \sum_{i_d=1}^{N_d} \prod_{l=1}^d (a_l^{i_l} - a_l^{i_l-1}) = \sum_{k=1}^n |R_k|. \tag{1.1.2}
\end{aligned}$$

For the general case, we subdecompose each rectangle R_i via the extended faces of the rectangles $R_1, \dots, R_n$ into rectangles $R_i = R_i^1 \cup \dots \cup R_i^{N_i}$. This subdecomposition is a grid which means that by the previous case

$$|R_i| = \sum_{j=1}^{N_i} |R_i^j|.$$

Moreover, the rectangles $\{R_i^j : j = 1, \dots, N_i, i = 1, \dots, n\}$ form a grid

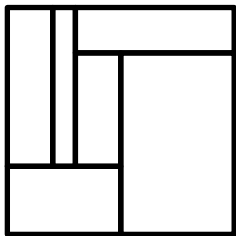


Figure 1.2: The general case.

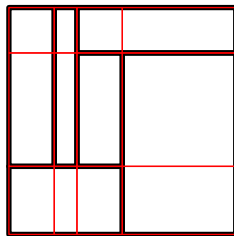


Figure 1.3: Subdecomposition into grid.

for R , so that

$$|R| = \sum_{i=1}^n \sum_{j=1}^{N_i} |R_i^j| = \sum_{i=1}^n |R_i|. \quad (1.1.3)$$

□

Lemma 1.1.3. Let $n \in \mathbb{N}$ and let $R, R_1, \dots, R_n$ be rectangles with

$$R \subset R_1 \cup \dots \cup R_n$$

Then

$$|R| \leq \sum_{k=1}^n |R_k|.$$

Note, that Lemma 1.1.3 also holds if $R_1, \dots, R_n$ are open, because $R_i \subset \overline{R_i}$ and $|\overline{R_i}| = |R_i|$.

Proof. This follows by the same proof as Lemma 1.1.2, except that some rectangles of the grid that decomposes R may belong to more

than one of the rectangles $R_1, \dots, R_n$. More precisely, the last equality in (1.1.2) and the first equality in (1.1.3) become inequalities. $\square$

We want to use a similar idea of writing a set in terms of sets whose volume we know in order to define its volume.

Definition 1.1.4. A **cube** $Q \subset \mathbb{R}^d$ is a rectangle whose sidelengths are all identical, i.e. for $a_1, \dots, a_d \in \mathbb{R}^d$ and $r > 0$ it is of the form

$$Q = (a_1, a_1 + r) \times \dots \times (a_d, a_d + r).$$

Its volume thus is $|Q| = r^d$.

Definition 1.1.5. For any set $E \subset \mathbb{R}^d$ we define its **outer Lebesgue measure** by

$$\mathcal{L}_*(E) = \inf \left\{ \sum_{Q \in \mathcal{Q}} |Q| : \right.$$

$\mathcal{Q}$ is a countable set of closed cubes

with $E \subset \bigcup \mathcal{Q}\}$.

Lemma 1.1.6. For each closed cube Q we have

$$\mathcal{L}_*(Q) = |Q|.$$

Proof. Since $\{Q\}$ is a cover of Q we have $\mathcal{L}_*(Q) \leq |Q|$. For the reverse inequality let $\varepsilon > 0$. Then there exists a countable cover $\mathcal{Q}$ of Q such that

$$\sum_{P \in \mathcal{Q}} |P| \leq \mathcal{L}_*(Q) + \varepsilon.$$

Let $\delta > 0$ and for each $P \in \mathcal{Q}$ denote by $\tilde{P}$ the open cube with the same center as P and volume $1 + \delta$ times the volume of P . Then $\tilde{P} \supset P$ which means that $\tilde{\mathcal{P}} := \{\tilde{P} : P \in \mathcal{Q}\}$ is an open cover of the compact set Q and thus has a finite subcover $\mathcal{P}$. By Lemma 1.1.3 we can conclude

$$|Q| \leq \sum_{\tilde{P} \in \mathcal{P}} |\tilde{P}| = (1 + \delta) \sum_{\tilde{P} \in \mathcal{P}} |P|$$

$$\leq (1 + \delta) \sum_{P \in Q} |P| \leq (1 + \delta)(\mathcal{L}_*(Q) + \varepsilon).$$

Since $\varepsilon, \delta > 0$ were arbitrarily small we can conclude $|Q| \leq \mathcal{L}_*(Q)$ and finish the proof. $\square$

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Remark 1.1.7. We need to allow countable sequences of cubes in the definition of the outer measure. If we only allowed finite sequences then unbounded sets would always have infinite outer Lebesgue measure. However, sets like $\mathbb{N} \subset \mathbb{R}$ should have zero volume.

For a set Ω denote by 2^Ω the set of all subsets of Ω .

Definition 1.1.8. We say that a set function $\mu_* : 2^\Omega \rightarrow [0, \infty]$ is an **outer measure** if it has the following two properties:

- (i) (empty set) $\mu_*(\emptyset) = 0$.

- (ii) (countable subadditivity) For each $E, E_1, E_2, \dots \subset \Omega$ with $E \subset E_1 \cup E_2 \cup \dots$ we have

$$\mu_*(E) \leq \sum_{n=1}^{\infty} \mu_*(E_n).$$

Observe that countable subadditivity implies the **monotonicity** property, that for each $E_0 \subset E_1 \subset \Omega$ we have $\mu_*(E_0) \leq \mu_*(E_1)$.

Proposition 1.1.9. Lebesgue outer measure is an outer measure on $\mathbb{R}^d$.

Proof. In order to show the empty set property it suffices to observe that the empty cover $\emptyset$ is a cover of $\emptyset$, and that an empty sum equals zero. In order to prove the countable subadditivity let $\varepsilon > 0$. Then for each $n = 1, 2, \dots$ exists a cover $\mathcal{Q}_n$ of E_n such that

$$\sum_{Q \in \mathcal{Q}_n} |Q| \leq \mathcal{L}_*(E_n) + 2^{-n} \varepsilon.$$

Then $\mathcal{Q} := \mathcal{Q}_1 \cup \mathcal{Q}_2 \cup \dots$ is a cover of $E_1 \cup E_2 \cup \dots$ and thus

$$\begin{aligned} & \mathcal{L}_*\left(\bigcup_{n=1}^{\infty} E_n\right) \\ & \leq \sum_{Q \in \mathcal{Q}_1 \cup \mathcal{Q}_2 \cup \dots} |Q| = \sum_{n=1}^{\infty} \sum_{Q \in \mathcal{Q}_n} |Q| \\ & \leq \sum_{n=1}^{\infty} (\mathcal{L}_*(E_n) + 2^{-n}\varepsilon) = \varepsilon + \sum_{n=1}^{\infty} \mathcal{L}_*(E_n). \end{aligned}$$

Since $\varepsilon > 0$ was arbitrarily small this finishes the proof. $\square$

For Lebesgue outer measure to represent a reasonable notion of volume, it should be true that if we divide a set into parts, the volumes of the parts should sum up to the volume of the original set. This property is called **additivity** if we ask it to hold for a division into finitely many parts, and **countable additivity** for countably

many. In Proposition 1.1.9 we have only proven countable subadditivity for Lebesgue outer measure, i.e. that the volumes of the parts sum up to *at least* the volume of the original set. Unfortunately, we cannot strengthen this to countable additivity. More precisely, we cannot prove that for any sequence $E_1, E_2, \dots \subset \mathbb{R}^d$ of disjoint sets we have

$$\mathcal{L}_*\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mathcal{L}_*(E_n). \quad (1.1.4)$$

In fact, we will see in Section 1.2.2 that this property can indeed fail if we assume the **axiom of choice**.

1.2 Measurable sets

1.2.1 Carathéodory's theorem

2

As we will see, (1.1.4) actually does hold for a vast amount of sets. To determine a sufficient class of those sets we elevate to a more abstract setting.

Definition 1.2.1. We say that a collection $\mathcal{M} \subset 2^\Omega$ of sets is a **σ -algebra** if

- (i) (empty set) $\emptyset \in \mathcal{M}$,
- (ii) (complement) for each $E \in \mathcal{M}$ we have $\Omega \setminus E \in \mathcal{M}$, and

²This section follows the more abstract Section 6.1.1 from [SS05].

(iii) (countable union) for each $E_1, E_0, \dots \in \mathcal{M}$ we have

$$\bigcup_{n=1}^{\infty} E_n \in \mathcal{M}.$$

Definition 1.2.2. Let $\mathcal{M}$ be a σ -algebra. A set function $\mu : \mathcal{M} \rightarrow [0, \infty]$ is called a **measure** if it has the following two properties.

(i) (empty set) $\mu(\emptyset) = 0$.

(ii) (countable additivity) For all disjoint $E_0, E_1, \dots \in \mathcal{M}$ we have

$$\mu\left(\bigcup_{n=0}^{\infty} E_n\right) = \sum_{n=0}^{\infty} \mu(E_n).$$

A triple $(\Omega, \mathcal{M}, \mu)$ of a set Ω , a σ -algebra $\mathcal{M}$ and a measure μ is called a **measure space**. Given $E \subset \Omega$ we also say that E is $(\mu$ -)measurable if $E \in \mathcal{M}$.

Note, that countable additivity is our missing property (1.1.4). That means we call an outer measure μ_* on a selected collection of sets a measure, if on those sets it is not only countable subadditive but countable additive. But how do we find that selected collection? As we will see soon, the following criterion will do.

Definition 1.2.3. Given an outer measure μ_* , we say that a set $A \subset \Omega$ is **Carathéodory measurable** if for all $B \subset \Omega$ we have

$$\mu_*(B) = \mu_*(B \cap A) + \mu_*(B \setminus A). \quad (1.2.1)$$

For brevity we will just say **measurable** instead of Carathéodory measurable. As we will see, for Lebesgue outer measure, essentially all the sets that we care about in analysis satisfy this Carathéodory criterion.

Note, that we ask (1.2.1) to hold for all sets $B \subset \Omega$, which in the end may include sets that do not belong to the collection $\mathcal{M}$ of measurable sets.

Note, that

$$\mu_*(B) \leq \mu_*(B \cap A) + \mu_*(B \setminus A)$$

always holds by subadditivity of an outer measure. That means (1.2.1) is equivalent to

$$\mu_*(B) \geq \mu_*(B \cap A) + \mu_*(B \setminus A).$$

Theorem 1.2.4. *Given an outer measure μ_* on Ω , the set $\mathcal{M}$ of all Carathéodory measurable subsets of Ω forms a σ -algebra.*

Theorem 1.2.5. *Given an outer measure μ_* on Ω , the map μ_* restricted to the set $\mathcal{M}$ of all Carathéodory measurable subsets of Ω is a measure.*

Proof of Theorems 1.2.4 and 1.2.5. We have to show the following properties: Let $E_1, E_2, \dots \subset \Omega$ be measurable. Then

(i) $\emptyset$ is measurable,

- (ii) $\Omega \setminus E_1$ is measurable,
- (iii) $G = E_1 \cup E_2 \cup \dots$ is measurable, and
- (iv) if $E_1, E_2, \dots$ are disjoint then

$$\mu_*(G) = \sum_{n=1}^{\infty} \mu_*(E_n).$$

Items (i) to (iii) imply Theorem 1.2.4, and (iv) together with the fact that μ_* is an outer measure implies Theorem 1.2.5.

$E = \emptyset$ satisfies (1.2.1), and since E_1 satisfies (1.2.1) also $\Omega \setminus E$ does. That proves (i) and (ii).

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Let $A \subset \Omega$. Then

$$\mu(A) = \mu(A \cap E_1) + \mu(A \setminus E_1)$$

$$\begin{aligned}
&= \mu(A \cap E_1 \cap E_2) + \mu(A \cap E_1 \setminus E_2) \\
&\quad + \mu((A \setminus E_1) \cap E_2) + \mu(A \setminus (E_1 \cup E_2)) \\
&\geq \mu(A \cap (E_1 \cup E_2)) + \mu(A \setminus (E_1 \cup E_2)).
\end{aligned}$$

That means $E_1 \cup E_2$ is measurable, and by induction we can conclude that for any $n \in \mathbb{N}$ the set $G_n := E_1 \cup \dots \cup E_n$ is measurable.

Set $\tilde{E}_1 = E_1$ and for each $n \geq 2$ set $\tilde{E}_n = E_n \setminus G_{n-1}$. Then

$$\tilde{E}_n = \Omega \setminus [(\Omega \setminus E_n) \cup G_{n-1}]$$

is measurable and $G_n = \tilde{E}_1 \cup \dots \cup \tilde{E}_n$, $G = \tilde{E}_1 \cup \tilde{E}_2 \cup \dots$. Then

$$\begin{aligned}
\mu(A \cap G_n) &= \mu(A \cap G_n \cap \tilde{E}_n) + \mu(A \cap G_n \setminus \tilde{E}_n) \\
&= \mu(A \cap \tilde{E}_n) + \mu(A \cap G_{n-1})
\end{aligned}$$

and by induction we can conclude

$$\mu(A \cap G_n) = \sum_{k=1}^n \mu(A \cap \tilde{E}_k).$$

Therefore

$$\mu(A) = \mu(A \cap G_n) + \mu(A \setminus G_n) \geq \sum_{k=1}^n \mu(A \cap \tilde{E}_k) + \mu(A \setminus G)$$

and letting $n \rightarrow \infty$ we obtain

$$\begin{aligned} \mu(A) &\geq \sum_{k=1}^{\infty} \mu(A \cap \tilde{E}_k) + \mu(A \setminus G) \\ &\geq \mu(A \cap G) + \mu(A \setminus G) \geq \mu(A). \end{aligned}$$

That means the previous inequality is an equality, which implies (iii). If the $E_1, E_2, \dots$ are disjoint then $E_n = \tilde{E}_n$. Setting $A = G$ in the previous equality thus implies (iv). $\square$

Definition 1.2.6. A measure space $(\Omega, \mathcal{M}, \mu)$ is called **complete** if for every $A \in \mathcal{M}$ with $\mu(A) = 0$ and every $E \subset A$ we have $E \in \mathcal{M}$.

It follows from the definition that any set $E \subset \Omega$ with $\mu_*(E) = 0$ is measurable as

$$\mu_*(A \cap E) + \mu_*(A \setminus E) = \mu_*(A \setminus E) \leq \mu(A).$$

This means the restriction of μ_* to its measurable sets is a complete measure.

Lemma 1.2.7. Let $(\Omega, \mathcal{M}, \mu)$ be a measure space and let $E_1, E_2, \dots \in \mathcal{M}$. If $E_1 \subset E_2 \subset \dots$ then

$$\lim_{n \rightarrow \infty} \mu(E_n) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right),$$

where both sides may be infinite. If $\mu(E_1) < \infty$ and $E_1 \supset E_2 \supset \dots$ then

$$\lim_{n \rightarrow \infty} \mu(E_n) = \mu\left(\bigcap_{n=1}^{\infty} E_n\right).$$

Proof. See exercise sheet 2. □

1.2.2 A non-measurable set

Lemma 1.2.8 (Translation invariance of Lebesgue measure). Let $E \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$ and denote

$$E + x = \{y + x : y \in E\}.$$

Then

$$\mathcal{L}_*(E + x) = \mathcal{L}_*(E).$$

Proof. It suffices to show $\mathcal{L}_*(E + x) \leq \mathcal{L}_*(E)$ because from that we also obtain $\mathcal{L}_*(E) = \mathcal{L}_*(E + x - x) \leq \mathcal{L}_*(E + x)$.

For any $\varepsilon > 0$ exists a cover $\mathcal{Q}$ of E with

$$\mathcal{L}_*(E) \leq \varepsilon + \sum_{Q \in \mathcal{Q}} |Q|.$$

Then $\tilde{\mathcal{Q}} = \{Q + x : Q \in \mathcal{Q}\}$ is a cover of $E + x$ and thus since

$|Q - x| = |Q|$ we obtain

$$\begin{aligned}\mathcal{L}_*(E + x) &\leq \sum_{\tilde{Q} \in \tilde{\mathcal{Q}}} |\tilde{Q}| = \sum_{\tilde{Q} \in \tilde{\mathcal{Q}}} |\tilde{Q} - x| = \sum_{Q \in \mathcal{Q}} |Q| \\ &\leq \mathcal{L}_*(E) + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitraly small we obtain $\mathcal{L}_*(E + x) \leq |E|$ and finish the proof. $\square$

Theorem 1.2.9. *Assume the axiom of choice holds. Then there exist disjoint sets $E_0, E_1, \dots \subset \mathbb{R}$ for which (1.1.4) fails.*

Proof. For $x, y \in [0, 1]$ denote $x \sim y$ if $x - y$ is a rational number. Then $\sim$ is an equivalence relation. That means there is a decomposition $\mathcal{A}$ of $[0, 1]$, i.e. the union

$$[0, 1] = \bigcup \mathcal{A}$$

is disjoint, such that for any $x, y \in [0, 1]$ we have $x \sim y$ if and only if x and y belong to the same set $A \in \mathcal{A}$. By the axiom of choice

there exists a set E that contains exactly one element from each set $A \in \mathcal{A}$. That means for each $x \in [0, 1]$ exists a $y \in E$ and a rational $q \in \mathbb{Q} \cap [-1, 1]$ such that $x = y + q$. Moreover, for each $x \in E$ and $q \in \mathbb{Q}$ we have $x + q \notin E$. We can conclude that the sets in $\{E + q : q \in \mathbb{Q} \cap [-1, 1]\}$ form a countable disjoint cover of $[0, 1]$ and belong to $[-1, 2]$.

By Lemma 1.2.8 we have $\mathcal{L}_*(E + q) = \mathcal{L}_*(E)$. Then by

$$1 = \mathcal{L}_*([0, 1]) \leq \sum_{q \in \mathbb{Q} \cap [-1, 1]} \mathcal{L}_*(E + q) = \sum_{q \in \mathbb{Q} \cap [-1, 1]} \mathcal{L}_*(E).$$

we must have $\mathcal{L}_*(E) > 0$. This however means

$$\begin{aligned} \sum_{q \in \mathbb{Q} \cap [-1, 1]} \mathcal{L}_*(E + q) &= \sum_{q \in \mathbb{Q} \cap [-1, 1]} \mathcal{L}_*(E) = \infty \\ &> 3 \geq \mathcal{L}_*([-1, 2]) \\ &\geq \mathcal{L}_*\left(\bigcup_{q \in \mathbb{Q} \cap [-1, 1]} E + q\right), \end{aligned}$$

i.e. (1.1.4) fails. □

By Theorem 1.2.9, Lebesgue outer measure is not a measure on the σ -algebra $2^{\mathbb{R}}$ if we assume the axiom of choice. In the subsequent section we will see that still essentially all the sets that we care about in analysis are Lebesgue measurable.

1.2.3 Metric measures

Our definition of measurability works in a very general setting of a mere set Ω . Since the fundamental domain in this course is $\mathbb{R}^d$ we allow ourselves to assume a bit more structure.

Definition 1.2.10. Given a set Ω a map $d : \Omega \times \Omega \rightarrow [0, \infty)$ is called a **metric** if

- (i) for all $x \in \Omega$ we have $d(x, x) = 0$,
- (ii) (symmetry) for all $x, y \in \Omega$ we have $d(x, y) = d(y, x)$, and

- (iii) (triangle inequality) for all $x, y, z \in \Omega$ we have $d(x, z) \leq d(x, y) + d(y, z)$.

The pair (Ω, d) is called a **metric space**.

Definition 1.2.11. Given a metric d , we can define an **(open) ball** centered in $x \in \Omega$ with radius $r > 0$,

$$B(x, r) = \{y \in \Omega : d(x, y) < r\}.$$

We say that a set $A \subset \Omega$ is **open** if for every $x \in A$ exists an $r > 0$ such that $B(x, r) \subset A$. We say that $A \subset \Omega$ is **closed** if $\Omega \setminus A$ is open. We define the **Borel σ -algebra** $\mathcal{B}_\Omega$ to be the smallest σ algebra that contains all open sets $A \subset \Omega$. Its members $E \in \mathcal{B}_\Omega$ are called **Borel** sets.

See exercise sheet 1 for what we mean precisely by the smallest σ algebra that contains all open sets $A \subset \Omega$.

We want to show that all Borel subsets of $\mathbb{R}^d$ are Lebesgue measurable. This will be a consequence of the fact that Lebesgue outer measure is a metric outer measure.

We extend the metric to $d : 2^\Omega \times 2^\Omega \rightarrow [0, \infty)$ by defining

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

Definition 1.2.12. An outer measure μ_* is called a **metric outer measure** if for all $A, B \subset \Omega$ with $d(A, B) > 0$ we have

$$\mu_*(A \cup B) = \mu_*(A) + \mu_*(B).$$

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Lemma 1.2.13. Lebesgue outer measure is a metric outer measure.

