1. Prove Lemma 3.2.6: Show, that a Radon measure μ on $\mathbb{R}$ is absolutely continuous with respect to Lebesgue measure if and only if the map $F : \mathbb{R} \to \mathbb{R}$ given by /6

$$F(x) = \begin{cases} \mu((0,x]) & x \geq 0 \\ -\mu((x,0]) & x < 0 \end{cases}$$

is an absolutely continuous map.

Solution: Let F be absolutely continuous and let $a < b$. let $E \subset [a,b]$ with $\mathcal{L}(E) = 0$. Let $\varepsilon > 0$. Take δ from the absolute continuity of F. Then there exist a countable set $\mathcal{B}$ of half open intervals $(a,b]$ with $E \subset \bigcup \mathcal{B}$ and

$$\sum_{(a,b] \in \mathcal{B}} b - a < \delta.$$

Let $(a_1, b_1], ..., (a_n, b_n] \in \mathcal{B}$. Then we can write

$$\bigcup_{k=1}^{n} (a_k, b_k]$$

as a finite union of disjoint intervals $(\tilde{a}_1, \tilde{b}_n], ..., (\tilde{a}_m, \tilde{b}_m]$ and

$$\sum_{k=1}^{m} \tilde{b}_k - \tilde{a}_k \leq \sum_{k=1}^{n} b_k - a_k < \delta.$$

We can conclude

$$\mu\Big(\bigcup_{k=1}^{n} (a_k, b_k]\Big) = \mu\Big(\bigcup_{k=1}^{m} (\tilde{a}_k, \tilde{b}_k]\Big) = \sum_{k=1}^{m} F(\tilde{b}_k) - F(\tilde{a}_k) < \varepsilon.$$

Letting $n \to \infty$ we obtain from the measure continuity of μ that

$$\mu(E) \leq \mu(\bigcup \mathcal{B}) < \varepsilon.$$

Now, for $E \subset \mathbb{R}$ with $\mathcal{L}(E) = 0$ we can conclude for every $n \in \mathbb{N}$ that $\mu(E \cap [-n, n]) = 0$ and hence $\mu(E) = 0$.

For the converse implication, assume that F is not absolutely continuous. Then there exists $a < b$ and an $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ exist disjoint intervals $\mathcal{B}_n = (a_1, b_1], ..., (a_{m_n}, b_{m_n}]$ with $\bigcup \mathcal{B} \subset (a,b]$ and $\mathcal{L}(\bigcup \mathcal{B}_n) < 2^{-n}$ but $\mu(\bigcup \mathcal{B}_n) \geq \varepsilon$. Define

$$B_n = \bigcup_{k > n} \bigcup \mathcal{B}_n.$$

Then $B_{n+1} \subset B_n$, $\mathcal{L}(B_n) < 2^{-n}$, and $\mu(B_n) \geq \varepsilon$. Thus, for $B = \bigcap_{n=1}^{\infty}$ we have $\mathcal{L}(B) = 0$, and since $B_n \subset (a,b]$ and $\mu((a,b]) = F(b) - F(a) < \infty$ we can conclude by the measure continuity lemma that $\mu(B) \geq \varepsilon$. That means μ is not absolutely continuous with respect to Lebesgue measure.

2. (a) For $i = 0, 1$ let $E_i \subset \mathbb{R}_i$ be $\mathcal{L}^1$-measurable. Show, that /3

$$\mathcal{L}^1(E_0) \cdot \mathcal{L}^1(E_1) \geq \mathcal{L}^2_*(E_0 \times E_1),$$

with the interpretation $0 \cdot \infty = 0$. Note, that we have not yet shown that $E_0 \times E_1$ is $\mathcal{L}^2_*$-measurable.

Hint: Work directly with the definition of Lebesgue outer measure.

(b) Conclude that $\mathcal{L}^1 \times \mathcal{L}^1$ and $\mathcal{L}^2_*$ agree as outer measures on $\mathbb{R}^2$. /3

Hint: Use that we know from the lecture that for E_0, E_1 measurable we know $(\mathcal{L}^1 \times \mathcal{L}^1)(E_0 \times E_1) = \mathcal{L}^1(E_0) \cdot \mathcal{L}^1(E_1)$.

Solution: We show the statement for $\mathcal{L}^{n_0}$ and $\mathcal{L}^{n_1}$ instead of $\mathcal{L}^1$, as promised in the lecture.

(i) Let $\varepsilon > 0$. Then for $i = 0, 1$ there exist cubes $Q_i^1, Q_i^2, ... \subset \mathbb{R}^{n_i}$ such that $E_i \subset Q_i^1 \cup Q_i^2 \cup ...$ and

$$\mathcal{L}^{n_i}(E_i) \leq \sum_{k=1}^{\infty} |Q_i^k|_{n_i}.$$

Thus, $\mathcal{Q} = \{Q_0^k \times Q_1^m : k, m \in \mathbb{N}\}$ is a cover of $E_0 \times E_1$ and

$$\begin{aligned}
\mathcal{L}^2_*(E_0 \times E_1) \leq \sum_{Q \in \mathcal{Q}} |Q|_{n_0+n_1} &= \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} |Q_0^k \times Q_1^m|_{n_0+n_1} \\
&= \sum_{k=1}^{\infty} |Q_0^k|_{n_0} \sum_{m=1}^{\infty} |Q_1^m|_{n_1} \\
&\leq (\mathcal{L}^{n_0}(E_0) + \varepsilon)(\mathcal{L}^{n_1}(E_1) + \varepsilon) \\
&= \mathcal{L}^{n_0}(E_0)\mathcal{L}^{n_1}(E_1) + \varepsilon(\mathcal{L}^{n_0}(E_0) + \mathcal{L}^{n_1}(E_1) + \varepsilon),
\end{aligned}$$

finishing the proof in every case except when $\mathcal{L}^{n_0}(E_0) = 0$ and $\mathcal{L}^{n_1}(E_1) = \infty$ (or vice versa). In that case, we have to show $\mathcal{L}^{n_0+n_1}_*(E_0 \times E_1) = 0$. Since for $N > 0$ and $E_1^R = E_1 \cap B(0, N)$ we have $\mathcal{L}^{n_1}(E_1^R) < \infty$ we obtain $\mathcal{L}^{n_0+n_1}_*(E_0 \times E_1) = 0$ from the previos case, and we can finish the proof using the (outer) measure continuity lemma.

(ii) It follows directly from the definition that $(\mathcal{L}^{n_0} \times \mathcal{L}^{n_1})(E) \leq \mathcal{L}^2_*(E)$, since the infimum in the first one is over a larger set. For the reverse inequality let $\varepsilon > 0$ and take measurable $A_k^i \subset \mathbb{R}^{n_i}$ such that

$$\begin{aligned}
(\mathcal{L}^{n_0} \times \mathcal{L}^{n_1})(E) + \varepsilon &\geq \sum_{k=1}^{\infty} \mathcal{L}^{n_0}(A_k^0)\mathcal{L}^{n_1}(A_k^1) \\
&\geq \sum_{k=1}^{\infty} \mathcal{L}^{n_0+n_1}(A_k^0 \times A_k^1) \\
&\geq \mathcal{L}^{n_0+n_1}\Big(\bigcup_{k=1}^{\infty} A_k^0 \times A_k^1\Big) \\
&\geq \mathcal{L}^{n_0+n_1}(E).
\end{aligned}$$