1. Find a bounded Lebesgue integrable function on $[0,1]$ which is not Riemann integrable and prove these properties. /3

   *Hint: A function of the form $1_E$ will do.*

2. Let $E \subset \mathbb{R}^d$ be $\mathcal{L}$-measurable. Show, that there exist compact sets $K_1, K_2, \ldots \subset E$ with $\mathcal{L}(K_n) \to \mathcal{L}(E)$. /1

   **Solution:** We already know that there exist closed sets $C_1, C_2, \ldots \subset E$ with $\mathcal{L}(C_n) \to \mathcal{L}(E)$. For each $C_n$ and $k \in \mathbb{N}$ the set $K_{n,k} = C_n \cap \overline{B(0,k)} \subset C_n \subset E$ is compact with $\mathcal{L}(K_{n,k}) \to \mathcal{L}(C_n)$ as $k \to \infty$. Taking a diagonal sequence $K_n = K_{n,k_n}$ with $k_n$ large enough we get what we want.

3. Let $f : \Omega \to [0,\infty]$ be $\mu$-measurable.

   (a) Show, that the area below the graph, $\{(x,t) \in \Omega \times \mathbb{R} : 0 \le t < f(x)\}$, is $\mu \times \mathcal{L}$-measurable. *Hint: It can be written as a countable union of "rectangles" $E \times [0,t)$ for certain measurable $E \subset \Omega$ and $t \in \mathbb{R}$.* /3

   (b) Show the **layer cake formula / Cavalieri's principle**, /2

   $$\int_\Omega f \,\mathrm{d}\mu = \int_0^\infty \mu(\{f > \lambda\}) \,\mathrm{d}\lambda.$$

   *Hint: Fubini*

   (c) For $1 \le p < \infty$ prove the $L^p$-variant /3

   $$\int_\Omega f^p \,\mathrm{d}\mu = p \int_0^\infty \lambda^{p-1} \mu(\{f > \lambda\}) \,\mathrm{d}\lambda$$

   and conclude **Chebyshev's inequality**:

   $$\mu(\{f > \lambda\}) \le \frac{1}{\lambda^p} \int f^p \,\mathrm{d}\mu.$$

   (d) The space of functions $f$ for which $\sup_{\lambda \ge 0} \lambda^p \mu(\{|f| > \lambda\}) < \infty$ is called $L^{p,\infty}(\Omega,\mu)$, or **weak-**$L^p(\Omega,\mu)$. Part (c) implies $L^p(\Omega,\mu) \subset L^{p,\infty}(\Omega,\mu)$. /2

   Show that the inclusion is strict for $\Omega = \mathbb{R}$ and $\mu = \mathcal{L}$, i.e. that there exists a measurable $f : \mathbb{R} \to [0,\infty]$ with $\sup_{\lambda \ge 0} \lambda^p \mathcal{L}(\{f > \lambda\}) < \infty$ but $\|f\|_{L^p(\mathbb{R},\mathcal{L})} = \infty$.

4. For $\lambda > 0$ and $E \subset \mathbb{R}^d$ denote $\lambda E = \{\lambda x : x \in E\}$.

   (a) Show, that /2

   $$\mathcal{L}_*(\lambda E) = \lambda^d \mathcal{L}_*(E).$$

   (b) Show, that $E$ is Lebesgue measurable if and only if $\lambda E$ is. /2