Proof. Let $A_0, A_1 \subset \mathbb{R}^d$ with $\delta := d(A_0, A_1) > 0$. It suffices to prove

$$\mathcal{L}(A_0 \cup A_1) \geq \mathcal{L}(A_0) + \mathcal{L}(A_1).$$

Let $\varepsilon > 0$. Then there exists a countable cover $\mathcal{Q}$ of $A_0 \cup A_1$ with closed cubes such that

$$\sum_{Q \in \mathcal{Q}} |Q| \leq \mathcal{L}(A_0 \cup A_1) + \varepsilon.$$

There is a subdivision $\tilde{\mathcal{Q}}$ of the cubes in $\mathcal{Q}$ in cubes with diameter less than δ and. That means the cover $\bigcup \mathcal{Q}$ and by Lemma 1.1.2 we have

$$\sum_{Q \in \mathcal{Q}} |Q| = \sum_{Q \in \tilde{\mathcal{Q}}} |Q|.$$

Since $d(A_0, A_1) = \delta$, any $Q \in \tilde{\mathcal{Q}}$ cannot intersect both A_0 and A_1 . That means the sets $\tilde{\mathcal{Q}}_i = \{Q \in \tilde{\mathcal{Q}} : Q \cap A_i \neq \emptyset\}$ are disjoint for $i = 0, 1$. Moreover, $\tilde{\mathcal{Q}}_i$ is a cover of A_i . We can conclude

$$\mathcal{L}(A_0 \cup A_1) + \varepsilon \geq \sum_{Q \in \tilde{\mathcal{Q}}} |Q| \geq \sum_{Q \in \tilde{\mathcal{Q}}_0} |Q| + \sum_{Q \in \tilde{\mathcal{Q}}_1} |Q|$$

$$\geq \mathcal{L}(A_0) + \mathcal{L}(A_1).$$

Since $\varepsilon > 0$ was arbitrarily small this finishes the proof. $\square$

Theorem 1.2.14. *Let μ_* be a metric outer measure. Then all Borel sets are μ_* -measurable.*

Proof. By the definition of Borel sets and Theorem 1.2.4 it suffices to show that all closed sets are μ_* -measurable. To that end it suffices to show that for all $A \subset \Omega$ and all closed $B \subset \Omega$ we have

$$\mu_*(A) \geq \mu_*(A \cap B) + \mu_*(A \setminus B). \quad (1.2.2)$$

For $n \in \mathbb{N}$ denote

$$A_n = \{x \in A \setminus B : d(B, \{x\}) \geq 1/n\}.$$

Let $x \in \Omega$ with $d(\{x\}, B) = 0$. Then there exists a sequence $x_1, x_2, \dots \in B$ with $d(x_n, x) \rightarrow 0$. Since B is closed this means

$x \in B$. Therefore, for any $n \in \mathbb{N}$

$$\bigcup_{k=n+1}^{\infty} A_k = \{x \in A \setminus B : d(\{x\}, B) > 0\} = A \setminus B.$$

Moreover, for each n we have $d(B, A_n) \geq 1/n$ and since μ_* is a metric outer measure this means

$$\mu_*(A) \geq \mu_*(A \cap B \cup A_n) = \mu_*(A \cap B) + \mu_*(A_n). \quad (1.2.3)$$

Set $C_n = A_{n+1} \setminus A_n$. Then

$$d(C_{n+1}, A_{n-1}) \geq \frac{1}{n(n+1)}.$$

Since μ_* is a metric outer measure, by induction this implies

$$\mu_*(A) \geq \mu_*\left(\bigcup_{k=1}^n C_{2k}\right) = \sum_{k=1}^n \mu_*(C_{2k})$$

and similarly

$$\mu_*(A) \geq \mu_*\left(\bigcup_{k=1}^n C_{2k-1}\right) = \sum_{k=1}^n \mu_*(C_{2k-1}).$$

Since it suffices to consider $\mu_*(A) < \infty$ (why?), we can conclude

$$\sum_{k=1}^{\infty} \mu_*(C_k) < \infty$$

Therefore,

$$\mu_*(A_n) \leq \mu_*(A \setminus B) \leq \mu_*(A_n) + \sum_{k=n+1}^{\infty} \mu_*(C_k).$$

Letting $n \rightarrow \infty$ this means $\mu_*(A_n) \rightarrow \mu_*(A \setminus B)$ and thus from (1.2.3) we can conclude (1.2.2) and finish the proof. $\square$

Definition 1.2.15. An outer measure for which all open, or, equivalently, all Borel sets, are measurable is also called a **Borel outer measure**. By Lemma 1.2.13 and Theorem 1.2.14 Lebesgue measure is a Borel measure.

Conversely, a Borel outer measure μ on $\mathbb{R}^d$ is called **Borel regular** if for every measurable set A exists a Borel set $B \subset A$ with $\mu(A) = \mu(B)$. A Borel regular outer measure μ such that for all compact $K \subset \mathbb{R}^d$ we have $\mu(K) < \infty$ is called a **Radon outer measure**.

A measure is called Borel if its σ -algebra $\mathcal{M}$ contains all Borel sets, and it is called Borel regular or Radon measure respectively if the above properties hold for all sets $A \in \mathcal{M}$. That means the restriction of a Borel/Borel regular/Radon outer measure to its measurable sets yields a Borel/Borel regular/Radon measure.

Definition 1.2.16. For $n \in \mathbb{Z}$ recall the **dyadic numbers**

$$\mathbb{D}_n = \{k2^n : k \in \mathbb{Z}\}, \quad \mathbb{D} = \bigcup_{n \in \mathbb{Z}} \mathbb{D}_n \subset \mathbb{R}.$$

We define the set of **dyadic cubes** by

$$\mathcal{D}_n = \left\{ \prod_{l=1}^d 2^n [k_l, k_l + 1) : k_1, \dots, k_d \in \mathbb{Z} \right\},$$

$$\mathcal{D} = \bigcup_{n \in \mathbb{Z}} \mathcal{D}_n.$$

Then for any $n \in \mathbb{Z}$ and $\mathcal{Q} \subset \bigcup_{k=-\infty}^n \mathcal{D}_k$, we have

$$\bigcup \mathcal{Q} = \bigcup \{Q \in \mathcal{Q} : \forall P \in \mathcal{Q} \neg Q \subsetneq P\}.$$

The latter is called the **maximal disjoint subset** of $\mathcal{Q}$. Any two dyadic cubes in that collection are disjoint. Moreover, for any $x \in \mathbb{R}^d$ and $r > 0$ exists a $Q \in \mathbb{D}$ with $x \in Q \subset B(x, r)$. Combining these two facts, we arrive at the following.

Fact 1.2.17. Any open set $U \subset \mathbb{R}^d$ can be written as a countable disjoint union of dyadic cubes.

For a proof in one dimension see exercise sheet 2.

For two sets $A, B \subset \Omega$ define their **symmetric difference** by

$$A \Delta B = (A \setminus B) \cup (B \setminus A).$$

Proposition 1.2.18. Let $E \subset \mathbb{R}^d$.

(i)

$$\mathcal{L}_*(E) = \inf\{\mathcal{L}(U) : U \subset \mathbb{R}^d \text{ open, } E \subset U\}.$$

If E is measurable then for every $\varepsilon > 0$ exists an open set $U \supset E$ with

$$\mathcal{L}_*(U \setminus E) < \varepsilon.$$

(ii) If E is Lebesgue measurable then for every $\varepsilon > 0$ exists a closed set $C \subset E$ with

$$\mathcal{L}(E \setminus C) < \varepsilon.$$

(iii) There exists a Borel set $B \supset E$ such that

$$\mathcal{L}(B) = \mathcal{L}(E).$$

If E is Lebesgue measurable then there exists a Borel set $B \supset E$ such that

$$\mathcal{L}_*(B \setminus E) = 0.$$

(iv) If E is Lebesgue measurable then there exists a Borel set $B \subset E$ such that

$$\mathcal{L}(E \setminus B) = 0.$$

(v) If E is Lebesgue measurable with $\mathcal{L}(E) < \infty$ then for every $\varepsilon > 0$ exists a finite collection $\mathcal{Q}$ of disjoint dyadic cubes with

$$\mathcal{L}(E \Delta \bigcup \mathcal{Q}) < \varepsilon.$$

In particular, Lebesgue outer measure is a Radon outer measure.

Proof. We only prove that (i) implies (ii). For the remaining see exercise sheet 3.

First, assume that there exists a ball B with $E \subset \overline{B}$. Then by (i) there exists an open set $U \supset \overline{B} \setminus E$ with $\mathcal{L}(U \setminus (\overline{B} \setminus E)) < \varepsilon$. Define $C = \overline{B} \setminus U$, which is closed. Then $C \subset E$ and

$$\begin{aligned} E \setminus C &= (\overline{B} \cap E) \setminus \overline{B} \setminus U = \overline{B} \cap (E \cap U) \\ &= (\overline{B} \cap U) \setminus (\overline{B} \setminus E) \subset U \setminus (\overline{B} \setminus E) \end{aligned}$$

and thus we can conclude $\mathcal{L}(E \setminus C) < \varepsilon$.

Finally, for a general set $E \subset \mathbb{R}^d$ and $n \in \mathbb{N}$ abbreviate the annulus by

$$A_n := \overline{B(0, n)} \setminus B(0, n-1).$$

By the previous case exists a closed set $C_n \subset E \cap A_n$ with $\mathcal{L}(E \cap A_n \setminus C_n) < 2^{-n}\varepsilon$. Since C_n and A_n are closed, also their intersection is closed. Note, that the annuli have no accumulation point, i.e. no sequence $x_1, x_2, \dots \in \mathbb{R}^d$ with $x_n \in A_n$ has a convergent subsequence.

As a consequence, the set

$$C = \bigcup_{n=1}^{\infty} C_n \cap A_n$$

is closed and satisfies

$$\mathcal{L}(E \setminus C) \leq \sum_{n=0}^{\infty} \mathcal{L}(E \cap A_n \setminus C_n) \leq 2\varepsilon.$$

□

Corollary 1.2.19. For each Lebesgue measurable set $E \subset \mathbb{R}^d$ exists Borel sets B_0, B_1 with $B_0 \subset E \subset B_1$ such that $\mathcal{L}(B_1 \setminus B_0) = 0$.

Conversely, for B_0, B_1 Borel with $\mathcal{L}(B_1 \setminus B_0) = 0$, each $B_0 \subset E \subset B_1$ is measurable.

In this sense the set of Lebesgue measurable sets is the completion of the Borel σ -algebra with respect to sets with Lebesgue outer measure zero.

Proof. For $E \subset \mathbb{R}^d$ measurable and $\varepsilon > 0$ denote by $U_\varepsilon, C_\varepsilon$ the open and closed sets from Proposition 1.2.18. Then

$$B_0 := \bigcap_{n \in \mathbb{N}} U_{1/n}, \quad B_1 := \bigcup_{n \in \mathbb{N}} C_{1/n}$$

have the required properties.

For the other implication, E is the union of the Borel set B_0 and the set $E \setminus B_0$ which has zero Lebesgue measure, making E a union of two measurable sets and thus measurable itself. $\square$

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Proposition 1.2.18 also holds for Radon measures in general.

Proposition 1.2.20. Let μ be a Radon measure. Then for any measurable set E and any $\varepsilon > 0$ exists an open set $U \supset E$ and a closed set $C \subset E$ such that

$$\mu(U \setminus E) < \varepsilon, \quad \mu(E \setminus C) < \varepsilon.$$

If μ is a Radon outer measure then in addition for every $E \subset \mathbb{R}^d$ we have

$$\mu(E) = \inf\{\mu(U) : U \text{ open, } E \subset U\}.$$

Before we can prove Proposition 1.2.20 we need two preliminary results.

Lemma 1.2.21. Let μ be a Borel measure such that for every ball B we have $\mu(B) < \infty$. Then for any Borel set E and any $\varepsilon > 0$ exists an open set $U \supset E$ and a closed set $C \subset E$ such that

$$\mu(U \setminus E) < \varepsilon, \quad \mu(E \setminus C) < \varepsilon.$$

Proof. First consider the case that E is a countable union of closed sets $E = C_1 \cup C_2 \cup \dots$. Since finite unions of closed sets are closed, it suffices to consider the case that the sequence is increasing, i.e. $C_1 \subset C_2 \subset \dots$. Take any $x \in \Omega$, let $\varepsilon > 0$ and $n \in \mathbb{N}$ and abbreviate the annulus by

$$A_n := \overline{B(x, n)} \setminus B(x, n-1).$$

Then since $\mu(A_n)$ is finite by assumption, by Lemma 1.2.7 there exists an $N(n)$ such that

$$\mu([E \setminus C_{N(n)}] \cap A_n) \leq 2^{-n}\varepsilon.$$

Since $C_{N(n)}$ and A_n are closed, also their intersection is closed. Note, that the annuli have no accumulation point, i.e. no sequence $x_1, x_2, \dots \in \Omega$ with $x_n \in A_n$ has a convergent subsequence. As a consequence, the set

$$C = \bigcup_{n=1}^{\infty} C_{N(n)} \cap A_n$$

is closed and satisfies

$$\mu(E \setminus C) \leq \sum_{n=1}^{\infty} 2^{-n}\varepsilon = \varepsilon.$$

We now proceed essentially by induction. Denote by $\mathcal{M}$ the set of sets E that satisfy the conclusion of the lemma. It suffices to

show that $\mathcal{M}$ is a σ -algebra which contains all open sets. So, let U be open. Then we can write U as a countable union of closed sets,

$$U = \bigcup_{n=1}^{\infty} \{x \in U : d(\{x\}, \Omega \setminus U) \geq 1/n\}.$$

By the previous argument we can conclude $U \in \mathcal{M}$.

Now we show that $\mathcal{M}$ is a σ -algebra. Since open and closed sets are complements it follows that $\mathcal{M}$ is closed under complement. It remains to show that $\mathcal{M}$ is closed under countable union. So, let $E_1, E_2, \dots \in \mathcal{M}$ and set $E = E_1 \cup E_2 \cup \dots$ and let $\varepsilon > 0$. By inductive assumption for each $n \in \mathbb{N}$ exists an open set U_n and a closed set C_n with $\mu(U_n \setminus E_n) < 2^{-n}\varepsilon$ and $\mu(E_n \setminus C_n) < 2^{-n}\varepsilon$. That means $U = U_1 \cup U_2 \cup \dots$ and $K = C_1 \cup C_2 \cup \dots$ satisfy

$$\mu(U \setminus E) \leq \sum_{n=1}^{\infty} \mu(U_n \setminus E_n) \leq \sum_{n=1}^{\infty} 2^{-n}\varepsilon = \varepsilon,$$

$$\mu(E \setminus K) \leq \sum_{n=1}^{\infty} \mu(E_n \setminus C_n) \leq \sum_{n=1}^{\infty} 2^{-n} \varepsilon = \varepsilon.$$

The set U is open, but K might not be. However, K is a countable union of closed sets and thus by the first argument there exists a closed set $C \subset K$ with $\mu(K \setminus C) < \varepsilon$. We can conclude

$$\mu(E \setminus C) \leq \mu(E \setminus K) + \mu(K \setminus C) \leq 2\varepsilon$$

and finish the proof. □

Proposition 1.2.22. Let μ be a Borel regular (outer) measure on $\mathbb{R}^d$. Let $A \subset \mathbb{R}^d$ be measurable with $\mu(A) < \infty$. Then $\mu \llcorner A$ given by

$$\mu \llcorner A(E) = \mu(A \cap E)$$

is a Radon (outer) measure.

Proof. We prove the statement for a measure and an outer measure in parallel. In the measure case let $\mathcal{M}$ be the σ algebra of μ , and in the outer measure case we denote $\mathcal{M} = 2^{\mathbb{R}^d}$.

Since $\mu(A) < \infty$, for all compact sets K we have $\mu \llcorner A(K) < \infty$. Since $A \in \mathcal{M}$ is μ -measurable, each μ -measurable set E is also $\mu \llcorner A$ -measurable (why?). This is in particular true for Borel sets E . Thus, it remains to show Borel regularity.

By the Borel regularity of μ exists a Borel set $B \supset A$ with $\mu(A) = \mu(B)$, and since A is measurable,

$$\mu(B \setminus A) = \mu(B) - \mu(A) = 0.$$

Thus, for any $C \in \mathcal{M}$ we have

$$\begin{aligned} \mu(B \cap C) &= \mu(B \cap C \cap A) + \mu(B \cap C \setminus A) \\ &\leq \mu(C \cap A) + \mu(B \setminus A) = \mu(C \cap A) \\ &\leq \mu(B \cap C), \end{aligned}$$

so the inequalities are equalities and $\mu \llcorner A(C) = \mu \llcorner B(C)$. That means it suffices to consider the case that $A = B$, i.e. that A is Borel. It remains to show that for every $C \in \mathcal{M}$ exists a Borel set

$D \supset C$ with $\mu \ll A(D) = \mu \ll A(C)$. By the Borel regularity of μ , there exists a Borel set $E \supset A \cap C$ with $\mu(E) = \mu(A \cap C)$. Define $D = E \cup (\mathbb{R}^d \setminus A)$. Then D is Borel and $C \subset D$. Finally, since $D \cap A = E \cap A$ we can conclude

$$\begin{aligned}\mu \ll A(D) &= \mu(E \cap D) = \mu(E \cap A) \\ &\leq \mu(E) = \mu(A \cap C) = \mu \ll A(C),\end{aligned}$$

and the reverse inequality follows from $C \subset D$. □

Proof of Proposition 1.2.20. By definition of a Radon outer measure, for each $E \subset \mathbb{R}^d$ exists a Borel set $B \supset E$ with $\mu(E) = \mu(B)$. By Lemma 1.2.21 we can conclude

$$\begin{aligned}\mu(E) &= \mu(B) = \inf\{\mu(U) : U \text{ open}, B \subset U\} \\ &\geq \inf\{\mu(U) : U \text{ open}, E \subset U\} \geq \mu(E).\end{aligned}$$

If E is measurable and μ is a Radon measure, then similarly by

Lemma 1.2.21 for each $\varepsilon > 0$ exists an open set $U \supset B \supset E$ with

$$\mu(U \setminus E) = \mu(U \setminus B) + \mu(B \setminus E) < \varepsilon.$$

It remains to show that a measurable set E can be approximated by closed sets. First consider the case $\mu(E) < \infty$. By Proposition 1.2.22, the restriction $\mu \llcorner E$ is a Radon (outer) measure. Then by the previous argument, for each $\varepsilon > 0$ exists an open set $U \supset \mathbb{R}^d \setminus E$ with $\mu \llcorner E(U) < \varepsilon$, which means $C = \mathbb{R}^d \setminus U$ is closed with $C \subset E$ and $\mu(E \setminus C) = \mu(U \cap E) < \varepsilon$.

If $\mu(E) = \infty$ we proceed as in the proof of Lemma 1.2.21: For each $n \in \mathbb{N}$ define $E_n = E \cap B(0, n+1) \setminus B(0, n)$. Since μ is Radon, we have $\mu(E_n) < \infty$, and thus by the finite case exists a closed set $C_n \subset E_n \subset B(0, n+1) \setminus B(0, n)$ with $\mu(E_n \setminus C_n) < 2^{-n}\varepsilon$. Then $C := C_0 \cup C_1 \cup \dots \subset E$ is closed and

$$\mu(E \setminus C) \leq \sum_{n=0}^{\infty} \mu(E_n \setminus C_n) = 2\varepsilon.$$

1.3 Measurable functions

1.3.1 Definition and extent of the class

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space. We consider functions with values in the extended real line, $f : \Omega \rightarrow [-\infty, \infty]$, i.e. real valued functions that can also attain the values $\pm\infty$.

Definition 1.3.1. A function $f : \Omega \rightarrow [-\infty, \infty]$ is called **(μ) -measurable** if for every $a \in \mathbb{R}$ the set

$$\{f < a\} := f^{-1}([-\infty, a)) := \{x \in \Omega : f(x) \in [-\infty, a)\}$$

is (μ) -measurable, i.e. belongs to $\mathcal{M}$.

Lemma 1.3.2. Let $f : \Omega \rightarrow [-\infty, \infty]$. Then following are equivalent to f being measurable

- (i) For every $a \in \mathbb{R}$ the set $\{f \geq a\}$ is measurable.
- (ii) For every $a \in \mathbb{R}$ the set $\{f > a\}$ is measurable.
- (iii) For every $a \in \mathbb{R}$ the set $\{f \leq a\}$ is measurable.
- (iv) The function $-f$ is measurable.

