ICTP Real Analysis 2025
Julian Weigt
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Exercise 8
Due: Thursday, 2025-11-06

Total: /12

1. Prove Lemma 3.2.6: Show, that a Radon measure μ on $\mathbb{R}$ is absolutely continuous with respect to Lebesgue measure if and only if the map $F : \mathbb{R} \to \mathbb{R}$ given by /6

$$F(x) = \begin{cases} \mu((0,x]) & x \geq 0 \\ -\mu((x,0]) & x < 0 \end{cases}$$

is an absolutely continuous map.

2. (a) For $i = 0, 1$ let $E_i \subset \mathbb{R}_i$ be $\mathcal{L}^1$-measurable. Show, that /3

$$\mathcal{L}^1(E_0) \cdot \mathcal{L}^1(E_1) \geq \mathcal{L}^2_*(E_0 \times E_1),$$

with the interpretation $0 \cdot \infty = 0$. Note, that we have not yet shown that $E_0 \times E_1$ is $\mathcal{L}^2_*$-measurable.

Hint: Work directly with the definition of Lebesgue outer measure.

(b) Conclude that $\mathcal{L}^1 \times \mathcal{L}^1$ and $\mathcal{L}^2_*$ agree as outer measures on $\mathbb{R}^2$. /3

Hint: Use that we know from the lecture that for E_0, E_1 measurable we know $(\mathcal{L}^1 \times \mathcal{L}^1)(E_0 \times E_1) = \mathcal{L}^1(E_0) \cdot \mathcal{L}^1(E_1)$.