1. Let $(\Omega, \mathcal{M}, \mu)$ be a measure space with $\mu(\Omega) < \infty$ and let $1 \le p \le \infty$. For any μ-measurable $f : \Omega \to [0, \infty]$ define

$$\|f\|_{L^p(\Omega,\mathcal{M},\mu)} = \Big(\int f^p \,\mathrm{d}\mu\Big)^{\frac{1}{p}}, \qquad 1 \le p < \infty,$$
$$\|f\|_{L^\infty(\Omega,\mathcal{M},\mu)} = \inf\{\lambda \ge 0 : \mu(\{f > \lambda\}) = 0\} \qquad .$$

For $1 \le p \le \infty$ let $p' = \frac{p}{p-1}$ so that $\frac{1}{p} + \frac{1}{p'} = 1$, where $1' = \infty$ and $\infty' = 1$.

(a) Prove **homogeneity**: For any $a \ge 0$ we have $\|af\|_{L^p(\Omega,\mu)} = a\|f\|_{L^p(\Omega,\mu)}$. /1

(b) Prove **Young's inequality**: For $1 < p < \infty$ and any $a, b \ge 0$ we have /3

$$ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'}.$$

Hint: First reduce to the case that $ab = 1$.

(c) Conclude **Hölder's inequality**: For $1 \le p \le \infty$ and any $f, g : \Omega \to [0, \infty]$ we have /3

$$\|fg\|_{L^1(\Omega,\mu)} \le \|f\|_{L^p(\Omega,\mu)}\|g\|_{L^{p'}(\Omega,\mu)}.$$

Hint: First reduce to the case that $\|f\|_{L^p(\Omega,\mu)} = \|g\|_{L^{p'}(\Omega,\mu)} = 1$.

(d) Conclude the **triangle inequality**: For $1 \le p \le \infty$ and any $f, g \in L^p(\Omega, \mathcal{M}, \mu)$ we have /4

$$\|f + g\|_{L^p(\Omega,\mu)} \le \|f\|_{L^p(\Omega,\mu)} + \|g\|_{L^p(\Omega,\mu)}$$

Hint: For $p < \infty$ *factor* $(f(x) + g(x))^p = f(x)(f(x) + g(x))^{p-1} + g(x)(f(x) + g(x))^{p-1}$ *and apply Hölder's inequality.*

(e) Find a measurable $f : \mathbb{R}^d \to [0, \infty]$ that is not zero everywhere but with $\|f\|_{L^p(\mathbb{R}^d,\mathcal{L})} = 0$. /1

Remark. Homogeneity and the triangle inequality make $\|\cdot\|_{L^p(\mathbb{R}^d,\mathcal{L}}$ a **seminorm**. Part (e) shows that it is not a **norm**. But we can make it a norm by considering equivalence classes instead, see the lecture notes.

(To be precise, a norm is defined on a vector space, and the set of all nonnegative functions is not a linear space. However, all results that we have proven here directly extend to general integrable functions, which are a vector space.)

2. Prove Proposition 2.1.17 for $\Omega = \mathbb{R}^d$ and $\mu = \mathcal{L}$: Let $f : \mathbb{R}^d \to [0, \infty]$ with $\int f \,\mathrm{d}\mathcal{L} < \infty$ and let $\varepsilon > 0$.

(a) Show, that there exists a measurable set $B \subset \mathbb{R}^d$ with $\mathcal{L}(B) < \infty$ such that /3

$$\int_{\mathbb{R}^d \setminus B} f \,\mathrm{d}\mathcal{L} < \varepsilon.$$

Hint: Use the monotone convergence theorem applied to appropriate restrictions of f.

(b) Show, that there exists a $\delta > 0$ such that for all measurable $E \subset \mathbb{R}^d$ with $\mathcal{L}(E) < \delta$ we have /4

$$\int_E f \,\mathrm{d}\mathcal{L} < \varepsilon.$$

Hint: Find $\lambda \ge 0$ *such that* $\mathcal{L}(\{f > \lambda\}) < \delta$ *and prove the result for* $f1_{\{f>\lambda\}}$ *and* $f1_{\{f\le\lambda\}}$ *separately.*