If $f : \Omega \rightarrow (-\infty, \infty)$ then measurability is also equivalent to each of the following:

- (v) For every $a, b \in \mathbb{R}$ the set $\{a < f < b\}$ is measurable. Equivalently we can replace $<$ by $\leq$ in either instance.
- (vi) For every open $U \subset \mathbb{R}$ the set $f^{-1}(U)$ is measurable.
- (vii) For every closed $C \subset \mathbb{R}$ the set $f^{-1}(C)$ is measurable.

(viii) For every Borel $B \subset \mathbb{R}$ the set $f^{-1}(B)$ is measurable.

The latter also apply to functions $f : \Omega \rightarrow [-\infty, \infty]$ if in addition we require $f^{-1}(\{-\infty\})$ and $f^{-1}(\{\infty\})$ to be measurable.

Proof. For every $a \in \mathbb{R}$ we have $\{f \geq a\} = \Omega \setminus \{f < a\}$. That means $\{f \geq a\}$ is measurable if and only if $\{f < a\}$ is measurable. This proves (i).

The remaining items are exercises. □

Lemma 1.3.3. Let Ω be a metric space and μ be a Borel measure on Ω , i.e. its σ -algebra $\mathcal{M}$ contains the Borel σ -algebra. Then every continuous $f : \Omega \rightarrow (-\infty, \infty)$ is measurable.

Proof. A map $f : \Omega \rightarrow (-\infty, \infty)$ is continuous if and only if for every open subset $U \subset \mathbb{R}$, also $f^{-1}(U)$ is open. Thus the result follows from Lemma 1.3.2(vi). □

Lemma 1.3.4. Let $f_1, f_2, \dots : \Omega \rightarrow [-\infty, \infty]$ be measurable. Then the following functions are measurable

- (i) $x \mapsto \sup_n f_n(x)$
- (ii) $x \mapsto \inf_n f_n(x)$
- (iii) $x \mapsto \limsup_n f_n(x)$
- (iv) $x \mapsto \liminf_n f_n(x)$
- (v) If $(f_n)_n$ converges pointwise, then $x \mapsto \lim_n f_n(x)$ is measurable.

Proof. Denote $f = \sup_n f_n$. Then

$$\{f > a\} = \bigcup_n \{f_n > a\},$$

which is a union of measurable sets and thus measurable itself. This proves (i).

The proof of the remaining items are exercises. □

Lemma 1.3.5. Let $f : \Omega \rightarrow (-\infty, \infty)$ be measurable and $g : \mathbb{R} \rightarrow \mathbb{R}$ be continuous. Then $g \circ f$ is measurable.

Proof. Exercise. □

Lemma 1.3.6. Let $f, g : \Omega \rightarrow (-\infty, \infty)$ be measurable. Then for any $c \in \mathbb{R}$ the maps cf , $f + g$ and fg are measurable.

Proof. It suffices to consider $c > 0$. Then for any $a \in \mathbb{R}$ we have

$$\{cf < a\} = \{f < a/c\},$$

which is measurable.

Assume $f(x) + g(x) < a$. Then there exists an $\varepsilon > 0$ with $f(x) + g(x) < a + \varepsilon$ and an $r > 0$ and an $r \in \mathbb{Q}$ with $f(x) < r < f(x) + \varepsilon$. Thus, $g(x) < a - f(x) + \varepsilon < a - r$. Conversely, if $f(x) < r$ and $g(x) < a - r$ then $f(x) + g(x) < a$. We can conclude

$$\{f + g < a\} = \bigcup_{r \in \mathbb{Q}} \{f < r\} \cup \{g < a - r\},$$

which is a countable union of measurable sets and thus measurable.

Moreover,

$$\{f^2 < a\} = \{f < a^{1/2}\} \setminus \{f \leq -a^{1/2}\}$$

is measurable, and therefore

$$fg = \frac{(f+g)^2 - (f-g)^2}{2}$$

is measurable, too. □

Definition 1.3.7. We say that a statement that involves $x \in \Omega$ holds **(μ -)almost everywhere** if the set of all $x \in \Omega$ for which the statement fails is μ -measurable and has zero μ -measure.

For example, given $f, g : \Omega \rightarrow [-\infty, \infty]$, we say that $f(x) = g(x)$ for μ -almost everywhere x if $\mu(\{x \in \Omega : f(x) \neq g(x)\}) = 0$.

Lemma 1.3.8. Let μ be a complete measure and let $f, g : \Omega \rightarrow [-\infty, \infty]$ such that f is measurable and $f(x) = g(x)$ for μ -almost every x . Then g is μ -measurable.

Proof. Exercise. □

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1.3.2 Approximation

Definition 1.3.9. The most basic measurable functions are **characteristic functions**, which are functions of the form

$$1_E(x) := \begin{cases} 1 & x \in E, \\ 0 & x \notin E, \end{cases}$$

for a measurable set E . For $a_1, \dots, a_n \in \mathbb{R}$, we call a weighted sum

$$f = \sum_{k=1}^n a_k 1_{E_k}$$

a **simple function**. If the $E_1, \dots, E_n$ are disjoint rectangles then f is called a **step functions**.

Recall, that the Riemann integral is given in terms of **step functions**. The Lebesgue integral will be defined in terms of the more general simple functions. Because simple functions can approximate the large class of measurable functions, the Lebesgue integral will generalize the Riemann integral.

Definition 1.3.10. Let $a \in (0, \infty)$. Take $N \in \mathbb{Z}$ maximal with $a < 2^{-N}$ and for all $n \in \mathbb{Z}$ with $n \leq N$ set $a_n = 0$. For $n > N$ we define a_n inductively as follows: Set $a_{n+1} = 0$ if

$$a < \sum_{k=N+1}^n a_k 2^{-k} + 2^{-(n+1)}$$

and $a_{n+1} = 1$ otherwise.

For $a = 0$ set $a_n = 0$ for all n and for $a = \infty$ set $a_n = 1$ for all n .

The sequence $(a_n)_{n \in \mathbb{Z}}$ is called the **binary expansion** of $a \in [0, \infty]$.

Lemma 1.3.11. Let $a \in [0, \infty)$. Then for any $n \in \mathbb{Z}$ we have

$$0 \leq a - \sum_{k=-\infty}^n a_k 2^{-k} < 2^{-n}. \quad (1.3.1)$$

Proof. We proceed by induction. For $n \leq N$ (1.3.1) follows from

$$0 \leq a < 2^{-N} \leq 2^{-n}.$$

So, assume (1.3.1) holds for n . If $a_{n+1} = 0$ then by inductive assumption

$$\sum_{k=-\infty}^{n+1} a_k 2^{-k} = \sum_{k=-\infty}^n a_k 2^{-k} \leq a$$

and by the defining condition of $a_{n+1} = 0$ we have

$$a < \sum_{k=-\infty}^n a_k 2^{-k} + 2^{-(n+1)} = \sum_{k=-\infty}^{n+1} a_k 2^{-k} + 2^{-(n+1)}.$$

It remains to consider the case $a_{n+1} = 1$. Then by the defining condition of $a_{n+1} = 1$ we have

$$\sum_{k=-\infty}^{n+1} a_k 2^{-k} = \sum_{k=-\infty}^n a_k 2^{-k} + 2^{-(n+1)} \leq a$$

and by the inductive assumption and $2^{-n} = 2^{-(n+1)} + a_{n+1} 2^{-(n+1)}$, we have

$$a < \sum_{k=-\infty}^n a_k 2^{-k} + 2^{-n} = \sum_{k=-\infty}^{n+1} a_k 2^{-k} + 2^{-(n+1)}.$$

□

Corollary 1.3.12. For all $a \in [0, \infty]$ we have

$$a = \sum_{k=-\infty}^{\infty} a_k 2^{-k}.$$

Lemma 1.3.13. For $a \in [0, \infty)$ and any $n \in \mathbb{Z}$, the subsequence $(a_k)_{k=-\infty}^n$ of the binary expansion of a is the only sequence of numbers in $\{0, 1\}$ for which (1.3.1) holds.

Proof. Let $(\tilde{a}_k)_{k=-\infty}^n$ with $\tilde{a}_k \in \{0, 1\}$ for which (1.3.1) holds and which is different from $(a_k)_{k=-\infty}^n$. Then there exists an $N \in \mathbb{Z}$ such that for all $k \leq N$ we have $\tilde{a}_k = 0$, and we take N so small that also $a_k = 0$. So, take $K \in \mathbb{Z}$ minimal for which $\tilde{a}_K \neq a_K$. From $|\tilde{a}_k - a_k| 2^{-k} \leq 2^{-k}$ it follows by induction on $m \geq K$ that

$$\left| \sum_{k=-\infty}^m \tilde{a}_k 2^{-k} - \sum_{k=-\infty}^m a_k 2^{-k} \right| \geq 2^{-m}.$$

For $m = n$, this means (1.3.1) cannot hold for both $(\tilde{a}_k)_{k=-\infty}^n$ and $(a_k)_{k=-\infty}^n$ at the same time. $\square$

Theorem 1.3.14. *Let $f : \Omega \rightarrow [0, \infty]$ be measurable. Then there exist measurable sets $A_1, A_2, \dots$ and $a_1, a_2, \dots \geq 0$ such that for every $x \in \Omega$ we have*

$$f(x) = \sum_{n=1}^{\infty} a_n 1_{A_n}.$$

Proof. For $n \in \mathbb{Z}$ define the set

$$A_n = \{x \in \Omega : f(x)_n = 1\}$$

using the binary expansion of $f(x)$. Then by Corollary 1.3.12 we have

$$f(x) = \sum_{n \in \mathbb{Z}} 2^{-n} 1_{A_n}(x).$$

Given $a \in [0, \infty)$ by Lemmas 1.3.11 and 1.3.13 for each $n \in \mathbb{Z}$ we have $a_n = 1$ if and only if there exists an $N \leq n$ and a tuple

$(\tilde{a}_N, \tilde{a}_{N+1}, \dots, \tilde{a}_{n-1}) \in \{0, 1\}^{n-N}$ such that

$$0 \leq a - \sum_{k=-N}^{n-1} \tilde{a}_k 2^{-k} < 2^{-n}.$$

That means

$$A_n = \{f = \infty\} \cup$$

$$\bigcup_{N=-\infty}^{n-1} \bigcup_{(a_N, \dots, a_{n-1}) \in \{0, 1\}^{n-N}} \left\{ x \in \Omega : 0 \leq f(x) - \sum_{k=-N}^{n-1} a_k 2^{-k} < 2^{-n} \right\},$$

which is a countable union of measurable sets and thus measurable.

Reenumerating $(A_n)_{n \in \mathbb{Z}}$ as $(\tilde{A}_n)_{n \in \mathbb{N}}$ finishes the proof. $\square$

Corollary 1.3.15. Let $f : \Omega \rightarrow [0, \infty]$ be measurable. Then there exist simple functions $f_1, f_2, \dots : \Omega \rightarrow [0, \infty)$ such that for every $x \in \Omega$ we have $f_n(x) \leq f(x)$ and

$$f(x) = \lim_{n \rightarrow \infty} f_n(x).$$

Proof. With a_n, A_n from Theorem 1.3.18 take

$$f_n = \sum_{k=1}^n a_n 1_{A_n}.$$

□

Alternative independent proof. For $n \in \mathbb{N}$ and $0 \leq m < 2^{2n}$ set

$$A_n^m = \{x \in \Omega : 2^{-n}m \leq f(x) < 2^{-n}(m+1)\}, \quad A_n^{2^{2n}} = \{x \in \Omega : 2^n \leq f(x)\}.$$

Then A_n^m is measurable. Define the simple function

$$f_n = \sum_{m=0}^{2^{2n}} 2^{-n}m 1_{A_n^m}.$$

For every $x \in \Omega$ exists exactly one $0 \leq m \leq 2^{2n}$ with $x \in A_n^m$. This means

$$f_n(x) = 2^{-n}m \leq f(x) < 2^{-n}(m+1) = f_n(x) + 2^{-n},$$

in case $f(x) < 2^n$ and $m < 2^{2n}$. If $f(x) \geq 2^n$ then $m = 2^{2n}$ and $f_n^m(x) = 2^n$. We can conclude

$$\lim_{n \rightarrow \infty} f_n(x) = f(x).$$

□

Remark 1.3.16. The function f_n from Corollary 1.3.15 equals

$$\sum_{k=-n+1}^n 2^{-k} 1_{A_k}$$

from the proof of Theorem 1.3.14.

Definition 1.3.17. For a function $f : \Omega \rightarrow [-\infty, \infty]$ define its **positive** and **negative part** by

$$f^+(x) = \max\{f(x), 0\}, \quad f^-(x) = \max\{-f(x), 0\}.$$

Note, that $f = f^+ - f^-$.

Theorem 1.3.18. *Let $f : \Omega \rightarrow [-\infty, \infty]$ be measurable. Then there exists a sequence $\varphi_1, \varphi_2, \dots$ of simple functions with $|\varphi_n| \leq |\varphi_{n+1}|$ and such that*

$$f(x) = \lim_{n \rightarrow \infty} \varphi_n(x).$$

Proof. By Corollary 1.3.15 there exist simple functions $\varphi_1^+, \varphi_2^+, \dots$ and $\varphi_1^-, \varphi_2^-, \dots$ with $0 \leq \varphi_1^\pm \leq \varphi_2^\pm \leq \dots \leq f^\pm$ and

$$\lim_{n \rightarrow \infty} \varphi_n^\pm(x) = f^\pm(x).$$

Then $\varphi_n := \varphi_n^+ - \varphi_n^-$ have the required properties. □

Definition 1.3.19. We say, that a measure space $(\Omega, \mathcal{M}, \mu)$ is σ -finite if there exists a sequence $E_1, E_2, \dots \in \mathcal{M}$ with $\mu(E_n) < \infty$ such that $\Omega = E_1 \cup E_2 \cup \dots$.

For example $\mathbb{R}^d$ is σ -finite with respect to Lebesgue measure, since $\mathbb{R}^d = \bigcup_{n \in \mathbb{N}} B(0, n)$ and $\mathcal{L}(B(0, n)) < \infty$.

Lemma 1.3.20. Let $(\Omega, \mathcal{M}, \mu)$ be σ -finite. Let $f_1, f_2, \dots : \Omega \rightarrow [-\infty, \infty]$ that converge to $f : \Omega \rightarrow [-\infty, \infty]$ almost everywhere, and for each $n \in \mathbb{N}$ let $f_n^1, f_n^2, \dots : \Omega \rightarrow [-\infty, \infty]$ that converge to f_n almost everywhere. Then there exist $k_1, k_2, \dots$ and $n_1, n_2, \dots \in \mathbb{N}$ such that $f_{n_m}^{k_{n_m}}$ converge to f almost everywhere.

Proof. For $m \in \mathbb{N}$ and functions g, h denote

$$\begin{aligned} A_m(g, h) = & \{x \in \mathbb{R}^d : |h(x) - g(x)| > 2^{-m}\} \\ & \cup \{x \in \mathbb{R}^d : h(x) = \infty, g(x) < 2^m\} \\ & \cup \{x \in \mathbb{R}^d : h(x) = -\infty, g(x) > -2^m\}. \end{aligned}$$

Then

$$\{x \in \mathbb{R}^d : \lim_{n \rightarrow \infty} f_n(x) \neq f(x) \text{ or the limit does not exist.}\}$$

$$= \bigcup_{m \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} \bigcup_{n \geq k} A_m(f_n, f). \quad (1.3.2)$$

Recall $\Omega = E_1 \cup E_2 \cup \dots$. We may redefine those sets such that for all $n \in \mathbb{N}$ we have $E_n \subset E_{n+1}$. Fix $N \in \mathbb{N}$.

Since $f_n \rightarrow f$ almost everywhere, i.e. the left hand side of (1.3.2) has zero Lebesgue measure, by Lemma 1.2.7 for every $m \in \mathbb{N}$ exists an $n_m \geq m, N$ such that for all $n \geq n_m$ we have

$$\mathcal{L}(E_N \cap A_m(f_n, f)) \leq 2^{-m}.$$

Also $\psi_n = 1_{E_n} f_n$ converge to f almost everywhere and $\psi_n^k = 1_{E_n} f_n^k$ converge to ψ_n almost everywhere. That means similarly, for each n, m exists a $k_{n,m} \geq n$ such that for all $k \geq k_{n,m}$ we have

$$\mathcal{L}(A_m(\psi_n^k, \psi_n)) \leq 2^{-m}.$$

Then

$$|f(x) - g_m(x)| \leq |f(x) - f_{n_m}(x)| + |f_{n_m}(x) - f_{n_m}^{k_{n_m}, m}(x)|.$$

Note, that for every $l, m, k \in \mathbb{Z}$ and $n \geq N$ we have

$$E_N \cap A_l(f_n^k, f) \subset A_{l+1}(f_n, f) \cup A_{l+1}(\psi_n^k, \psi_n).$$

In particular, for all $m \geq l+1$ sufficiently large so that, $n_m \geq l+1, N$, we obtain

$$\begin{aligned} \mathcal{L}(E_N \cap (A_l(f_{n_m}, f) \cup A_l(\psi_{n_m}^{k_{n_m, m}}, \psi_{n_m}))) \\ \leq \mathcal{L}(E_N \cap (A_l(f_{n_m}, f) \cup A_l(\psi_{n_m}^{k_{n_m, m}}, \psi_{n_m}))) \\ \leq 2^{-m+1}. \end{aligned}$$

We can conclude

$$\begin{aligned} \mathcal{L}(\{x \in E_N : \lim_{m \rightarrow \infty} f_{n_m}^{k_{n_m, m}}(x) \neq f(x)\}) \\ \leq \mathcal{L}\left(\bigcup_{l \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} \bigcup_{m \geq k} E_N \cap (A_l(f_{n_m}, f) \cup A_l(\psi_{n_m}^{k_{n_m, m}}, \psi_{n_m}))\right) \\ \leq \sum_{l \in \mathbb{N}} \lim_{k \rightarrow \infty} \sum_{m \geq k} 2^{-m+1} = 0. \end{aligned}$$

That means the $f_{n_m}^{k_{n_m}, m}$ converge to f pointwise for almost every $x \in E_N$. Since N was arbitrary, they in fact converge for almost every $x \in \mathbb{R}^d$. $\square$

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Theorem 1.3.21. *Let $f : \mathbb{R}^d \rightarrow [-\infty, \infty]$ be measurable. Then there exist step functions $\psi_1, \psi_2, \dots$ such that for almost every $x \in \mathbb{R}^d$ we have*

$$\lim_{n \rightarrow \infty} \psi_n(x) = f(x).$$

Proof. We first consider the case that for $E \subset \mathbb{R}^d$ measurable with $\mathcal{L}(E) < \infty$ we have $f = 1_E$. The by an exercise, for every $n \in \mathbb{N}$ exists a finite set $\mathcal{Q}_n$ of disjoint dyadic cubes with $\mathcal{L}(E \Delta \bigcup \mathcal{Q}_n) < 2^{-n}$. Then

$$f_n = \sum_{Q \in \mathcal{Q}_n} 1_Q$$

is a step function such that

$$A_n := \{x \in \mathbb{R}^d : f_n(x) \neq 1_E(x)\}$$

satisfies $\mathcal{L}(A_n) < 2^{-n}$. We can conclude for

$$A :=$$

$$\{x \in \mathbb{R}^d : \lim_{n \rightarrow \infty} f_n(x) \neq 1_E(x) \text{ or the limit does not exist.}\}$$

by Lemma 1.2.7 and (1.3.2) that

$$\begin{aligned} \mathcal{L}(A) &\leq \mathcal{L}\left(\bigcap_{k \in \mathbb{N}} \bigcup_{n \geq k} A_n\right) \leq \lim_{k \rightarrow \infty} \sum_{n=k}^{\infty} \mathcal{L}(A_n) \\ &\leq \lim_{k \rightarrow \infty} \sum_{n=k}^{\infty} 2^{-n} = \lim_{k \rightarrow \infty} 2^{-k+1} = 0. \end{aligned}$$

Next, consider the case that $a_1, \dots, a_N \in \mathbb{R}$ and $A_1, \dots, A_N \subset \mathbb{R}^d$ are measurable with $\mathcal{L}(A_k) < \infty$ and $f = a_1 1_{A_1} + \dots + a_N 1_{A_N}$. Then

by the previous case for each $k = 1, \dots, N$ exists a sequence of step functions f_k^n with $f_k^n \rightarrow 1_{A_k}$ pointwise almost everywhere as $n \rightarrow \infty$. Thus, $a_1 f_1^n + \dots + a_N f_N^n$ are step functions that converge to f .

