1. Find an example of a sequence of functions $f_1, f_2, \ldots : (0,1) \to [0,\infty]$ such that for every $x \in (0,1)$ we have $\lim_{n\to\infty} f_n(x) = 0$, but $\int_{(0,1)} f_n \,\mathrm{d}\mathcal{L}$ does not converge to $0 = \int_{(0,1)} 0 \,\mathrm{d}\mathcal{L}$. /3

2. Let $f : \Omega \to [-\infty,\infty]$ be measurable and assume that there exists a $1 \le p < \infty$ such that $\|f\|_{L^p(\Omega,\mathcal{M},\mu)} < \infty$. Show, that /4
$$\lim_{p\to\infty} \|f\|_{L^p(\Omega,\mathcal{M},\mu)} = \|f\|_{L^\infty(\Omega,\mathcal{M},\mu)}.$$
Note, that $\|f\|_{L^\infty(\Omega,\mathcal{M},\mu)}$ may be infinite.
You receive partial points if you prove it under the assumption $\mu(\Omega) < \infty$. (In this case we do not need the assumption that there exists a $1 \le p < \infty$ such that $\|f\|_{L^p(\Omega,\mathcal{M},\mu)} < \infty$.)
Hint: First consider functions of the form $a1_E$, and then compare f with such functions.

3. Let $a, b \in \mathbb{R}$ with $a < b$ and $C > 0$. Recall one definition of the Riemann integral of a bounded function $f : [a,b] \to [0,C]$. We say that a finite set $P = \{x_0, \ldots, x_n\}$ is a partition if $a = x_0 < \ldots < x_n = b$. Given such a partition P define their lower and upper Riemann sums as
$$\underline{\mathcal{R}}_P(f) = \sum_{k=1}^{n} (x_n - x_{n-1}) \inf_{x_{n-1} \le x < x_n} f(x), \qquad \overline{\mathcal{R}}_P(f) = \sum_{k=1}^{n} (x_n - x_{n-1}) \sup_{x_{n-1} \le x < x_n} f(x).$$
We say that f is Riemann-integrable if
$$\sup_P \underline{\mathcal{R}}_P(f) = \inf_P \overline{\mathcal{R}}_P(f)$$
in which case, we define this value to be its Riemann-integral $\mathcal{R}(f)$.
Let $f : [a,b] \to [0,C]$ be Riemann integrable.
   (a) Given partition $P = \{x_0, \ldots, x_n\}$ denote their associated lower and upper step functions by /2
   $$\underline{S}_P(f) = \sum_{k=1}^{n} 1_{[x_{n-1},x_n)} \inf_{x_{n-1} \le x < x_n} f(x), \qquad \overline{S}_P(f) = \sum_{k=1}^{n} 1_{[x_{n-1},x_n)} \sup_{x_{n-1} \le x < x_n} f(x).$$
   Show, that for $P \subset Q$ we have
   $$\underline{S}_P(f) \le \underline{S}_Q(f), \qquad \overline{S}_P(f) \ge \overline{S}_Q(f).$$
   (b) Use part (a) to show that there exists a sequence of $P_1 \subset P_2 \subset \ldots$ of *nested* partitions with /3
   $$\lim_{n\to\infty} \underline{\mathcal{R}}_{P_n}(f) = \lim_{n\to\infty} \overline{\mathcal{R}}_{P_n}(f) = \mathcal{R}(f).$$
   (c) Show, that with P_n from part (b), that for every $a \le x \le b$ the limits /2
   $$\underline{f}(x) := \lim_{n\to\infty} \underline{S}_{P_n}(f)(x), \qquad \overline{f}(x) := \lim_{n\to\infty} \overline{S}_{P_n}(f)(x)$$
   exist, with
   $$\underline{f}(x) \le f(x) \le \overline{f}(x),$$
   and show that $\underline{f}$ and $\overline{f}$ are Lebesgue integrable with
   $$\int \underline{f} \,\mathrm{d}\mathcal{L} = \int \overline{f} \,\mathrm{d}\mathcal{L} = \mathcal{R}(f).$$
   (d) Use Theorem 2.1.9 (v) in order to conclude that f is measurable and Lebesgue integrable with /2
   $$\int f \,\mathrm{d}\mathcal{L} = \mathcal{R}(f).$$