1. Find an example of a sequence of functions $f_1, f_2, \ldots : (0,1) \to [0,\infty]$ such that for every $x \in (0,1)$ we have $\lim_{n\to\infty} f_n(x) = 0$, but $\int_{(0,1)} f_n \, \mathrm{d}\mathcal{L}$ does not converge to $0 = \int_{(0,1)} 0 \, \mathrm{d}\mathcal{L}$. /3

2. Let $f : \Omega \to [-\infty, \infty]$ be measurable and assume that there exists a $1 \le p < \infty$ such that $\|f\|_{L^p(\Omega,\mathcal{M},\mu)} < \infty$. Show, that /4

$$\lim_{p\to\infty} \|f\|_{L^p(\Omega,\mathcal{M},\mu)} = \|f\|_{L^\infty(\Omega,\mathcal{M},\mu)}.$$

Note, that $\|f\|_{L^\infty(\Omega,\mathcal{M},\mu)}$ may be infinite.

You receive partial points if you prove it under the assumption $\mu(\Omega) < \infty$. (In this case we do not need the assumption that there exists a $1 \le p < \infty$ such that $\|f\|_{L^p(\Omega,\mathcal{M},\mu)} < \infty$.)

Hint: First consider functions of the form $a1_E$, and then compare f with such functions.

Solution: Abbreviate $\lambda := \|f\|_\infty$. If $\lambda = 0$ then for almost every $x \in \Omega$ we have $f(x) = 0$ and thus also $\|f\|_p = 0$, so it remains to consider $\lambda > 0$.

We prove the equality as two inequalities. First we prove "$\ge$". To that end let $t < \lambda$ and set $A = \{x \in \Omega : |f(x)| > t\}$. Then $\mu(A) > 0$. Thus, for $t1_A$ we have

$$\|f\|_p \ge \|t1_A\|_p = t\mu(A)^{\frac{1}{p}} \to t$$

as $p \to \infty$. Since t was arbitrary this implies $\liminf_{p\to\infty} \|f\|_p \ge \|f\|_\infty$.

It remains to prove "$\le$". If $\lambda = \infty$ this is immediate and it remains to consider the case $0 < \lambda < \infty$. Set $A = \{x \in \Omega : |f(x)| > \lambda/2\}$. By assumption there exists a $p < \infty$ with $\|f\|_p < \infty$. Since $\lambda > 0$ this implies $\mu(A) < \infty$. Then

$$\lim_{q\to\infty} \|\lambda 1_A\|_q = \lim_{q\to\infty} \lambda\mu(A)^{\frac{1}{q}} = \lambda.$$

Therefore,

$$\lim_{q\to\infty} \frac{1}{\|\lambda 1_A\|_q^q} \int_{\Omega\setminus A} |f|^q \, \mathrm{d}\mu = \lim_{q\to\infty} \frac{1}{\lambda^q} \int_{\Omega\setminus A} |f|^q \, \mathrm{d}\mu \le \lim_{q\to\infty} \frac{1}{\lambda^q} (\lambda/2)^{q-p} \int_{\Omega\setminus A} |f|^p \, \mathrm{d}\mu$$
$$\le \lim_{q\to\infty} \frac{1}{2^{q-p}} \frac{1}{\lambda^p} \|f\|_p^p = 0.$$

Thus, for

$$g(x) = \begin{cases} \lambda & x \in A \\ |f(x)| & x \notin A \end{cases}$$

we have

$$\frac{\|g\|_q}{\|\lambda 1_A\|_q} = \frac{1}{\|\lambda 1_A\|_q} \Big(\int_{\Omega\setminus A} |f|^q \, \mathrm{d}\mu + \|\lambda 1_A\|_q^q \Big)^{\frac{1}{q}} = \Big(\frac{1}{\|\lambda 1_A\|_q^q} \int_{\Omega\setminus A} |f|^q \, \mathrm{d}\mu + 1 \Big)^{\frac{1}{q}} \to 1$$

as $q \to \infty$ by the above arguments, which implies $\|g\|_q \to \lambda$. Since for almost every $x \in \Omega$ we have $0 \le |f(x)| \le g(x) \le \lambda$ which implies

$$\limsup_{p\to\infty} \|f\|_p \le \limsup_{p\to\infty} \|g\|_p \le \|f\|_\infty.$$

3. Let $a, b \in \mathbb{R}$ with $a < b$ and $C > 0$. Recall one definition of the Riemann integral of a bounded function $f : [a,b] \to [0,C]$. We say that a finite set $P = \{x_0, \ldots, x_n\}$ is a partition

if $a = x_0 < ... < x_n = b$. Given such a partition P define their lower and upper Riemann sums as

$$\underline{\mathcal{R}}_P(f) = \sum_{k=1}^{n} (x_n - x_{n-1}) \inf_{x_{n-1} \le x < x_n} f(x), \qquad \overline{\mathcal{R}}_P(f) = \sum_{k=1}^{n} (x_n - x_{n-1}) \sup_{x_{n-1} \le x < x_n} f(x).$$

We say that f is Riemann-integrable if

$$\sup_P \underline{\mathcal{R}}_P(f) = \inf_P \overline{\mathcal{R}}_P(f)$$

in which case, we define this value to be its Riemann-integral $\mathcal{R}(f)$.

Let $f : [a, b] \to [0, C]$ be Riemann integrable.

(a) Given partition $P = \{x_0, ..., x_n\}$ denote their associated lower and upper step functions by /2

$$\underline{S}_P(f) = \sum_{k=1}^{n} 1_{[x_{n-1}, x_n)} \inf_{x_{n-1} \le x < x_n} f(x), \qquad \overline{S}_P(f) = \sum_{k=1}^{n} 1_{[x_{n-1}, x_n)} \sup_{x_{n-1} \le x < x_n} f(x).$$

Show, that for $P \subset Q$ we have

$$\underline{S}_P(f) \le \underline{S}_Q(f), \qquad \overline{S}_P(f) \ge \overline{S}_Q(f).$$

(b) Use part (a) to show that there exists a sequence of $P_1 \subset P_2 \subset ...$ of *nested* partitions with /3

$$\lim_{n\to\infty} \underline{\mathcal{R}}_{P_n}(f) = \lim_{n\to\infty} \overline{\mathcal{R}}_{P_n}(f) = \mathcal{R}(f).$$

(c) Show, that with P_n from part (b), that for every $a \le x \le b$ the limits /2

$$\underline{f}(x) := \lim_{n\to\infty} \underline{S}_{P_n}(f)(x), \qquad \overline{f}(x) := \lim_{n\to\infty} \overline{S}_{P_n}(f)(x)$$

exist, with

$$\underline{f}(x) \le f(x) \le \overline{f}(x),$$

and show that $\underline{f}$ and $\overline{f}$ are Lebesgue integrable with

$$\int \underline{f} \, \mathrm{d}\mathcal{L} = \int \overline{f} \, \mathrm{d}\mathcal{L} = \mathcal{R}(f).$$

(d) Use Theorem 2.1.9 (v) in order to conclude that f is measurable and Lebesgue integrable with /2

$$\int f \, \mathrm{d}\mathcal{L} = \mathcal{R}(f).$$