Finally, let $f : \mathbb{R}^d \rightarrow [-\infty, \infty]$ be measurable. By Theorem 1.3.18 there exist simple functions $\varphi_1, \varphi_2, \dots$ that converge to f everywhere. Then $\psi_n = 1_{B(0,n)} \varphi_n$ also converge to f everywhere, and ψ_n is a finite sum of characteristic functions with finite measure. That means, by the previous case there exist step functions $\psi_n^1, \psi_n^2, \dots$ that converge to ψ_n almost everywhere. An application of Lemma 1.3.20 concludes the proof. $\square$

Theorem 1.3.22 (Egorov). *Let μ be a measure on Ω with $\mu(\Omega) < \infty$. Let $f, f_1, f_2, \dots : \Omega \rightarrow (-\infty, \infty)$ measurable s.t. $f_n \rightarrow f$ a.e.. Then for every $\varepsilon > 0$ exists a set C with $\mu(\Omega \setminus C) < \varepsilon$ such that $f_n \rightarrow f$ uniformly on C .*

If μ is a Radon measure then we can take C closed.

Proof. By Lemma 1.2.7 and (1.3.2) for every $m \in \mathbb{N}$ exists a $k_m \in \mathbb{N}$

such that

$$\mathcal{L}\left(\bigcup_{n \geq k_m} A_m(f_n, f)\right) \leq 2^{-m}.$$

Take $N \in \mathbb{N}$ such that

$$\sum_{m \geq N} 2^{-m} < \varepsilon/2$$

and define

$$A = \bigcup_{m \geq N} \bigcup_{n \geq k_m} A_{n,m}$$

which means $\mathcal{L}(A) < \varepsilon$. If μ is a Radon measure then by Proposition 1.2.20 there exists a closed set $C \subset \Omega \setminus A$ with $\mathcal{L}(\Omega \setminus C) < \varepsilon$. For a general measure just set $C = \Omega \setminus A$.

Let $m \geq N$. Then for all $n \geq k_m$ and $x \in C$ we have $x \notin A_{n,m}$ which means $|f_n(x) - f(x)| < 2^{-m}$. This means $f_n \rightarrow f$ uniformly. $\square$

Theorem 1.3.23 (Lusin). *Let $E \subset \mathbb{R}^d$ be measurable with $\mathcal{L}(E) < \infty$ and $f : E \rightarrow (-\infty, \infty)$. Then for every $\varepsilon > 0$ exists a closed set $C \subset E$ with $\mathcal{L}(E \setminus C) < \varepsilon$ such that $f : C \rightarrow (-\infty, \infty)$ is continuous.*

This does not mean that $f : E \rightarrow (-\infty, \infty)$ is continuous in every $x \in C$.

Proof. By Theorem 1.3.21 exists a sequence $f_1, f_2, \dots$ of step functions that converge to f almost everywhere. For any $n \in \mathbb{N}$ exists a set E_n such that $\mathcal{L}(E_n) < 2^{-n}\varepsilon$ and f_n is continuous on $E \setminus E_n$. By Egorov's theorem exists a set A with $\mathcal{L}(E \setminus A) < \varepsilon$ on which $f_n \rightarrow f$ uniformly. Then on

$$B = A \setminus \bigcup_{n \in \mathbb{N}} E_n$$

f is a uniform limit of continuous functions, and thus continuous itself. By Proposition 1.2.18 exists a closed set $C \subset B$ with $\mathcal{L}(B \setminus C) < \varepsilon$. Then f is continuous on C and $\mathcal{L}(E \setminus C) < 3\varepsilon$. $\square$

Chapter 2

Integration

2.1 The Lebesgue integral

In this section we assume $(\Omega, \mathcal{M}, \mu)$ to be a measure space. Our goal is to define and integral for a large class of measurable functions

$f : \Omega \rightarrow [-\infty, \infty]$ from Ω to the extended real line.

2.1.1 Simple functions

In Section 1.3.2 we saw that any measurable function can be approximated pointwise by simple functions. The latter have a convenient definition of their integral.

Definition 2.1.1. Let f be a simple function. Then there exists an $n \in \mathbb{N} \cup \{0\}$ such that f assumes n unique distinct $a_1 < \dots < a_n$ with $a_k \in [-\infty, \infty]$. Define $E_k = f^{-1}(\{a_k\})$. We call those numbers $a_1, \dots, a_n$ and sets $E_1, \dots, E_k$ the **canonical form** of f .

Note, that

$$f = \sum_{k=1}^n a_k 1_{E_k}.$$

Definition 2.1.2. In the context of this definition we set $0 \cdot \infty = 0$.

Let f be a nonnegative simple function with canonical form $a_1, \dots, a_n, E_1, \dots, E_n$. We call f **μ -integrable** if there are no k, m with $a_k \mu(E_k) = \infty$ and $a_m \mu(E_m) = -\infty$.

For a simple function f that is μ -integrable we define its **Lebesgue integral** by

$$\int f \, d\mu = \sum_{k=1}^n a_k \mu(E_k). \quad (2.1.1)$$

For $E \subset \Omega$ measurable define

$$\int_E f \, d\mu = \int f 1_E \, d\mu.$$

Note, that f is μ -integrable if and only if the sum in (2.1.1) is well defined.

Proposition 2.1.3. Let

$$f = \sum_{k=1}^n a_k 1_{E_k}$$

be a simple function such that there are no k, m with $a_k \mu(E_k) = \infty$ and $a_m \mu(E_m) = -\infty$. Then

$$\int f \, d\mu = \sum_{k=1}^n a_k \mu(E_k),$$

where we again set $0 \cdot \infty = 0$.

That means (2.1.1) holds not only for the canonical form but for any expression that yields the same simple function, provided the sum in (2.1.1) is well defined.

Proof. Let $\tilde{a}_1, \dots, \tilde{a}_{\tilde{n}}$ and $\tilde{E}_1, \dots, \tilde{E}_{\tilde{n}}$ be the canonical form of f , i.e.

$$f = \sum_{k=1}^{\tilde{n}} \tilde{a}_k 1_{E_k}.$$

First, assume that the sets $E_1, \dots, E_n$ are disjoint. Then for each $\tilde{k} = 1, \dots, \tilde{n}$ the set $E_{\tilde{k}}$ equals the disjoint union of all E_k for which $a_k = \tilde{a}_{\tilde{k}}$ which means

$$\mu(E_{\tilde{k}}) = \sum_{k: a_k = \tilde{a}_{\tilde{k}}} \mu(E_k)$$

and thus,

$$\sum_{k=1}^n a_k \mu(E_k) = \sum_{k=1}^{\tilde{n}} \tilde{a}_k \mu(\tilde{E}_k).$$

It remains to consider the general case that the $E_1, \dots, E_n$ may

overlap. For $A \subset \{1, \dots, n\}$ define the collection of sets

$$E_A = \bigcap_{k \in A} E_k \setminus \bigcup_{m \notin A} E_m.$$

Let $A, B \in \{1, \dots, n\}$ with $A \neq B$. Assume $\#A \leq \#B$. Then there exists a k with $k \in B \setminus A$. That means $E_B \subset E_k$ and $E_A \cap E_k = \emptyset$, and therefore we have $E_A \cap E_B = \emptyset$. Next, let $k \in \{1, \dots, n\}$. Then for each A with $k \in A$ we have $E_A \subset E_k$. Conversely, let $x \in E_k$ and let A be the set of those $m \in \{1, \dots, n\}$ with $x \in E_m$. Then $x \in E_A$ and $k \in A$. That means $E_k = \bigcup_{A \ni k} E_A$ and we can conclude

$$\mu(E_k) = \sum_{A \in \{1, \dots, n\}: k \in A} \mu(E_A).$$

Thus,

$$\sum_{k=1}^n a_k \mu(E_k) = \sum_{A \subset \{1, \dots, n\}} \sum_{k \in A} a_k \mu(E_A).$$

Since

$$f = \sum_{A \subset \{1, \dots, k\}} \left(\sum_{m \in A} a_m \right) 1_{E_A}$$

writes f as a simple function with disjoint sets, by the previous case we can conclude

$$\int f \, d\mu = \sum_{k=1}^n a_k \mu(E_k).$$

□

We collect a few basic properties of the Lebesgue integral of simple functions.

Proposition 2.1.4. (i) (linearity) Let $a, b \in [-\infty, \infty]$ and f, g be simple functions. Then

$$\int af + bg \, d\mu = a \int f \, d\mu + b \int g \, d\mu,$$

provided the right hand side is well defined, i.e. not of the form $\infty - \infty$.

- (ii) (additivity) Let $E, F \subset \Omega$ be disjoint and $f1_{E \cup F}$ be simple and nonnegative. Then

$$\int_{E \cup F} f \, d\mu = \int_E f \, d\mu + \int_F f \, d\mu.$$

- (iii) (monotonicity) Let f, g be simple and integrable such that $f \leq g$. Then

$$\int f \, d\mu \leq \int g \, d\mu.$$

Proof. (i) This follows from the linearity of the formula in Proposition 2.1.3.

- (ii) This follows from linearity and $f1_{E \cup F} = f1_E + f1_F$.

- (iii) Since $g - f \geq 0$, the assumptions of the linearity conditions are satisfied for the summands f and $g - f$. Thus the result follows from linearity and $\int g - f \, d\mu \geq 0$.

□

2.1.2 Nonnegative functions

Definition 2.1.5. For $f : \Omega \rightarrow [0, \infty]$ measurable define its Lebesgue integral by

$$\int f \, d\mu := \sup \left\{ \int g \, d\mu : g : \Omega \rightarrow [0, \infty] \text{ simple, } g \leq f \right\}.$$

Remark 2.1.6. By Proposition 2.1.4(iii), for a simple function f this definition coincides with the definition of the Lebesgue integral of a simple function.

Proposition 2.1.4 carries over to nonnegative functions. Before we are able to prove that pointwise convergence implies convergence

of the integral in a bounded setting.

Lemma 2.1.7. Let $\mu(\Omega) < \infty$ and let $N \geq 0$. Let $f : \Omega \rightarrow [0, N]$ be measurable and $f_1, f_2, \dots : \Omega \rightarrow [0, N]$ be simple functions such that $f_n(x) \rightarrow f(x)$ for a.e. $x \in \Omega$. Then

$$\lim_{n \rightarrow \infty} \int f_n \, d\mu = \int f \, d\mu.$$

Proof. First, we show that the limit exists. Abbreviate $I_n = \int f_n \, d\mu$ and let $\varepsilon > 0$. By Egorov's theorem exists a measurable set $C \subset \Omega$ with $\mu(\Omega \setminus C) < \varepsilon$ such that $f_n \rightarrow f$ uniformly on C . Then for n, m sufficiently large, for all $x \in C$ we have $|f_n(x) - f_m(x)| \leq \varepsilon$. Thus,

$$\begin{aligned} |I_n - I_m| &\leq \int_C |f_n - f_m| \, d\mu + 2N\mu(\Omega \setminus C) \\ &\leq \int_C 2\varepsilon \, d\mu + 2N\varepsilon \leq 2(N + \mu(\Omega))\varepsilon. \end{aligned}$$

Since ε was arbitrarily small, this means $(I_n)_n$ is a Cauchy sequence and hence converges.

In particular, this means any above sequence $f_1, f_2, \dots$ converges to the same limit, because otherwise we could be interlacing two sequences with different limits that would not converge. That means it suffices to find one sequence of functions f_n that satisfies the assumptions of this lemma and whose integrals converge to $\int f \, d\mu$.

To that end, take φ_n from Corollary 1.3.15 and a maximizing sequence g_n from the definition of $\int f \, d\mu$, i.e. such that $\int g_n \, d\mu \rightarrow \int f \, d\mu$. Then $f_n(x) := \max\{\varphi_n(x), g_n(x)\} \leq f$ is a step function for which $\psi_n \rightarrow f$ pointwise and $\int \psi_n \, d\mu \rightarrow \int f \, d\mu$. $\square$

Definition 2.1.8. For $f : \Omega \rightarrow [-\infty, \infty]$ the **support** of f is

$$\text{spt}(f) := f^{-1}([-\infty, \infty] \setminus \{0\}),$$

i.e. the set of points $x \in \Omega$ for which $f(x) \neq 0$.

Theorem 2.1.9. *Let μ be a measure on Ω .*

(i) (linearity) Let $a, b \geq 0$ and $f, g \geq 0$ be measurable. Then

$$\int af + bg \, d\mu = a \int f \, d\mu + b \int g \, d\mu.$$

(ii) (additivity) Let $E, F \subset \Omega$ be disjoint and $f \geq 0$ be measurable. Then

$$\int_{E \cup F} f \, d\mu = \int_E f \, d\mu + \int_F f \, d\mu.$$

(iii) (monotonicity) Let $f, g \geq 0$ be measurable such that $f \leq g$. Then

$$\int f \, d\mu \leq \int g \, d\mu.$$

(iv) If $f \geq 0$ is measurable and $\int f \, d\mu < \infty$ then for a.e. $x \in \Omega$ we have $f(x) < \infty$.

(v) Let $f \geq 0$ be measurable. Then $\int f \, d\mu = 0$ if and only if a.e. $x \in \Omega$ we have $f(x) = 0$.

Proof. We first prove linearity. Let φ, ψ be simple functions with $\varphi \leq f$ and $\psi \leq g$. Then $a\varphi + b\psi$ is a simple function with $a\varphi + b\psi \leq af + bg$, and thus by linearity and monotonicity of the Lebesgue integral of a simple function we can conclude

$$a \int f \, d\mu + b \int g \, d\mu \leq \int af + bg \, d\mu.$$

For the reverse inequality it suffices to consider the case that $a \int f \, d\mu, b \int g \, d\mu < \infty$. That means for every $\lambda > 0$ we have $\mu(\{af + bg > \lambda\}) \leq \mu(\{af > \lambda/2\}) + \mu(\{bg > \lambda/2\}) < \infty$. Let $\eta \leq af + bg$ be a simple function. Then by the previous sentence we have $\mu(\text{spt}(\eta)) < \infty$ and $\int \eta \, d\mu < \infty$. Define $\eta_1 = \min\{\eta, af\} \leq af$ and $\eta_2 = \eta - \eta_1 \leq bg$. Then

for $i = 1, 2$ we have $\text{spt}(\eta_i) \subset \text{spt}(\eta)$ which implies $\mu(\text{spt}(\eta_i)) < \infty$. Thus, by Corollary 1.3.15 and Lemma 2.1.7, for $i = 1, 2$ exist sequences $\varphi_1^i, \varphi_2^i, \dots$ of simple functions with $0 \leq \varphi_n^1 + \varphi_n^2 \leq \eta$ and $\int \varphi_n^1 + \varphi_n^2 \, d\mu \rightarrow \int \eta \, d\mu$. Since by definition we have

$$\begin{aligned} \frac{1}{a} \int \varphi_n^1 \, d\mu &= \int \frac{\varphi_n^1}{a} \, d\mu \leq \int f \, d\mu, \\ \frac{1}{b} \int \varphi_n^2 \, d\mu &= \int \frac{\varphi_n^2}{b} \, d\mu \leq \int g \, d\mu, \end{aligned}$$

We can conclude

$$a \int f \, d\mu + b \int g \, d\mu \geq \lim_{n \rightarrow \infty} \int \varphi_n^1 \, d\mu + \int \varphi_n^2 \, d\mu = \int \eta \, d\mu.$$

Since $\eta \leq af + bg$ was an arbitrary simple function this implies

$$a \int f \, d\mu + b \int g \, d\mu \geq \int af + bg \, d\mu.$$

Additivity follows from linearity and $f1_{E \cup F} = f1_E + f1_F$.

Monotonicity follows from the fact that for any simple functions $\varphi \leq f$ we also have $\varphi \leq g$.

Denote $E = \{x \in \Omega : f(x) = \infty\}$. If E has positive measure then the simple function $\infty \cdot E$ is pointwise bounded by f and $\int \infty \cdot E \, d\mu = \infty \cdot \mu(E) = \infty$. Therefore $\int f \, d\mu = \infty$.

Assume $\mu(\{f > 0\}) > 0$ and for $n \in \mathbb{N}$ let $E_n = \{x \in \Omega : f(x) \geq 1/n\}$. Then $\{x \in \Omega : f(x) > 0\} = E_1 \cup E_2 \cup \dots$. That means there exists an $n \in \mathbb{N}$ for which $\mu(E_n) > 0$. Now, the simple function $1_{E_n}/n$ is pointwise bounded by f and thus $\int f \, d\mu \geq \int 1_{E_n}/n \, d\mu = \mu(E_n)/n > 0$.

Assume $\mu(\{f > 0\}) = 0$. And let $a_1, \dots, a_n, E_1, \dots, E_n$ be the canonical form of a simple function $\varphi \leq f$. Then for any k with $a_k > 0$ we must have $\mu(E_k) = 0$. Thus by definition $\int \varphi \, d\mu = 0$ $\square$

The convergence result Lemma 2.1.7 had the two boundedness assumptions that the domain has finite measure, and that all func-

tions are bounded from below and from above. Next investigate what happens if we relax those conditions.

Example 2.1.10. For $n \in \mathbb{N}$ let $f_n = 1_{[n, n+1]}$. Then for every $x \in \mathbb{R}$ we have $f_n(x) \rightarrow 0$ and $0 \leq f_n \leq 1$, but

$$\lim_{n \rightarrow 0} \int f_n \, d\mathcal{L} = 1 > 0 = \int \lim_{n \rightarrow \infty} f_n(x) \, d\mathcal{L}(x).$$

That means the pointwise limit of nonnegative functions can have a strictly smaller integral than the limit of the integrals. However, it cannot be strictly larger.

Theorem 2.1.11 (Fatou's lemma). *Let μ be a measure on Ω and let $f_1, f_2, \dots : \Omega \rightarrow [0, \infty]$. Then*

$$\int \liminf_{n \rightarrow \infty} f_n \, d\mu \leq \liminf_{n \rightarrow \infty} \int f_n \, d\mu.$$

Proof. Let

$$0 \leq \varphi = \sum_{j=1}^k a_j 1_{A_j} \leq \liminf_{n \rightarrow \infty} f_n$$

be a simple function. As in the proof of Proposition 2.1.3 we can make $A_1, \dots, A_k$ disjoint.

Let $0 < t < 1$. Define

$$B_{j,n} = \{x \in A_j : \forall m \geq n \ f_m(x) > ta_j\}.$$

Then for any j we have

$$B_{j,1} \subset B_{j,2} \subset \dots \subset A_j, \quad \bigcup_{n=1}^{\infty} B_{j,n} = A_j$$

and for any n we have

$$f_n > \sum_{j=1}^k ta_j 1_{B_{j,n}} \quad \Rightarrow \quad \int f_n \, d\mu > \sum_{j=1}^k ta_j \mu(B_{j,n}).$$

Thus, by Lemma 1.2.7

$$\liminf_{n \rightarrow \infty} \int f_n \, d\mu \geq \sum_{j=1}^k t a_j \liminf_{n \rightarrow \infty} \mu(B_{j,n}) = t \int \varphi \, d\mu.$$

Since $0 < t < 1$ and $\varphi \leq \liminf_{n \rightarrow \infty} f_n$ were arbitrary this finishes the proof. $\square$

Corollary 2.1.12. Let $f, f_1, f_2, \dots : \Omega \rightarrow [0, \infty]$ with $f_n \leq f$ and $f_n \rightarrow f$ almost everywhere. Then

$$\int f \, d\mu = \lim_{n \rightarrow \infty} \int f_n \, d\mu.$$

Proof. We have

$$\limsup_{n \rightarrow \infty} \int f_n \, d\mu \leq \int f \, d\mu \leq \liminf_{n \rightarrow \infty} \int f_n \, d\mu.$$

$\square$

Corollary 2.1.13 (Monotone convergence theorem). Let $f_1, f_2, \dots : \Omega \rightarrow [0, \infty]$ with $f_1 \leq f_2 \leq \dots$. Then

$$\int \lim_{n \rightarrow \infty} f_n \, d\mu = \lim_{n \rightarrow \infty} \int f_n \, d\mu.$$

2.1.3 Measurable functions

For $f : \Omega \rightarrow [-\infty, \infty]$ recall its positive and negative parts f^+ and f^- from Definition 1.3.17. Then $f^\pm \geq 0$, so its Lebesgue integral is defined.

Definition 2.1.14. We say that f is **integrable** if $\int f^+ \, d\mu < \infty$ or $\int f^- \, d\mu < \infty$, in which case we define its Lebesgue measure by

$$\int f \, d\mu := \int f^+ \, d\mu - \int f^- \, d\mu,$$

using the definition of the Lebesgue measure of the nonnegative functions $f^\pm$.

Lemma 2.1.15. Let $f_1, f_2 \geq 0$ be measurable such that for at least one $i \in \{1, 2\}$ we have $\int f_i \, d\mu < \infty$. Then

$$\int f \, d\mu = \int f_1 \, d\mu - \int f_2 \, d\mu.$$

Proof. We have

$$f^+ = (f_1 - f_2)1_{\{f_1 \geq f_2\}}, \quad f^- = (f_2 - f_1)1_{\{f_1 < f_2\}}.$$

By the linearity of the integral for nonnegative function we have

$$\begin{aligned} \int f_1 1_{\{f_1 \geq f_2\}} \, d\mu &= \int (f_1 - f_2) 1_{\{f_1 \geq f_2\}} + f_2 1_{\{f_1 \geq f_2\}} \, d\mu \\ &= \int (f_1 - f_2) 1_{\{f_1 \geq f_2\}} \, d\mu + \int f_2 1_{\{f_1 \geq f_2\}} \, d\mu \end{aligned}$$

which means

$$\int (f_1 - f_2) 1_{\{f_1 \geq f_2\}} \, d\mu = \int f_1 1_{\{f_1 \geq f_2\}} \, d\mu - \int f_2 1_{\{f_1 \geq f_2\}} \, d\mu.$$

Argueing similarly for f^- we have

$$\int (f_2 - f_1)1_{\{f_1 < f_2\}} d\mu = \int f_2 1_{\{f_1 < f_2\}} d\mu - \int f_1 1_{\{f_1 < f_2\}} d\mu$$

and we can conclude

$$\begin{aligned} \int f d\mu &= \int f^+ d\mu - \int f^- d\mu \\ &= \int f_1 1_{\{f_1 \geq f_2\}} d\mu - \int f_2 1_{\{f_1 \geq f_2\}} d\mu \\ &\quad - \int f_2 1_{\{f_1 < f_2\}} d\mu + \int f_1 1_{\{f_1 < f_2\}} d\mu \\ &= \int f_1 d\mu - \int f_2 d\mu. \end{aligned}$$

□

Proposition 2.1.16. Let μ be a measure on Ω .

- (i) (linearity) Let $a, b \in [-\infty, \infty]$ and $f, g : \Omega \rightarrow [-\infty, \infty]$ be integrable. Then

$$\int af + bg \, d\mu = a \int f \, d\mu + b \int g \, d\mu,$$

provided the right hand side is well defined, i.e. not of the form $\infty - \infty$.

- (ii) (additivity) Let $E, F \subset \Omega$ be disjoint and f be integrable. Then

$$\int_{E \cup F} f \, d\mu = \int_E f \, d\mu + \int_F f \, d\mu.$$

- (iii) (monotonicity) Let f, g be integrable such that $f \leq g$. Then

$$\int f \, d\mu \leq \int g \, d\mu.$$

(iv) (triangle inequality) Let f be integrable. Then $|f|$ is integrable and

$$\left| \int f \, d\mu \right| \leq \int |f| \, d\mu.$$

Proof. Since $(-f)^+ = f^-$ and $(-f)^- = f^+$ it follows that $\int (-f) \, d\mu = -\int f \, d\mu$. Similarly we can prove $\int af \, d\mu = a \int f \, d\mu$. Thus, in order to prove linearity it remains to show $\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu$, provided the right hand side is well defined. Now, $(f+g)^+ \leq f^+ + g^+$ and $(f+g)^- \leq f^- + g^-$, which means that if $\int f \, d\mu + \int g \, d\mu$ is well defined then so is $\int f + g \, d\mu$. Now, $f + g = f^+ + g^+ - (f^- + g^-)$. Hence by Lemma 2.1.15 and linearity for nonnegative functions we can conclude

$$\begin{aligned} \int f + g \, d\mu &= \int f^+ + g^+ \, d\mu - \int f^- + g^- \, d\mu \\ &= \int f^+ \, d\mu + \int g^+ \, d\mu - \int f^- \, d\mu - \int g^- \, d\mu \end{aligned}$$

$$= \int f \, d\mu + \int g \, d\mu.$$

As before, additivity follows from linearity.

For monotonicity we observe $f^+ \leq g^+$ and $f^- \geq g^-$ and monotonicity follows from monotonicity of the integral of nonnegative functions.

The triangle inequality follows from $f \leq |f|$, $-f \leq |f|$ linearity and monotonicity. $\square$

Proposition 2.1.17. Let $(\Omega, \mathcal{M}, \mu)$ be σ -finite and $f : \Omega \rightarrow [0, \infty]$ with $\int f \, d\mu < \infty$ and let $\varepsilon > 0$.

- (i) There exists a set $B \in \mathcal{M}$ with $\mu(B) < \infty$ such that

$$\int_{\Omega \setminus B} f \, d\mu < \varepsilon.$$

- (ii) There exists a $\delta > 0$ such that for all $E \in \mathcal{M}$ with $\mu(E) < \delta$ we have

$$\int_E f \, d\mu < \varepsilon.$$

Proof. Exercise. □

Theorem 2.1.18 (Dominated convergence). *Let $g \geq 0$ with $\int g \, d\mu < \infty$ and $f, f_1, f_2, \dots : \Omega \rightarrow [-\infty, \infty]$ such that for all n we have $|f_n| \leq g$ almost everywhere and $f_n \rightarrow f$ almost everywhere. Then*

$$\lim_{n \rightarrow \infty} \int |f_n - f| \, d\mu = 0.$$

Proof. By Fatou's lemma Theorem 2.1.11 we have

$$\begin{aligned} \int 2g \, d\mu &= \int \liminf_{n \rightarrow \infty} (2g - |f - f_n|) \, d\mu \\ &\leq \liminf_{n \rightarrow \infty} \int (2g - |f - f_n|) \, d\mu \end{aligned}$$

$$= \int 2g \, d\mu - \limsup_{n \rightarrow \infty} \int |f - f_n| \, d\mu$$

which means

$$\limsup_{n \rightarrow \infty} \int |f - f_n| \, d\mu = 0.$$

□

2025-10

2.2 L^p -spaces

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space and let $1 \leq p \leq \infty$. Then for any μ -measurable $f : \Omega \rightarrow [-\infty, \infty]$, the function $|f|^p$ is measurable and nonnegative and thus integrable. Define

$$\|f\|_{L^p(\Omega, \mathcal{M}, \mu)} = \left(\int |f|^p \, d\mu \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

$$\|f\|_{L^\infty(\Omega, \mathcal{M}, \mu)} = \inf\{\lambda \geq 0 : \mu(\{|f| > \lambda\}) = 0\} \quad .$$

For brevity we omit $\mathcal{M}$ in the notation. If $g = f$ μ -almost everywhere then $\|g\|_{L^p(\Omega, \mu)} = \|f\|_{L^p(\Omega, \mu)}$ and $\|f - g\|_{L^p(\Omega, \mu)} = 0$. For f measurable denote by $[f]$ the equivalence class of functions g that equal f μ -almost everywhere. We define

$$L^p(\Omega, \mathcal{M}, \mu) = \{[f] : f : \Omega \rightarrow [-\infty, \infty] \text{ measurable, } \|f\|_{L^p(\Omega, \mathcal{M}, \mu)} < \infty\}.$$

For $[f] \in L^p(\Omega, \mathcal{M}, \mu)$ we will also slightly abuse notation and write $f \in L^p(\Omega, \mathcal{M}, \mu)$.

You will show in an exercise that $L^p(\Omega, \mathcal{M}, \mu)$ is a normed space with norm $\|\cdot\|_{L^p(\Omega, \mathcal{M}, \mu)}$. In fact, the normed space is complete, making it a Banach space. You will probably see it in the functional analysis course.

Part of being a norm is the triangle inequality,

$$\|f + g\|_{L^p(\Omega, \mathcal{M}, \mu)} \leq \|f\|_{L^p(\Omega, \mathcal{M}, \mu)} + \|g\|_{L^p(\Omega, \mathcal{M}, \mu)}.$$

For $1 \leq p \leq \infty$ denote $p' = \frac{p}{1-p}$ so that $\frac{1}{p} + \frac{1}{p'} = 1$. Note, that $(p')' = p$. You will also show Hölder's inequality, that for $f \in L^p(\Omega, \mathcal{M}, \mu)$ and $g \in L^{p'}(\Omega, \mathcal{M}, \mu)$ we have $fg \in L^1(\Omega, \mathcal{M}, \mu)$ with

$$\|fg\|_{L^1(\Omega, \mathcal{M}, \mu)} \leq \|f\|_{L^p(\Omega, \mathcal{M}, \mu)} \|g\|_{L^{p'}(\Omega, \mathcal{M}, \mu)}.$$

Proposition 2.2.1. Let $1 \leq p \leq \infty$, $f \in L^p(\Omega, \mathcal{M}, \mu)$ and $\varepsilon > 0$. Then

- (i) exists a simple function φ with $\|\varphi - f\|_{L^p(\Omega, \mathcal{M}, \mu)} < \varepsilon$
- (ii) if $\Omega = \mathbb{R}^d$ and $p < \infty$ exists a step function φ with $\|\varphi - f\|_{L^p(\mathbb{R}^d, \mathcal{L})} < \varepsilon$
- (iii) if $\Omega = \mathbb{R}^d$ and $p < \infty$ exists a compactly supported continuous function φ with $\|\varphi - f\|_{L^p(\mathbb{R}^d, \mathcal{L})} < \varepsilon$.

Proof. (i) For $x \in \Omega$ define by $\sigma(x) = 1$ if $f(x) \geq 0$ and $\sigma(x) = -1$ if $f(x) < 0$. We first consider $p = \infty$. Apply Corollary 1.3.15

to $|f|$, take f_n from the proof of Corollary 1.3.15. Then $g_n = \sigma f_n$ is a simple function. If n such that $2^n \geq \|f\|_{L^\infty(\Omega, \mathcal{M}, \mu)}$ then $\mu(A_n^{2^{2n}}) = 0$, and thus for μ -almost every $x \in \Omega$ we have

$$|g_n(x) - f(x)| = |f(x)| - f_n(x) \leq 2^{-n}.$$

Hence $\|g_n - f\|_{L^\infty(\Omega, \mathcal{M}, \mu)} \leq 2^{-n}$.

Now consider $1 \leq p < \infty$. By Corollary 1.3.15 applied to $|f|^p$ exist a sequence of simple functions $f_1, f_2, \dots$ that monotonously converge to $|f|^p$ from below. Define the step function $g_n = \sigma(f_n)^{1/p}$. Then $|f - g_n|^p$ goes to zero μ -almost everywhere and $0 \leq |f - g_n|^p \leq |f|^p$. Thus, by dominated convergence we have $\|f - g_n\|_{L^p(\Omega, \mathcal{M}, \mu)} \rightarrow 0$.

- (ii) Take φ from (i). Then φ is a weighted sum of characteristic functions of sets with finite measure. By the triangle inequality it thus suffices to show that for each $E \subset \mathbb{R}^d$ with $\mathcal{L}(E) < \infty$

exists a finite set $\mathcal{Q}$ of disjoint rectangles with

$$\left\| 1_E - \sum_{Q \in \mathcal{Q}} 1_Q \right\|_{L^p(\mathbb{R}^d, \mathcal{L})} < \varepsilon.$$

Since $\sum_{Q \in \mathcal{Q}} 1_Q = 1_{\bigcup \mathcal{Q}}$ and

$$\|1_E - 1_{\bigcup \mathcal{Q}}\|_{L^p(\mathbb{R}^d, \mathcal{L})} = \mathcal{L}(E \Delta \bigcup \mathcal{Q})^{\frac{1}{p}},$$

the existence of such $\mathcal{Q}$ is a consequence of Proposition 1.2.18 (v).

(iii) Let $f = 1_Q$ for some cube Q . Let Then $\varphi_{Q, \delta}$ given by

$$\varphi_{Q, \delta}(x) = \min\{1, \inf\{|x - y| : y \in \mathbb{R}^d \setminus Q\}/\delta\}$$

is continuous, supported on Q , and, for δ sufficiently small satisfies

$$\|1_Q - \varphi_{Q, \delta}\|_{L^p(\mathbb{R}^d, \mathcal{L})} \leq \|1_Q - 1_{\{x: \inf\{|x - y| : y \in \mathbb{R}^d \setminus Q\} < \delta\}}\|_{L^p(\mathbb{R}^d, \mathcal{L})} < \varepsilon$$

by an explicit calculation or the monotone convergence theorem. By the triangle inequality, the proof of (ii), and since finite sums of compactly supported continuous functions are compactly supported and continuous, this suffices to conclude the proof.

□

2.3 Fubini's theorem

Definition 2.3.1. For $i = 0, 1$ let μ_i be an outer measure on Ω_i . Define the **product (outer) measure** of μ_0 and μ_1 , $\mu_0 \times \mu_1 : 2^{\Omega_0 \times \Omega_1} \rightarrow [0, \infty]$, by

$$(\mu_0 \times \mu_1)(E) = \inf \left\{ \sum_{n=1}^{\infty} \mu_0(A_0^n) \mu_1(A_1^n) : A_i^n \subset \Omega_i \text{ } \mu_i\text{-measurable, } E \subset \bigcup_{n=1}^{\infty} A_0^n \times A_1^n \right\}$$

Definition 2.3.2. Let $(\Omega, \mathcal{M}, \mu)$ be a complete measure space and $E \subset \Omega$ with $\mu(E) = 0$. Then for $f : \Omega \setminus E \rightarrow [-\infty, \infty]$ we say that f is measurable or integrable if

$$\bar{f}(x) = \begin{cases} f(x) & x \in \Omega \setminus E, \\ 0 & x \in E \end{cases}$$

is measurable or integrable respectively and we define

$$\int f \, d\mu = \int \bar{f} \, d\mu.$$

Theorem 2.3.3. *Let μ_0 and μ_1 be σ -finite and complete.*

(i) *Then $\mu_0 \times \mu_1$ is a σ -finite measure and for each μ_i -measurable sets $A_i \subset 2^{\Omega_i}$ their product $A_0 \times A_1$ is $\mu_0 \times \mu_1$ -measurable with*

$$(\mu_0 \times \mu_1)(A_0 \times A_1) = \mu_0(A_0)\mu_1(A_1).$$

(ii) For any $\mu_0 \times \mu_1$ -measurable $E \subset 2^{\Omega_0 \times \Omega_1}$ their cross-sections

$$E_{i,x} = \{a_i : (a_0, a_1) \in E, a_{1-i} = x\}$$

are μ_i -measurable for μ_{1-i} -almost every $x \in \Omega_{1-i}$. Moreover, the maps $x \mapsto \mu_i(E_{i,x})$ are μ_{1-i} -measurable and

$$(\mu_0 \times \mu_1)(E) = \int \mu_i(E_{i,x}) d\mu_{1-i}(x).$$

Note, that here we need to make use of Definition 2.3.2.

(iii) For $f : \Omega_0 \times \Omega_1 \rightarrow [-\infty, \infty]$ and $x \in \Omega_{1-i}$ denote $f_{i,x}(y) = f(x, y)$. If f is $\mu_0 \times \mu_1$ -integrable then for μ_{1-i} -almost every $x \in \Omega_{1-i}$ the map $f_{i,x} : \Omega_i \rightarrow [-\infty, \infty]$ is μ_i -integrable, the map

$$x \mapsto \int f_{i,x} d\mu_i$$

is μ_{1-i} -integrable and

$$\int f \, d(\mu_0 \times \mu_1) = \int \left[\int f_{i,x}(y) \, d\mu_i(y) \right] d\mu_{1-i}(x).$$

In particular, (iii) means

$$\int \int f(x, y) \, d\mu_0(x) \, d\mu_1(y) = \int \int f(x, y) \, d\mu_1(y) \, d\mu_0(x).$$

Before we start with the proof we consider the case of Lebesgue measure. Denote by $\mathcal{L}_*^n$ Lebesgue outer measure on $\mathbb{R}^n$. You will show in an exercise that $\mathcal{L}_*^n \times \mathcal{L}_*^k = \mathcal{L}_*^{n+k}$. In particular, they have the same measurable sets and also agree as measures. That means for an $\mathcal{L}^{n+k}$ -integrable function $f : \mathbb{R}^{n+k} \rightarrow \mathbb{R}$ we have

$$\int_{\mathbb{R}^{n+k}} f \, d\mathcal{L}^{n+k} = \int_{\mathbb{R}^k} \int_{\mathbb{R}^n} f(x, y) \, d\mathcal{L}^n(x) \, d\mathcal{L}^k(y)$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^k} f(x, y) \, d\mathcal{L}^k(y) \, d\mathcal{L}^n(x).$$

Proof. Define

$$\mathcal{P}_0 = \{A_0 \times A_1 : A_i \subset \Omega_i \text{ } \mu_i\text{-measurable}\}$$

$$\mathcal{P}_1 = \left\{ \bigcup_{n=1}^{\infty} E_n : E_n \in \mathcal{P}_0 \right\}$$

$$\mathcal{P}_2 = \left\{ \bigcap_{n=1}^{\infty} E_n : E_n \in \mathcal{P}_1 \right\}.$$

For $i, n = 0, 1$ let $A_i^n \subset \Omega_i$ be μ_i -measurable. Then

$$\begin{aligned} (A_0^0 \times A_1^0) \cap (A_0^1 \times A_1^1) &= (A_0^0 \cap A_0^1) \times (A_1^0 \cap A_1^1), \\ (A_0^0 \times A_1^0) \setminus (A_0^1 \times A_1^1) &= ([A_0^0 \setminus A_0^1] \times A_1^0) \cup (A_0^1 \times [A_1^0 \setminus A_1^1]) \end{aligned}$$

$$\cup ([A_0^0 \cap A_0^1] \times [A_1^0 \setminus A_1^1]).$$

That means both intersections and set differences of two elements from $\mathcal{P}_0$ can be written as finite disjoint unions of elements from $\mathcal{P}_0$. Let $E_1, E_2, \dots \in \mathcal{P}_0$.

Claim 1. Each element in $\mathcal{P}_1$ can be written as a countable disjoint union of elements from $\mathcal{P}_0$.

Proof. We show by induction that for each $n \in \mathbb{N}$ exists a collection $\mathcal{A}_n$ of disjoint elements from $\mathcal{P}_0$ such that

$$E_1 \cup \dots \cup E_n = \bigcup \mathcal{A}_n$$

and $A_n \subset A_{n+1}$. Then we can conclude that $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \dots$ is a countable set of disjoint elements from $\mathcal{P}_0$ and $\bigcup \mathcal{A} = E_1 \cup E_2 \cup \dots$

For $n \in \mathbb{N}$ take $\mathcal{A}_n = \{F_1, \dots, F_N\}$ from the inductive hypothesis. First we prove by an induction argument within that for each $k =$

$0, \dots, N$ exists a finite set $\mathcal{B}_k$ of disjoint elements from $\mathcal{P}_0$ with

$$\bigcup \mathcal{B}_k = E_{n+1} \cup F_1 \cup \dots \cup F_k,$$

$$\mathcal{B}_k \supset \{F_1, \dots, F_k\} :$$

Take $\mathcal{B}_k$ from the inductive hypothesis. For each $E \in \mathcal{B}_k$ we may write $E \cup F_{k+1}$ as the disjoint union $F_{k+1} \cup (E \setminus F_{k+1})$. By the previous argument this disjoint union can be written as a finite disjoint union of elements from $\mathcal{P}_0$. We collect in $\mathcal{B}_{k+1}$ all the resulting sets. They are all disjoint, cover $F_1 \cup \dots \cup F_{k+1}$. Among its members are F_{k+1} , and also F_m for all $m \leq k$ because they appear as $F_m \setminus F_{k+1} = F_m$. This proves the inner induction argument. In particular, the conclusion holds for $k = N$ and we can set $\mathcal{A}_{n+1} = \mathcal{B}_N$, finishing the outer induction. $\square$

Let $\mathcal{F}$ be the collection of sets $E \subset \Omega_0 \times \Omega_1$ for which

$$x \mapsto 1_E(x, y)$$

is μ_0 -measurable for μ_1 -almost every $y \in \Omega_1$, and

$$y \mapsto \int 1_E(x, y) \, d\mu_0(x)$$

is μ_1 -measurable. That means $\mathcal{P}_0 \subset \mathcal{F}$. For $E \in \mathcal{F}$ define

$$\rho(E) = \int \left[\int 1_E(x, y) \, d\mu_0(x) \right] d\mu_1(y).$$

Claim 2. Let $E, F \in \mathcal{F}$ with $E \subset F$.

- (i) Then $\rho(E) \leq \rho(F)$.
- (ii) Let $E_1 \cup E_2 \cup \dots \in \mathcal{P}_0$. Then $E_1 \cup E_2 \cup \dots \in \mathcal{F}$ and

$$\rho(E_1 \cup E_2 \cup \dots) \leq \sum_{n=1}^{\infty} \rho(E_n).$$

In particular, $\mathcal{P}_1 \subset \mathcal{F}$.

(iii) If $E_1, E_2, \dots \in \mathcal{P}_0$ are disjoint then

$$\rho(E_1 \cup E_2 \cup \dots) = \sum_{n=1}^{\infty} \rho(E_n).$$

(iv) If $E_1, E_2, \dots \in \mathcal{P}_1$ with $\rho(E_1) < \infty$ then $E_1 \cap E_2 \cap \dots \in \mathcal{F}$ with

$$\rho(E_1 \cap E_2 \cap \dots) = \lim_{n \rightarrow \infty} \rho(E_1 \cap \dots \cap E_n).$$

In particular, $\mathcal{P}_2 \subset \mathcal{F}$.

Proof. If $E \subset F$ then for μ_1 -almost every $y \in \Omega_1$ we have $\{x : (x, y) \in E\} \subset \{x : (x, y) \in F\}$ and monotonicity follows from monotonicity of the integrals.

Let $E_1, E_2, \dots \in \mathcal{P}_0$ and let $n \in \mathbb{N}$. Then for $y \in \Omega_1$ the map $x \mapsto \sum_{k=1}^n 1_{E_k}(x, y)$ is a finite sum of μ_0 -measurable maps and thus measurable, and converges pointwise monotoneously to

$x \mapsto \sum_{k=1}^{\infty} 1_{E_k}(x, y)$. By linearity and monotone convergence the same is true for

$$y \mapsto \int \sum_{k=1}^n 1_{E_k}(x, y) d\mu_0(x) \rightarrow \int \sum_{k=1}^{\infty} 1_{E_k}(x, y) d\mu_0(x)$$

and again by linearity and monotone convergence we obtain

$$\int \int \lim_{n \rightarrow \infty} \sum_{k=1}^n 1_{E_k}(x, y) d\mu_0(x) d\mu_1(y) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \rho(E_k). \quad (2.3.1)$$

If $E_1, E_2, \dots$ are disjoint then

$$1_{E_1 \cup E_2 \cup \dots}(x, y) = \lim_{n \rightarrow \infty} \sum_{k=1}^n 1_{E_k}(x, y)$$

and we can conclude

$$\rho(E_1 \cup E_2 \cup \dots) = \sum_{k=1}^{\infty} \rho(E_k),$$

finishing the proof in the disjoint case.

If they are not disjoint, we can make them disjoint, thereby establishing the required measurability properties from the previous case. Moreover, since $1_{E_1 \cup E_2 \cup \dots} \leq \sum_{k=1}^{\infty} 1_{E_k}$ we obtain from (2.3.1) that

$$\rho(E_1 \cup E_2 \cup \dots) \leq \sum_{k=1}^{\infty} \rho(E_k).$$

Finally, let $E_1, E_2, \dots \in \mathcal{P}_1$. Then for each n exist disjoint rectangles $F_n^1, F_n^2, \dots \in \mathcal{P}_0$ with $E_n = F_n^1 \cup F_n^2 \cup \dots$. That means $E_1 \cap \dots \cap E_n$ equals the union of all sets of the form $F_1^{k_1} \cap \dots \cap F_n^{k_n}$ with $k_1, \dots, k_n \in \mathbb{N}$, which is a countable union of disjoint rectangles. By the previous case this establishes the measurability of

$$x \mapsto 1_{E_1 \cap \dots \cap E_n}(x, y) \rightarrow 1_{E_1 \cap E_n \cap \dots}(x, y)$$

and

$$y \mapsto \int 1_{E_1 \cap \dots \cap E_n}(x, y) \, d\mu_0(x)$$

Since $\rho(E_1) < \infty$ then for μ_1 -almost every $y \in \Omega_1$ we have

$$\int 1_{E_1}(x, y) \int \mu_0(x) < \infty.$$

Similarly as in the previous case we can now argue by dominated convergence or the measure continuity lemma that

$$y \mapsto \int 1_{E_1 \cap E_2 \cap \dots}(x, y) d\mu_0(x) = \lim_{n \rightarrow \infty} \int 1_{E_1 \cap \dots \cap E_n}(x, y) d\mu_0(x)$$

is μ_1 -measurable and

$$\begin{aligned} \rho(E_1 \cap E_2 \cap \dots) &= \int \lim_{n \rightarrow \infty} \int 1_{E_1 \cap \dots \cap E_n}(x, y) d\mu_0(x) d\mu_1(y) \\ &= \lim_{n \rightarrow \infty} \rho(E_1 \cap \dots \cap E_n). \end{aligned}$$

□

Claim 3. For each $E \subset \Omega_0 \times \Omega_1$ we have

$$\begin{aligned}(\mu_0 \times \mu_1)(E) &= \inf\{(\mu_0 \times \mu_1)(G) : E \subset G \in \mathcal{P}_1\} \\ &= \inf\{\rho(G) : E \subset G \in \mathcal{P}_1\}.\end{aligned}$$

Proof. By Claim 1 for any $G \in \mathcal{P}_1$ exist measurable $A_i^n \subset \Omega_i$ such that

$$G = \bigcup_{n=1}^{\infty} A_0^n \times A_1^n,$$

where the union is disjoint. Thus

$$(\mu_0 \times \mu_1)(G) \leq \sum_{n=1}^{\infty} \mu_0(A_0^n) \mu_1(A_1^n) = \rho(G).$$

Since $\mu_0 \times \mu_1$ is monotone we can conclude the claim with " $\leq$ " instead of equalities.

In order to prove " $\geq$ " let $\varepsilon > 0$. Then there exist A_i^n with $E \subset \bigcup_{n=1}^{\infty} A_0^n \times A_1^n \in \mathcal{P}_1$ and

$$(\mu_0 \times \mu_1)(E) + \varepsilon \geq \sum_{n=1}^{\infty} \mu_0(A_0^n) \mu_1(A_1^n) \geq \rho\left(\bigcup_{n=1}^{\infty} A_0^n \times A_1^n\right).$$

□

Let $A_0 \times A_1 \in \mathcal{P}_0$ and let $\varepsilon > 0$. Then by Claim 3 exists $G \in \mathcal{P}_1$ with

$$\begin{aligned} (\mu_0 \times \mu_1)(A_0 \times A_1) &\leq \mu_0(A_0) \mu_1(A_1) = \rho(A_0 \times A_1) \\ &\leq \rho(G) \leq (\mu_0 \times \mu_1)(A_0 \times A_1) + \varepsilon. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ we obtain

$$(\mu_0 \times \mu_1)(A_0 \times A_1) = \mu_0(A_0) \mu_1(A_1).$$

In order to finish the proof of (i) it remains to show that $A_0 \times A_1$ is $\mu_0 \times \mu_1$ -measurable. To that end let $E \subset \Omega_0 \times \Omega_1$ and $\varepsilon > 0$. Then there exists a $G \in \mathcal{P}_1$ with $G \supset E$ and

$$\rho(G) \leq (\mu_0 \times \mu_1)(E) + \varepsilon.$$

Since $G \cap (A_0 \times A_1)$ and $G \setminus (A_0 \times A_1)$ are disjoint and belong to $\mathcal{P}_1$ we have

$$\begin{aligned} & (\mu_0 \times \mu_1)(E \cap (A_0 \times A_1)) + (\mu_0 \times \mu_1)(E \setminus (A_0 \times A_1)) \\ & \leq \rho(G \cap (A_0 \times A_1)) + \rho(G \setminus (A_0 \times A_1)) \\ & = \rho(G) \end{aligned}$$

and letting $\varepsilon \rightarrow 0$ we can conclude that $A_0 \times A_1$ is $\mu_0 \times \mu_1$ -measurable and finish the proof of (i).

Claim 4. We have $\mathcal{P}_2 \subset \mathcal{F}$ and for each $E \subset \Omega_0 \times \Omega_1$ with $(\mu_0 \times \mu_1)(E) < \infty$ exists a $E \subset G \in \mathcal{P}_2$ with

$$(\mu_0 \times \mu_1)(E) = \rho(G) = (\mu_0 \times \mu_1)(G).$$

Proof. By Claim 3 for each $n \in \mathbb{N}$ exists $E \subset G_n, F_n \in \mathcal{P}_1$ with

$$\begin{aligned}(\mu_0 \times \mu_1)(E) &\leq \rho(G_n), (\mu_0 \times \mu_1)(F_n) \\ &\leq (\mu_0 \times \mu_1)(E) + 2^{-n} < \infty.\end{aligned}$$

Set $G = G_1 \cap F_1 \cap G_2 \cap F_2 \cap \dots \in \mathcal{P}_2$. It follows from the monotonicity of $(\mu_0 \times \mu_1)$ that $(\mu_0 \times \mu_1)(E) = (\mu_0 \times \mu_1)(G)$, and

$$(\mu_0 \times \mu_1)(E) \leq \rho(G) \leq \lim_{n \rightarrow \infty} \rho(G_1 \cap \dots \cap G_n) = (\mu_0 \times \mu_1)(E).$$

□

Let $E \subset \Omega_0 \times \Omega_1$ be $\mu_0 \times \mu_1$ -measurable with $(\mu_0 \times \mu_1)(E) < \infty$. Then take $G \supset E$ from Claim 4 so that $(\mu_0 \times \mu_1)(G \setminus E) = 0$. Then there exists $G \setminus E \subset F \in \mathcal{P}_2$ with $\rho(F) = 0$. That means for μ_1 -almost every $y \in \Omega_1$ we have $\mu_0(F_{0,y}) = 0$. Since $G_{0,y}$ is μ_0 -measurable and μ_0 is complete this means that also $E_{0,y}$ is μ_0 -measurable with

$$\mu_0(G_{0,y}) = \mu_0(E_{0,y})$$

and thus

$$\begin{aligned}(\mu_0 \times \mu_1)(E) &= (\mu_0 \times \mu_1)(G) = \int \mu_0(G_{0,y}) \, d\mu_1(y) \\ &= \int \mu_0(E_{0,y}) \, d\mu_1(y),\end{aligned}$$

proving (ii) in case $(\mu_0 \times \mu_1)(E) < \infty$. If $(\mu_0 \times \mu_1)(E) = \infty$ then since μ_i is σ -additive so is $\mu_0 \times \mu_1$, which means we can decompose E into countably many pieces with finite measure and apply (ii) to each piece. By monotone convergence this implies (ii) for E .

Observe, that (iii) is (ii) in the case $f = 1_E$. If f is nonnegative we can hence deduce (iii) from (ii), Theorem 1.3.14, linearity of the integral and the monotone convergence theorem. If f is integrable we can deduce (iii) from $f = f^+ - f^-$ and the nonnegative case. $\square$

Chapter 3

Differentiation and integration on $\mathbb{R}$

3.1 The Lebesgue differentiation theorem

Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable¹³⁰ everywhere with continuous derivative $f = F'$. The fundamental theorem of calculus states that for

any $a < b$ we have

$$F(b) - F(a) = \int_{[a,b]} f \, d\mathcal{L}.$$

Conversely, if $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous then the map $F : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$F(x) = \begin{cases} \int_0^x f \, d\mathcal{L} & x \geq 0, \\ -\int_x^0 f \, d\mathcal{L} & x < 0 \end{cases} \quad (3.1.1)$$

is differentiable with $F' = f$. The latter is straightforward to prove: Let $x \in \mathbb{R}$. Then for any $\varepsilon > 0$ exists a $\delta > 0$ such that for all $0 < \gamma \leq \delta$ we have $|f(y) - f(x)| \leq \varepsilon$. As a consequence,

$$\begin{aligned} \frac{F(x+h) - F(x)}{h} &= \frac{1}{h} \int_{[x, x+h]} f \, d\mathcal{L} \in \left[\inf_{y \in [x, x+h]} f(y), \sup_{y \in [x, x+h]} f(y) \right] \\ &\subset [f(x) - \varepsilon, f(x) + \varepsilon]. \end{aligned}$$

Hence $(F(x+h) - F(x))/h \rightarrow f(x)$.

We want to push this. First, we can generalize to higher dimensions. For a continuous function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, by a similar argument we have

$$\lim_{r \rightarrow 0} \frac{1}{\mathcal{L}(B(x, r))} \int_{B(x, r)} f \, d\mathcal{L} = f(x). \quad (3.1.2)$$

But we are now able to integrate much more general functions than continuous functions.

Definition 3.1.1. We say that a map $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is **locally L^1 -integrable**, or $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, if for every ball B we have $\int_B f \, d\mathcal{L} < \infty$.

That means for $f \in L^1_{\text{loc}}(\mathbb{R})$ we may define F as in (3.1.1). Can we still recover $F'(x) = f$? More generally, is (3.1.2) still true? In general, no, for example take $f = -1_{(-\infty, 0)} + 1_{[0, \infty)}$. Then F is not even differentiable in 0, and the averages of f around 0 are all 0 and do not converge to $f(0) = 1$.

However, we can prove the following:

Theorem 3.1.2 (Lebesgue differentiation theorem). *Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. Then for $\mathcal{L}$ -almost every $x \in \mathbb{R}^d$ we have*

$$\lim_{r \rightarrow 0} \frac{1}{\mathcal{L}(B(x, r))} \int_{B(x, r)} f \, d\mathcal{L} = f(x). \quad (3.1.3)$$

The proof will take a while. We denote by M the **Hardy-Littlewood maximal operator** which maps a function $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ to $Mf : \mathbb{R}^d \rightarrow [0, \infty]$ given by

$$Mf(x) = \sup_{r > 0} \frac{1}{\mathcal{L}(B(x, r))} \int_{B(x, r)} |f| \, d\mathcal{L}.$$

Theorem 3.1.3 (Hardy-Littlewood maximal function theorem). *Let $f \in L^1(\mathbb{R}^d, \mathcal{L})$. Then $Mf : \mathbb{R}^d \rightarrow \mathbb{R}$ is measurable and there exists a $C_d \in \mathbb{R}$ such that for every $\lambda > 0$ we have*

$$\mathcal{L}(\{Mf > \lambda\}) \leq C_d \frac{\|f\|_{L^1(\mathbb{R}^d, \mathcal{L})}}{\lambda}.$$

Letting $\lambda \rightarrow \infty$ we in particular obtain that Mf is finite almost everywhere.

Remark 3.1.4. The conclusion of Theorem 3.1.3 can be written as the weak bound

$$\|Mf\|_{L^{1,\infty}(\mathbb{R}^d,\mathcal{L})} \leq C_d \|f\|_{L^1(\mathbb{R}^d,\mathcal{L})}.$$

Moreover, it is straightforward to see, that if $f \in L^\infty(\mathbb{R}^d, \mathcal{L})$ then for every $x \in \mathbb{R}^d$ we have $Mf(x) \leq \|f\|_{L^\infty(\mathbb{R}^d, \mathcal{L})}$, which means

$$\|Mf\|_{L^\infty(\mathbb{R}^d,\mathcal{L})} \leq \|f\|_{L^\infty(\mathbb{R}^d,\mathcal{L})}.$$

By the **Marcinkiewicz interpolation theorem** this implies that for every $1 < p \leq \infty$ exists a $C_{d,p} \in \mathbb{R}$ such that

$$\|Mf\|_{L^p(\mathbb{R}^d,\mathcal{L})} \leq C_{d,p} \|f\|_{L^p(\mathbb{R}^d,\mathcal{L})}.$$

For the proof of Theorem 3.1.3 we use the

Lemma 3.1.5 (Vitali covering lemma). Let $\mathcal{B}$ be a finite set of balls in $\mathbb{R}^d$. Then there exists a subset $\mathcal{C} \subset \mathcal{B}$ such that the balls in $\mathcal{C}$ are disjoint and

$$\bigcup \mathcal{B} \subset \bigcup_{B \in \mathcal{C}} 3B.$$

Here, $3B$ is the ball with the same center as B and three times its radius.

Proof. We inductively find balls $B_1, \dots, B_n \in \mathcal{B}$ as follows: For $k = 0, 1, \dots$ denote by $\mathcal{P}_k$ the set of balls $B \in \mathcal{B}$ which do not intersect any of the balls $B_1, \dots, B_k$. If $\mathcal{P}_k$ is empty, stop. Otherwise take B_{k+1} to be a ball in $\mathcal{P}_k$ with maximal radius.

This process will terminate at some number n since $\mathcal{B}$ is finite and we set $\mathcal{C} = \{B_1, \dots, B_n\}$. That means $\mathcal{P}_n$ is empty. Note, that $\mathcal{P}_0 = \mathcal{B}$. That means for each $B \in \mathcal{B}$ exists a k with $B \in \mathcal{P}_k \setminus \mathcal{P}_{k+1}$. That means B intersects B_{k+1} . Since B_{k+1} has maximal radius in $\mathcal{P}_k$ it has radius at least as large as B . This means $B \subset 3B_{k+1}$. $\square$

Proof of Theorem 3.1.3. Let $r > 0$ and $x \in \mathbb{R}^d$. Then

$$\lim_{y \rightarrow x} \mathcal{L}(B(x, r) \Delta B(y, r)) = 0.$$

As a consequence of Proposition 2.1.17 (ii) we can conclude that the map

$$x \mapsto \int_{B(x, r)} |f| \, d\mathcal{L}$$

is continuous.

For each $x \in \mathbb{R}^d$ with $Mf(x) > \lambda$ exists a ball $B_{x, \lambda} = B(x, r)$ with

$$\frac{1}{\mathcal{L}(B(x, r))} \int_{B(x, r)} |f| \, d\mathcal{L} > \lambda.$$

By the above continuity that means there exists an $\varepsilon > 0$ such that for all $y \in B(x, \varepsilon)$ we have

$$\frac{1}{\mathcal{L}(B(y, r))} \int_{B(y, r)} |f| \, d\mathcal{L} > \lambda.$$

This means $\{Mf > \lambda\}$ is open and in particular measurable.

Moreover,

$$\{Mf > \lambda\} \subset \bigcup_{x \in \mathbb{R}^d} B_{x,\lambda}.$$

Let $K \subset \{Mf > \lambda\}$ be compact. Then K has a finite subcover $\mathcal{B}$ of balls from $\{B_{x,\lambda} : x \in \mathbb{R}^d\}$. Take $\mathcal{C}$ from Lemma 3.1.5. Then

$$\begin{aligned} \mathcal{L}(K) &\leq \mathcal{L}\left(\bigcup \mathcal{B}\right) \leq \mathcal{L}\left(\bigcup_{B \in \mathcal{C}} 3B\right) \leq \sum_{B \in \mathcal{C}} \mathcal{L}(3B) = 3^d \sum_{B \in \mathcal{C}} \mathcal{L}(B) \\ &< \frac{3^d}{\lambda} \sum_{B \in \mathcal{C}} \int_B |f| \, d\mathcal{L} \leq \frac{3^d}{\lambda} \|f\|_{L^1(\mathbb{R}^d, \mathcal{L})}. \end{aligned}$$

By an exercise we can take compact sets $K_n \subset \{Mf > \lambda\}$ with $\mathcal{L}(K_n) \rightarrow \mathcal{L}(\{Mf > \lambda\})$, finishing the proof. $\square$

Finally, we can prove the Lebesgue differentiation theorem.

Proof of Theorem 3.1.2. For $\lambda > 0$ set

$$E_\lambda(f) = \left\{ x \in \mathbb{R}^d : \limsup_{r \rightarrow 0} \left| \frac{1}{\mathcal{L}(B(x, r))} \int_{B(x, r)} f \, d\mathcal{L} - f(x) \right| > \lambda \right\}.$$

Then we can write the set of points where (3.1.3) fails as $\bigcup_n E_{1/n}$. Thus it is enough to show that for every $\lambda > 0$ we have $\mathcal{L}(E_\lambda(f)) = 0$. To that end, it suffices to show that for each $n \in \mathbb{N}$ we have $\mathcal{L}(E_\lambda(f) \cap B(0, n)) = 0$. Since $E_\lambda(f) \cap B(0, n) = E_\lambda(f1_{B(0, n+1)}) \cap B(0, n)$ and $f1_{B(0, n+1)} \in L^1(\mathbb{R}^d, \mathcal{L})$ this means it suffices to consider the case that $f \in L^1(\mathbb{R}^d, \mathcal{L})$.

Let $\varepsilon > 0$. Then by Proposition 2.2.1 (iii) exists a continuous function g with compact support such that $\|f - g\|_{L^1(\mathbb{R}^d, \mathcal{L})} < \varepsilon$. As noted in (3.1.2), for every $x \in \mathbb{R}^d$ we have

$$\lim_{r \rightarrow 0} \frac{1}{\mathcal{L}(B(x, r))} \int_{B(x, r)} g \, d\mathcal{L} = g(x).$$

Since

$$\begin{aligned} & \frac{1}{\mathcal{L}(B(x, r))} \int_{B(x, r)} f \, d\mathcal{L} - f(x) \\ &= \frac{1}{\mathcal{L}(B(x, r))} \int_{B(x, r)} f - g \, d\mathcal{L} + \frac{1}{\mathcal{L}(B(x, r))} \int_{B(x, r)} g \, d\mathcal{L} - g(x) + g(x) - f(x), \end{aligned}$$

we can conclude, that

$$\limsup_{r \rightarrow 0} \left| \frac{1}{\mathcal{L}(B(x, r))} \int_{B(x, r)} f \, d\mathcal{L} - f(x) \right| \leq M(f - g)(x) + |f(x) - g(x)|.$$

That means

$$E_\lambda \subset \{M(f - g) > \lambda/2\} \cup \{|f - g| > \lambda/2\}.$$

By the Hardy-Littlewood maximal function theorem we have

$$\mathcal{L}(\{M(f - g) > \lambda/2\}) \leq 2C_d \frac{\|f - g\|_{L^1(\mathbb{R}^d, \mathcal{L})}}{\lambda}$$

and by Chebyshev's inequality (see exercise) we have

$$\mathcal{L}(|f - g| > \lambda/2) \leq 2 \frac{\|f - g\|_{L^1(\mathbb{R}^d, \mathcal{L})}}{\lambda}.$$

We can conclude $\mathcal{L}(E_\lambda) \leq 2(C_d + 1)\varepsilon$. Letting $\varepsilon \rightarrow 0$ concludes the proof. $\square$

We can in fact slightly strengthen Theorem 3.1.2.

Definition 3.1.6. We say that $x \in \mathbb{R}^d$ is a **Lebesgue point** of $f \in L^1(\mathbb{R}^d, \mathcal{L})$ if

$$\lim_{r \rightarrow 0} \frac{1}{\mathcal{L}(B(x, r))} \int_{B(x, r)} |f - f(x)| \, d\mathcal{L} = 0.$$

Corollary 3.1.7. Let $f \in L^1(\mathbb{R}^d, \mathcal{L})$. Then $\mathcal{L}$ -almost every $x \in \mathbb{R}^d$ is a Lebesgue point of f .

Proof. For $q \in \mathbb{Q}$ denote by E_q the set of points $x \in \mathbb{R}^d$ for which

$$\lim_{r \rightarrow 0} \frac{1}{\mathcal{L}(B(x, r))} \int_{B(x, r)} |f - q| \, d\mathcal{L} \neq |f(x) - q|$$

or the limit does not exist or $|f(x)| = \infty$. By the Lebesgue differentiation theorem $\mathcal{L}(E_q) = 0$. That means for $E = \bigcup_{q \in \mathbb{Q}} E_q$ also $\mathcal{L}(E) = 0$.

Let $x \in \mathbb{R}^d \setminus E$ and $\varepsilon > 0$. Then there exists $q \in \mathbb{Q}$ with $|f(x) - q| < \varepsilon$. We can conclude

$$\begin{aligned} & \limsup_{r \rightarrow 0} \frac{1}{\mathcal{L}(B(x, r))} \int_{B(x, r)} |f - f(x)| \, d\mathcal{L} \\ & \leq \limsup_{r \rightarrow 0} \frac{1}{\mathcal{L}(B(x, r))} \int_{B(x, r)} |f - q| \, d\mathcal{L} + |f(x) - q| \\ & = 2|f(x) - q| < 2\varepsilon. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ we obtain that x is a Lebesgue point of f . □

Note, that for measurable $E \subset \mathbb{R}^d$ we have $1_E \in L^1_{\text{loc}}(\mathbb{R}^d)$ and

$$\frac{1}{\mathcal{L}(B(x, r))} \int_{B(x, r)} 1_E \, d\mathcal{L} = \frac{\mathcal{L}(B(x, r) \cap E)}{\mathcal{L}(B(x, r))}.$$

Corollary 3.1.8. Let $E \subset \mathbb{R}^d$ be Lebesgue measurable. Then for $\mathcal{L}$ -almost every $x \in E$ we have

$$\lim_{r \rightarrow 0} \frac{\mathcal{L}(B(x, r) \cap E)}{\mathcal{L}(B(x, r))} = 1$$

and for $\mathcal{L}$ -almost every $x \in \mathbb{R}^d \setminus E$ we have

$$\lim_{r \rightarrow 0} \frac{\mathcal{L}(B(x, r) \cap E)}{\mathcal{L}(B(x, r))} = 0.$$

This can be seen as Lebesgue measurable sets being "clumpy". For example they cannot fill exactly half of every ball. Instead, in a

sense most small enough balls are either almost full of E or almost fully disjoint from E .

Coming back to our original question in one dimension, we can conclude that for any $f \in L^1_{\text{loc}}(\mathbb{R})$, the function $F : \mathbb{R} \rightarrow \mathbb{R}$ given as its integral, (3.1.1), is differentiable in $\mathcal{L}$ -almost every $x \in \mathbb{R}$ and $F' = f$. In particular

$$F(b) - F(a) = \int_{[a,b]} f \, d\mathcal{L} = \int_{[a,b]} F' \, d\mathcal{L}.$$

That means we have generalized the fundamental theorem of calculus to functions F given as integrals of $f \in L^1_{\text{loc}}(\mathbb{R})$.

Next, we consider more general functions $F : \mathbb{R} \rightarrow \mathbb{R}$.

3.2 Radon measures

For any measurable $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ with $f \geq 0$ define a map $f\mathcal{L}$ by

$$(f\mathcal{L})(E) = \int_E f \, d\mathcal{L}.$$

Observe, that $f\mathcal{L}$ is a measure on the σ -algebra of Lebesgue measurable sets and finite on bounded sets. This makes $f\mathcal{L}$ a Radon measure, with the property that for F given by

$$F(x) = \begin{cases} \int_0^x f \, d\mathcal{L} & x \geq 0, \\ -\int_x^0 f \, d\mathcal{L} & x < 0 \end{cases},$$

for any interval $(a, b]$ we have $(f\mathcal{L})((a, b]) = F(b) - F(a)$.

Definition 3.2.1. If μ, ν are measures on the same σ -algebra $\mathcal{M}$ and for every $E \in \mathcal{M}$ with $\nu(E) = 0$ we have $\mu(E) = 0$, we say that μ is **absolutely continuous** with respect to ν and write $\mu \ll \nu$.

For example, the measure $f\mathcal{L}$ is absolutely continuous with respect to $\mathcal{L}$. Not all Radon measures on $\mathbb{R}$ are absolutely continuous with respect to Lebesgue measure. We show next that there is a one to one correspondence between general Radon measures μ and general increasing maps F that do not necessarily arise as an integral of a function.

Definition 3.2.2. We say that $F : \mathbb{R} \rightarrow \mathbb{R}$ is **upper semicontinuous** in x if for every $\varepsilon > 0$ exists a $\delta > 0$ such that for all $y \in B(x, \delta)$ we have $f(y) < f(x) + \varepsilon$.

Theorem 3.2.3. *Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be nondecreasing and upper semicontinuous. Then there exists a unique Radon measure μ such that for all $a \leq b$ we have $\mu((a, b]) = F(b) - F(a)$.*

Conversely, if μ is a Radon measure on $\mathbb{R}$ then $F : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$F(x) = \begin{cases} \mu((0, x]) & x \geq 0 \\ -\mu((x, 0]) & x < 0 \end{cases}$$

is nondecreasing and upper semicontinuous.

Proof. We first show the second statement. Given a Radon measure μ , the function F is nondecreasing as a consequence of the monotonicity of μ . Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then $F(x + 1/n) - F(x) = \mu((x, x + 1/n])$, which is finite since μ is a Radon measure. Since $\emptyset = \bigcap_{n \in \mathbb{N}} (x, x + 1/n]$ we can conclude from the measure continuity lemma that $F(x + 1/n) \rightarrow F(x)$. Since F is nondecreasing this proves upper semicontinuity.

For the difficult direction let $F : \mathbb{R} \rightarrow \mathbb{R}$ be nondecreasing and upper semicontinuous and define

$$\mu(E) = \inf \left\{ \sum_{n=1}^{\infty} F(b_n) - F(a_n) : \right.$$

$$a_n < b_n, \quad E \subset \bigcup_{n=1}^{\infty} (a_n, b_n] \}.$$

Then μ is an outer measure on $\mathbb{R}$.

Claim 1. For any $a \leq b$ we have $\mu((a, b]) = F(b) - F(a)$.

Proof. Since $(a, b]$ covers $(a, b]$ follows " $\leq$ ". For the reverse inequality let $\varepsilon > 0$. Then there exist $a_n < b_n$ such that $(a, b] \subset \bigcup_{n=1}^{\infty} (a_n, b_n]$ and

$$\sum_{n=1}^{\infty} F(b_n) - F(a_n) \leq \mu((a, b]) + \varepsilon.$$

Then by the upper semicontinuity of F exists $a' > a$ with $F(a') \leq F(a) + \varepsilon$ and for each n exists $b'_n > b_n$ such that $F(b'_n) \leq F(b_n) + 2^{-n}\varepsilon$. Then the compact set $[a', b]$ is covered by the union of open intervals $\bigcup_{n=1}^{\infty} (a_n, b'_n)$, which hence has a finite subcover $\bigcup_{n=1}^N (a_n, b'_n)$. Inductively remove superfluous intervals, so that each (a_n, b'_n) intersects $[a', b] \setminus \bigcup_{k \in \{1, \dots, N\} \setminus n} (a_k, b'_k)$. Next, reorder the intervals so that

$a_n \leq a_{n+1}$. Then for each n we must have $b'_n > a_{n+1}$ for otherwise there would be a gap $[b'_n, a_{n+1}] \subset [a', b]$ that is not covered. Moreover, $a_1 < a'$ and $b < b'_N$. Since F is nondecreasing we can conclude

$$\begin{aligned}
F(b) - F(a) - \varepsilon &\leq F(b) - F(a') \\
&= F(a_n) - F(a') + \left(\sum_{n=1}^{N-1} F(a_{n+1}) - F(a_n) \right) + F(b) - F(a_N) \\
&\leq \sum_{n=1}^N F(b'_n) - F(a_n) \leq \varepsilon + \sum_{n=1}^N F(b_n) - F(a_n) \\
&\leq \mu((a, b]) + 2\varepsilon.
\end{aligned}$$

Letting $\varepsilon \rightarrow 0$ we finish the proof. $\square$

Now, let $E, F \subset \mathbb{R}$ with $d(E, F) > 0$. Similarly as for Lebesgue measure we may restrict the the definition of $\mu(E \cup F)$ to intervals with $b_n - a_n < d(E, F)/2$ in order to show $\mu(E \cup F) \geq \mu(E) + \mu(F)$,

which means that μ is a metric outer measure. By Theorem 1.2.14 this makes μ a Borel measure. Let $E \subset \mathbb{R}$. Then there exist sequences $(a_n^k)_{n,k \in \mathbb{N}}$ and $(b_n^k)_{n,k \in \mathbb{N}}$ such that $E \subset \bigcup_{n \in \mathbb{N}} (a_n^k, b_n^k]$ and

$$\begin{aligned} \mu(E) &= \lim_{k \rightarrow \infty} \sum_{n=1}^{\infty} F(b_n^k) - F(a_n^k) \\ &= \lim_{k \rightarrow \infty} \sum_{n=1}^{\infty} \mu((a_n^k, b_n^k]) \\ &\geq \lim_{k \rightarrow \infty} \mu\left(\bigcup_{n=1}^{\infty} (a_n^k, b_n^k]\right) \\ &\geq \mu(E). \end{aligned}$$

Therefore,

$$B = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} (a_n^k, b_n^k] \supset E$$

is a Borel set with $\mu(B) = \mu(E)$. This makes μ Borel regular. Since for any $-\infty < a < b < \infty$ we have $\mu((a, b]) = F(b) - F(a) < \infty$ the measure μ is finite on compact sets, which finally makes μ a Radon measure.

To show uniqueness, let μ_0, μ_1 be Radon measures with $\mu_i((a, b]) = F(b) - F(a)$. Now any open sets can be decomposed into disjoint dyadic intervals, which were intervals of the form $[2^nk, 2^n(k+1))$. By symmetry we can also decompose into intervals $(2^nk, 2^n(k+1)]$, and and we can conclude that for each open $U \subset \mathbb{R}$ we have $\mu_0(U) = \mu_1(U)$. That means by Proposition 1.2.20 we can conclude that μ_0 and μ_1 agree on all sets as soon as we show that they have the same measurable sets, or rather, more precisely, that they can be extended to a common σ -algebra of measurable sets.

To that end, define the outer measure

$$\mu_*(E) = \inf\{\mu_0(U) : E \subset U \text{ open}\}.$$

For each i , by Proposition 1.2.20 it agrees with μ_i all μ_i -measurable

E. By the same argument as Exercise 4, Question 1 applied to μ_i we can conclude that all μ_i -measurable sets satisfy the Caratheodory criterion for μ_* , which means that the σ -algebras of μ_i -measurable sets belongs to the σ -algebra of μ_* -Carathéodory measurable sets. That means μ_* restricted to all μ_* -Carathéodory measurable sets is an extension of μ_i . Since this is the case for both $i = 0, 1$, this is the common extension we wanted. $\square$

Corollary 3.2.4. Let $f \in L^1_{\text{loc}}(\mathbb{R})$ with $f \geq 0$ be measurable and

$$F(x) = \begin{cases} \int_0^x f \, d\mathcal{L} & x \geq 0, \\ -\int_x^0 f \, d\mathcal{L} & x < 0. \end{cases}$$

Then μ from Theorem 3.2.3 is also given by

$$\mu(E) = \int_E f \, d\mathcal{L}.$$

Proof. We already discussed that the map

$$E \mapsto \int_E f \, d\mathcal{L}$$

is a Radon measure. That means by the uniqueness assertion from Theorem 3.2.3 it equals μ . $\square$

Definition 3.2.5. We say that a map $F : \mathbb{R} \rightarrow \mathbb{R}$ is **absolutely continuous** if for any $a < b$ and any $\varepsilon > 0$ exists a $\delta > 0$ such that for any $a \leq a_1 \dots < a_{2n} \leq b$ with

$$\sum_{k=1}^n a_{2k} - a_{2k-1} < \delta$$

we have

$$\sum_{k=1}^n F(a_{2k}) - F(a_{2k-1}) < \varepsilon.$$

Lemma 3.2.6. A Radon measure μ on $\mathbb{R}$ is absolutely continuous with respect to Lebesgue measure if and only if the map $F : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$F(x) = \begin{cases} \mu((0, x]) & x \geq 0 \\ -\mu((x, 0]) & x < 0 \end{cases}$$

is absolutely continuous.

Proof. Exercise. □

Compare this with Proposition 2.1.17 (ii).

Remark 3.2.7. If $F : \mathbb{R} \rightarrow \mathbb{R}$ is absolutely continuous, then its derivative exists almost everywhere and

$$F(b) - F(a) = \int_a^b F' \, d\mathcal{L}.$$

We do not prove this. By Theorem 3.2.3 we can conclude that any Radon measure μ on $\mathbb{R}$ that is absolutely continuous with respect to

Lebesgue measure can be written as

$$\mu(E) = \int_E f \, d\mathcal{L}$$

for some $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.

In fact this is true in much generality.

Remark 3.2.8. For Radon measures μ, ν we have $\mu \ll \nu$ if and only if there exists $f \in L^1_{\text{loc}}(\nu)$ with $\mu = f\nu$. The map f is called the **Radon-Nikodym derivative** of μ with respect to ν .

3.3 The Cantor set

For example, we know that the counting measure $\mathcal{H}^0$ on $\mathbb{R}$, restricted to $\mathbb{N} \subset \mathbb{R}$ is a Radon measure. It arises from the function

$$F = \sum_{n \in \mathbb{N}} 1_{[n, \infty)} - \sum_{n \in \mathbb{N}} 1_{(-\infty, n]}.$$

This function has jumps, so it is very far from being absolutely continuous, and the counting measure is very far from Lebesgue measure because it is concentrated on single points. Next, we define the **Cantor function**, which gives rise to a measure "in between" these two extremes.

Inductively define $C_0 = [0, 1]$ and $C_{n+1} = \frac{1}{3}(C_n \cup (C_n + 2))$. It follows that $C_{n+1} \subset C_n$. The set

$$C = \bigcap_{n=0}^{\infty} C_n$$

is called the **Cantor set**. It is a subset of $[0, 1]$. Being an intersection of closed sets, it is closed itself, in particular Lebesgue measurable. Note, that $\mathcal{L}(C_{n+1}) = \frac{2}{3}\mathcal{L}(C_n)$, and by the measure continuity lemma it follows that $\mathcal{L}(C) = 0$.

Another way to write C is in terms of the ternary expansion of

numbers. Precisely,

$$C = \left\{ \sum_{k=1}^{\infty} a_k 3^{-k} : a_1, a_2, \dots \in \{0, 2\} \right\}. \quad (3.3.1)$$

We prove by induction that for each n the set C_n is the set of all real numbers that have ternary expansions with $a_1, \dots, a_n \in \{0, 2\}$. For $n = 0$ this is immediate. Assume this is true for C_n . Then

$$\begin{aligned} C_{n+1} &= \frac{1}{3}C_n \cup \frac{1}{3}(C_n + 2) \\ &= \left\{ \sum_{k=1}^{\infty} a_k 3^{-k} : a_1 = 0, a_2, \dots, a_{n+1} \in \{0, 2\} \right\} \\ &\quad \cup \left\{ \sum_{k=1}^{\infty} a_k 3^{-k} : a_1 = 2, a_2, \dots, a_{n+1} \in \{0, 2\} \right\} \\ &= \left\{ \sum_{k=1}^{\infty} a_k 3^{-k} : a_1, \dots, a_{n+1} \in \{0, 2\} \right\}. \end{aligned}$$

Now (3.3.1) follows from $C = \bigcap_{n=1}^{\infty} C_n$.

Note, that in this context we do allow expansions that for some N have $a_n = 2$ for all $n \geq N$. That means C contains also all numbers of the form

$$\sum_{k=1}^n a_k 3^{-k} + \sum_{k=n+1}^{\infty} 2 \cdot 3^{-k} = \sum_{k=1}^n a_k 3^{-k} + 3^{-n-1},$$

with $a_k \in \{0, 2\}$, i.e. those who have finite a ternary expansion that ends in 1, and with all other digits in $\{0, 2\}$.

Similarly, for $U_0 = \emptyset$, $U_1 = (1/3, 2/3)$ and $U_{n+1} = \frac{1}{3}(U_n \cup (U_n + 2))$ we have $U_n = [0, 1] \setminus C_n$ and

$$C = [0, 1] \setminus \bigcup_{n=1}^{\infty} U_n$$

and

$$U_n = \bigcup_{a_1, \dots, a_{n-1} \in \{0, 2\}, a_n = 1} (0, 3^{-n}) + \sum_{k=1}^n a_k 3^{-k}.$$

Now, C is very disconnected in the sense that for any $a, b \in C$ with $a < b$, there is an interval I of length a constant times $b - a$ that sits between a and b and belongs to the complement of C : Let $(a_n)_n$ and $(b_n)_n$ be their ternary expansion that witness $a, b \in C$. Let n be the least digit in which they differ. Then $a_n = 0$ and $b_n = 2$. That means the interval

$$(0, 3^{-n}) + \sum_{k=1}^{n-1} a_k 3^{-k} + 3^{-n}$$

sits between a and b and belongs to $U_n \subset [0, 1] \setminus C$.

Moreover, C does not contain any isolated point either, since for every $x \in C$ and $\varepsilon > 0$ exists a $y \in C$ with $|x - y| < \varepsilon$.

We define the **Cantor function** $F : C \rightarrow [0, 1]$ by

$$F\left(\sum_{n=1}^{\infty} a_n 3^{-n}\right) = \sum_{n=1}^{\infty} \frac{a_n}{2} 2^{-n}.$$

This map is well defined since the ternary expansion of a number that witnesses membership to the Cantor set is unique. Since it maps to any possible binary expansion of numbers in $[0, 1]$ it is surjective. This shows that C is uncountable. The Cantor function is also nondecreasing and continuous. Moreover, let (a, b) be one of the open intervals in U_n , i.e. there are $a_1, \dots, a_{n-1} \in \{0, 2\}$ such that

$$a = \sum_{k=1}^{n-1} a_k 3^{-k} + 3^{-n} = \sum_{k=1}^{n-1} a_k 3^{-k} + \sum_{k=n+1}^{\infty} 2 \cdot 3^{-k}$$

$$b = \sum_{k=1}^{n-1} a_k 3^{-k} + 2 \cdot 3^{-n}.$$

Then $F(a) = F(b)$. That means we can extend F to a continuous, nondecreasing map $F : [0, 1]$ by setting $F(x) = F(a)$ for all $a < x < b$. We further extend it to $\mathbb{R}$ by setting $F(x) = F(0) = 0$ for $x < 0$ and $F(x) = F(1) = 1$ for $x > 1$.

By Theorem 3.2.3 F gives rise to a measure μ_C .

Lemma 3.3.1. The measure μ_C has the following properties:

- (i) $\mu(C) = 1$.
- (ii) $\mu(\mathbb{R} \setminus C) = 0$.
- (iii) For every $x \in \mathbb{R}$ we have $\mu(\{x\}) = 0$.

That means μ_C is supported on a set with zero Lebesgue measure, but, unlike the counting measure, does not assign positive measure to any single point.

Proof. By the measure continuity lemma and the continuity of F we have

$$\mu(\{x\}) = \lim_{n \rightarrow \infty} \mu((x - 1/n, x]) = F(x) - F(x - 1/n) = 0.$$

Moreover,

$$\mu(\mathbb{R}) = \lim_{n \rightarrow \infty} F(n) - F(-n) = F(1) - F(0) = 1.$$

Similarly, we have $F([1, \infty)) = F((-\infty, 0]) = 0$. In order to finish the proof it remains to show $F([0, 1] \setminus C) = 0$. Since $[0, 1] \setminus C = \bigcup_{n=1}^{\infty} U_n$ it suffices to show that for every n we have $\mu_C(U_n) = 0$. By the way F was extended from C to $[0, 1]$ we have

$$\begin{aligned}\mu_C(U_n) &= \sum_{a_1, \dots, a_{n-1} \in \{0, 2\}, a_n = 1} \mu_C\left((0, 3^{-n}) + \sum_{k=1}^n a_k 3^{-k}\right) \\ &\leq \sum_{a_1, \dots, a_{n-1} \in \{0, 2\}, a_n = 1} F\left(3^{-n} + \sum_{k=1}^n a_k 3^{-k}\right) - F\left(\sum_{k=1}^n a_k 3^{-k}\right) \\ &= 0.\end{aligned}$$

□

3.4 Functions of bounded variation

For μ from Theorem 3.2.3 to be a measure we need F to be nondecreasing. What if we relax this condition? We can still assign the value $F(b) - F(a)$ to any interval $(a, b]$, but can we extend this map to all Lebesgue measurable sets?

Definition 3.4.1. Let $a < b$ and $F : [a, b] \rightarrow \mathbb{R}$. We define

$$\text{var}_{[a,b]}(F) = \sup \left\{ \sum_{k=1}^n |F(a_k) - F(a_{k-1})| : n \in \mathbb{N}, a_k \in [a, b], a_0 \leq \dots \leq a_n \right\}.$$

We say that F is of **bounded variation** if $\text{var}_{[a,b]}(F) < \infty$.

For example, if F is monotone and bounded by $M \geq 0$ then

$$\sum_{k=1}^n |F(a_k) - F(a_{k-1})| = \left| \sum_{k=1}^n F(a_k) - F(a_{k-1}) \right| = |F(a_n) - F(a_0)| \leq 2M,$$

so $\text{var}_{[a,b]}(F) \leq 2M$. Similarly, if F is nondecreasing and G is non-increasing and bounded by M then

$$\sum_{k=1}^n |(F+G)(a_k) - (F+G)(a_{k-1})| \leq \sum_{k=1}^n |F(a_k) - F(a_{k-1})| + \sum_{k=1}^n |G(a_k) - G(a_{k-1})| \leq 4M.$$

In fact, also the reverse is true: Any function with bounded variation is the sum of two monotone and bounded functions. To construct those functions, for an interval $I \subset [a, b]$ we define

$$\begin{aligned} \text{var}_I^+(F) &= \sup \left\{ \sum_{k=1}^n \max\{0, F(a_k) - F(a_{k-1})\} : n \in \mathbb{N}, a_k \in I, a_1 \leq \dots \leq a_n \right\}, \\ \text{var}_I^-(F) &= \sup \left\{ \sum_{k=1}^n \max\{0, F(a_{k-1}) - F(a_k)\} : n \in \mathbb{N}, a_k \in I, a_1 \leq \dots \leq a_n \right\}. \end{aligned}$$

Lemma 3.4.2. Let $F : [a, b] \rightarrow \mathbb{R}$ have bounded variation. Then

for every $x \in [a, b]$ we have

$$F(x) - F(a) = \text{var}_{[a,x]}^+(F) - \text{var}_{[a,x]}^-(F)$$

and

$$\text{var}_{[a,b]}(F) = \text{var}_{[a,b]}^+(F) + \text{var}_{[a,b]}^-(F).$$

Proof. Note, that $\text{var}_{[a,x]}^+(F), \text{var}_{[a,x]}^-(F) \leq \text{var}_{[a,x]}(F) < \infty$. Let $\varepsilon > 0$. Then there exist $a_0 < \dots < a_n$ such that

$$\text{var}_{[a,x]}^+(F) - \sum_{k=1}^n \max\{0, F(a_k) - F(a_{k-1})\} \leq \varepsilon.$$

Since the sum only increases if we refine the partition, i.e. add more points to $\{a_0, \dots, a_n\}$, we may take it such that $a_0 = a$, $a_n = x$ and also

$$\text{var}_{[a,x]}^-(F) - \sum_{k=1}^n \max\{0, F(a_{k-1}) - F(a_k)\} \leq \varepsilon.$$

Then

$$\begin{aligned} F(x) - F(a) &= \sum_{k=1}^n F(a_k) - F(a_{k-1}) \\ &= \sum_{k=1}^n \max\{0, F(a_k) - F(a_{k-1})\} \\ &\quad - \sum_{k=1}^n \max\{0, F(a_{k-1}) - F(a_k)\} \end{aligned}$$

and thus

$$|F(x) - F(a) - \overset{+}{\text{var}}_{[a,x]}(F) + \overset{-}{\text{var}}_{[a,x]}(F)| < 2\varepsilon.$$

Letting $\varepsilon \rightarrow 0$ finishes the proof of the first claim.

The second one is a consequence of

$$\sum_{k=1}^n |F(a_k) - F(a_{k-1})| = \sum_{k=1}^n \max\{0, F(a_k) - F(a_{k-1})\}$$

$$+ \sum_{k=1}^n \max\{0, F(a_{k-1}) - F(a_k)\}.$$

□

Chapter 4

Further topics

4.1 Signed measures

Definition 4.1.1. Let Ω be a set and $\mathcal{M} \subset 2^\Omega$ be a σ -algebra. A set function $\mu : \mathcal{M} \rightarrow (-\infty, \infty]$ is called a **signed measure** if for

disjoint $E_1, E_2, \dots \in \mathcal{M}$ we have

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

Remark 4.1.2. Assume we had a signed measure with $\mu(A) = -\infty$ and $\mu(B) = \infty$. Then by additivity $\mu(B \setminus A) = \infty$ or $\mu(B \cap A) = \infty$, and $\mu(A \setminus B) = -\infty$ or $\mu(A \cap B) = -\infty$. In either case there exist two disjoint sets C, D with $\mu(C) = \infty$ and $\mu(D) = -\infty$. But then $\mu(C \cup D) = \mu(C) + \mu(D)$ fails.

Of course we could also define signed measures to assume values in $[-\infty, \infty)$ instead.

Remark 4.1.3. The additivity for signed measures requires that the sum on the right hand side does not depend on the order of summation.

Example 4.1.4. (i) If μ_0 and μ_1 are measures on Ω with the

same σ -algebra and $\mu_1(\Omega) < \infty$ then $\mu_0 - \mu_1$ is a signed measure.

- (ii) If $f \in L^1(\Omega, \mu)$ then $E \mapsto \int_E f \, d\mu$ is a signed measure.
- (iii) And $F : \mathbb{R} \rightarrow \mathbb{R}$ which is continuous from the right, i.e.

$$\lim_{y \rightarrow x, y > x} F(y) = F(x)$$

and has bounded variation generates a signed measure μ with

$$F(b) - F(a) = \mu((a, b])$$

as follows: All of the maps $x \mapsto \text{var}_{[a, x]}(F)$, $x \mapsto \text{var}_{[a, x]}^+(F)$ and $x \mapsto \text{var}_{[a, x]}^-(F)$ are continuous from the right. Since they are also nondecreasing that makes them upper semicontinuous. That means by Theorem 3.2.3 they give rise to measures which we denote by $|\mu|$, μ^+ and μ^- , and by Lemma 3.4.2 we have $|\mu| = \mu^+ + \mu^-$. The map $\mu = \mu^+ - \mu^-$ gives a signed measure with and $|\mu(E)| \leq |\mu|(E)$.

Remark 4.1.5. If μ is given as $f\mathcal{L}$ for some $f \in L^1_{\text{loc}}(\mathbb{R})$ we say that f is the weak derivative of F . We call the space of functions $F \in L^p(\mathbb{R}, \mathcal{L})$ whose weak derivative also belongs to $L^p(\mathbb{R}, \mathcal{L})$ the **Sobolev space** $W^{1,p}(\mathbb{R})$. That means the space of functions with bounded variation can be seen as an extension of $W^{1,1}(\mathbb{R})$ to those functions whose derivative is a finite signed Radon measure. All those spaces can also be defined on $\mathbb{R}^d$.

Next, we show the converse of Example 4.1.4 (i), i.e. that any signed measure is the difference of two measures.

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Definition 4.1.6. Let μ be a signed measure. We define the **total variation measure** $|\mu|$ of μ by

$$|\mu|(E) = \sup \left\{ \sum_{n=1}^{\infty} |\mu(E_n)| : E_1, E_2, \dots \text{ are disjoint with } E = E_1 \cup E_2 \cup \dots \right\}.$$

Proposition 4.1.7. The total variation measure $|\mu|$ of a signed measure μ is in fact a measure with $|\mu(E)| \leq |\mu|(E)$.

Moreover, $|\mu|$ is the smallest such measure.

Proof. First, we show the minimality of $|\mu|$. For any E and disjoint $E_1, E_2, \dots$ with $E = E_1 \cup E_2 \cup \dots$ and any measure ν with $\nu(E_n) \geq |\mu(E_n)|$ we have

$$\nu(E) = \sum_{n=1}^{\infty} \nu(E_n) \geq \sum_{n=1}^{\infty} |\mu(E_n)|.$$

This shows $|\mu| \leq \nu$.

In order to prove the countable additivity of $|\mu|$ let $E_1, E_2, \dots \in \mathcal{M}$ be disjoint. Let $\varepsilon > 0$. Then for each n exist a partition $E_n^1, E_n^2, \dots$ of E_n with

$$|\mu|(E_n) - 2^{-n}\varepsilon \leq \sum_{k=1}^{\infty} |\mu(E_n^k)|.$$

Thus,

$$\begin{aligned}\sum_{n=1}^{\infty} |\mu|(E_n) &\leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\mu(E_n^k)| - \varepsilon \\ &\leq |\mu|\left(\bigcup_{n=1}^{\infty} E_n\right) - \varepsilon.\end{aligned}$$

Now we let $\varepsilon \rightarrow 0$.

For the reverse inequality let $F_1, F_2, \dots$ be a partition of E . Then for each n the sets $\{F_k \cap E_n : k \in \mathbb{N}\}$ are a partition of E_n and for each k the sets $\{F_k \cap E_n : n \in \mathbb{N}\}$ are a partition of F_k . Therefore,

$$\begin{aligned}\sum_{k=1}^{\infty} |\mu(F_k)| &\leq \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |\mu(F_k \cap E_n)| \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\mu(F_k \cap E_n)|\end{aligned}$$

$$\leq \sum_{n=1}^{\infty} |\mu|(E_n).$$

□

Now, we can define the **positive** and **negative parts** μ^+ and μ^- of a signed measure μ by

$$\mu^+ = \frac{1}{2}(|\mu| + \mu), \quad \mu^- = \frac{1}{2}(|\mu| - \mu).$$

Note, that μ^+ and μ^- are measures with $\mu = \mu^+ - \mu^-$. Since $|\mu|$ is the smallest measure with $|\mu|(E) \geq |\mu(E)|$ we can conclude that $\mu^\pm$ are the smallest measures with $\mu = \mu^+ - \mu^-$.

4.2 Convolution and approximation of the identity

Definition 4.2.1. Let $f, g : \mathbb{R}^d \rightarrow [-\infty, \infty]$ be measurable. We define their **convolution** by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x - y)g(y) \, d\mathcal{L}(y),$$

for those x for which the integrand is integrable.

There are different ways to ensure the convolution is defined almost everywhere. One way is assuming $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $g \in L^\infty_c(\mathbb{R}^d)$, as in the following situation.

Proposition 4.2.2. Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $g \in C^k_c(\mathbb{R}^d)$. Then $f * g \in C^k(\mathbb{R}^d)$.

Proof. Let $x, \nu \in \mathbb{R}^d$, $|\nu| = 1$ and $h > 0$. Then

$$\begin{aligned} \frac{(f * g)(x + h\nu) - (f * g)(x)}{h} &= \frac{1}{h} \left(\int_{\mathbb{R}^d} f(x - y)g(y) \, d\mathcal{L}(y) - \int_{\mathbb{R}^d} f(x - y)g(y + h\nu) \, d\mathcal{L}(y) \right) \\ &= \int_{\mathbb{R}^d} f(x - y) \frac{g(y) - g(y + h\nu)}{h} \, d\mathcal{L}(y) \\ &\rightarrow (f * \partial_\nu g)(x) \end{aligned}$$

by dominated convergence, as $\|\partial_\nu g\|_\infty < \infty$, g has compact support and $f \in L^1_{\text{loc}}$. $\square$

Theorem 4.2.3. *Let $1 \leq p, q, r \leq \infty$ such that $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. If $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ then $f * g \in L^r(\mathbb{R}^d)$ with*

$$\|f * g\|_{L^r(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}.$$

Proof. It suffices to consider the case that f, g are nonnegative.

Write

$$(f * g)(x) = \int [f(x-y)^{p/r} g(y)^{q/r}] f(x-y)^{p(1/p-1/r)} g(y)^{q(1/q-1/r)} d\mathcal{L}(y).$$

By Hölder's inequality with r , $1/p_1 = 1/p - 1/r$ and $1/p_2 = 1/q - 1/r$ applied to these three factors we obtain

$$(f * g)(x) \leq \left(\int f(x-y)^p g(y)^q d\mathcal{L}(y) \right)^{1/r} \|f\|_p^{p_1/p} \|g\|_q^{p_2/q},$$

and thus

$$\begin{aligned} \|f * g\|_{L^r(\mathbb{R}^d)} &\leq \left(\int f(x-y)^p g(y)^q d\mathcal{L}(y) d\mathcal{L}(x) \right)^{1/r} \|f\|_p^{p_1/p} \|g\|_q^{p_2/q} \\ &= \|f\|_p^{p_1/p + p/r} \|g\|_q^{p_2/q + q/r} = \|f\|_p \|g\|_q. \end{aligned}$$

□

Let $g : \mathbb{R}^d \rightarrow [-\infty, \infty]$ be integrable with $\int g \mathcal{L} = 1$. For $r > 0$ define

$$g_r(x) = \frac{g(x/r)}{r^d}.$$

Then $\int g_r \mathcal{L} = \int g \mathcal{L} = 1$.

Theorem 4.2.4. *Let $f \in L^1_{\text{loc}}$ and $g \in L^\infty_{\text{c}}$ with $\int g \mathcal{L} = 1$. Then for every Lebesgue point x of f we have*

$$\lim_{r \rightarrow 0} (f * g_r)(x) = f(x).$$

Proof. Let x be a Lebesgue point. Then

$$\begin{aligned} |f(x) - (f * g_r)(x)| &= \left| \int f(x)g_r(y) - f(x-y)g_r(y) \, d\mathcal{L}(y) \right| \\ &\leq \int |f(x) - f(x-y)| |g_r(y)| \, d\mathcal{L}(y) \\ &\leq \frac{\|g\|_\infty}{r^d} \int_{\text{rspt}(g)} |f(x) - f(x-y)| \, d\mathcal{L}(y) \end{aligned}$$

$$\leq \frac{\|g\|_\infty}{r^d} \int_{B(0, Cr)} |f(x) - f(x - y)| \, d\mathcal{L}(y) \rightarrow 0$$

as $r \rightarrow 0$. □

Theorem 4.2.4 has more general variants, without the assumption that g is compactly supported or uniformly bounded. However, often also the integrability assumption on f has to be strengthened.

Note, that taking $g \in C_c^\infty$, Theorem 4.2.4 combined with Proposition 4.2.2 gives an explicit pointwise approximation of any locally integrable function by smooth functions. We have not shown that a function $g \in C_c^\infty$ exists, but it does.

Under slightly different assumptions we also achieve approximation in L^p .

Theorem 4.2.5. *Let $1 \leq p < \infty$, $f \in L^p(\mathbb{R}^d)$ and $g \in L^1(\mathbb{R}^d)$ with $\int g \, d\mathcal{L} = 1$. Then*

$$\lim_{r \rightarrow 0} \|f - f * g_r\|_{L^p(\mathbb{R}^d)} = 0.$$

Since $L^p \subset L^1_{\text{loc}}$ this means we can also approximate f by smooth functions in L^p .

For the proof we will use Minkowski's integral inequality.

Theorem 4.2.6 (Minkowski's integral inequality). *For $i = 0, 1$ let (Ω_i, μ_i) be σ -finite measure spaces and let $F : \Omega_0 \times \Omega_1 \rightarrow [-\infty, \infty]$ be measurable. Then for every $1 \leq p \leq \infty$ we have*

$$\left[\int \left(\int F(x, y) \, d\mu_0(x) \right)^p \, d\mu_1(y) \right]^{\frac{1}{p}} \leq \int \left(\int F(x, y)^p \, d\mu_1(y) \right)^{\frac{1}{p}} \, d\mu_0(x).$$

This may look complicated, but in fact it is barely more general than the triangle inequality on L^p . For example, take $\Omega_0 = \{1, \dots, n\}$ and μ_0 the counting measure on Ω_0 . Then writing k instead of x and $f_n(y) = F(n, y)$, Minkowski's integral inequality becomes

$$\left\| \sum_{k=1}^n f_k \right\|_{L^p(\Omega_1, \mu_1)} \leq \sum_{k=1}^n \|f_k\|_{L^p(\Omega_1, \mu_1)}.$$

We also need the following.

Proposition 4.2.7. Let $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}^d)$. Then

$$\lim_{x \rightarrow 0} \|f(\cdot - x) - f\|_{L^p} = 0.$$

Proof. Let $\varepsilon > 0$. Then by Proposition 2.2.1 (ii) there exists a step function $\varphi = \sum_{k=1}^n a_k 1_{Q_k}$ with $\|f - \varphi\|_{L^p} < \varepsilon$. Thus, also $\|f(\cdot - x) - \varphi(\cdot - x)\|_{L^p} < \varepsilon$. Moreover, for each k we have $\|1_{Q_k}(\cdot - x) - 1_{Q_k}\|_p = \mathcal{L}((Q_k - x) \Delta Q_k)^{\frac{1}{p}} \rightarrow 0$ as $x \rightarrow 0$. We can conclude

$$\begin{aligned} \|f(\cdot - x) - f\|_{L^p} &\leq \|f(\cdot - x) - \varphi(\cdot - x)\|_{L^p} + \|\varphi(\cdot - x) - \varphi\|_{L^p} + \|f - \varphi\|_{L^p} \\ &\leq 2\varepsilon + \sum_{k=1}^n a_k \|1_{Q_k}(\cdot - x) - 1_{Q_k}\|_p \leq 3\varepsilon \end{aligned}$$

for all x small enough. □

Proof of Theorem 4.2.5. By a change of variables, $rz = y$ we have

$$\begin{aligned}(f * g_r)(x) - f(x) &= \int (f(x - y) - f(x))g_r(y) \, d\mathcal{L}(y) \\ &= \int (f(x - rz) - f(x))g(z) \, d\mathcal{L}(z).\end{aligned}$$

By Minkowski's integral inequality, we obtain

$$\|f * g_r - f\|_p \leq \int \|f(\cdot - rz) - f\|_p |g(z)| \, d\mathcal{L}(z).$$

Now, $\|f(\cdot - rz) - f\|_p \leq 2\|f\|_p$, and by Proposition 4.2.7, for every $z \in \mathbb{R}^d$ we have $\|f(\cdot - rz) - f\|_p \leq \rightarrow 0$ as $r \rightarrow 0$. Thus, by dominated convergence we can conclude $\|f * g_r - f\|_p \rightarrow 0$. $\square$

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