

1. Recall, that $\mathcal{L}_* : 2^{\mathbb{R}^d} \rightarrow [0, \infty]$ denotes Lebesgue outer measure. Let $\mathcal{M}$ be a σ -algebra such that $\mathcal{L}_*$ as a map $\mathcal{L}_* : \mathcal{M} \rightarrow [0, \infty]$ is a measure. Show, that all $A \in \mathcal{M}$ are in fact $\mathcal{L}_*$ -measurable, i.e. every $A \in \mathcal{M}$ satisfies the Caratheodory criterion.

Hint: In order to show that for all $A \in \mathcal{M}$ and $B \subset \mathbb{R}^d$ we have

$$\mathcal{L}_*(B) \geq \mathcal{L}_*(B \cap A) + \mathcal{L}_*(B \setminus A),$$

use that by Question 1 part 1 of exercise sheet 3, if $\mathcal{L}_(B) < \infty$ then there exists an open set $U \supset B$ with $\mathcal{L}_*(U) \leq \mathcal{L}_*(B) + \varepsilon$.*

This exercise is false. For a counterexample (assuming AC) take a nonmeasurable set E and the σ -algebra $\mathcal{M} = \{\emptyset, E, \mathbb{R} \setminus E, \mathbb{R}\}$.

However, it remains true if there are Caratheodory-measurable $E_1, E_2, \dots \in \mathcal{M}$ with $\mathcal{L}(E_n) < \infty$ and $\mathbb{R}^d = E_1 \cup E_2 \cup \dots$.

Solution: Let $n \in \mathbb{N}$ and let $A \in \mathcal{M}$ with $A \subset E_n$. For $k \in \mathbb{N}$ recall that $\mathcal{Q}_k$ denotes the set of dyadic cubes with sidelength 2^{-k} . For $Q \in \mathcal{Q}_k$ denote by $(1 - \varepsilon)Q$ the cube with the same center but sidelength $(1 - \varepsilon)2^{-k}$. Then since $\mathcal{L}_*$ is a metric outer measure we have

$$\begin{aligned} \mathcal{L}_*\left(\bigcup_{Q \in \mathcal{Q}_k} (1 - \varepsilon)Q \cap E_n\right) &= \sum_{Q \in \mathcal{Q}_k} \mathcal{L}_*((1 - \varepsilon)Q \cap E_n) \\ &\leq \sum_{Q \in \mathcal{Q}_k} \mathcal{L}_*((1 - \varepsilon)Q \cap A) + \mathcal{L}_*((1 - \varepsilon)Q \cap E_n \setminus A) \\ &= \mathcal{L}_*\left(\bigcup_{Q \in \mathcal{Q}_k} (1 - \varepsilon)Q \cap A\right) + \mathcal{L}_*\left(\bigcup_{Q \in \mathcal{Q}_k} (1 - \varepsilon)Q \cap E_n \setminus A\right) \\ &\leq \mathcal{L}_*(A) + \mathcal{L}_*(E_n \setminus A) = \mathcal{L}_*(E_n). \end{aligned}$$

Taking the limit $\varepsilon \rightarrow 0$, by outer measure continuity of Lebesgue outer measure the first term tends to $\mathcal{L}(E_n)$ and all the inequalities become equalities. In particular we get

$$\sum_{Q \in \mathcal{Q}} \mathcal{L}_*(Q \cap E_n) = \sum_{Q \in \mathcal{Q}} \mathcal{L}_*(Q \cap A) + \mathcal{L}_*(Q \cap E_n \setminus A).$$

Since for each $Q \in \mathcal{Q}_n$ we have $\mathcal{L}(Q \cap E_n) \leq \mathcal{L}_*(Q \cap A) + \mathcal{L}_*(Q \cap E_n \setminus A)$ this means in fact

$$\mathcal{L}_*(Q \cap E_n) = \mathcal{L}_*(Q \cap A) + \mathcal{L}_*(Q \cap E_n \setminus A).$$

Next, we show that A satisfies the Caratheodory criterion. To that end it suffices to consider $B \subset \mathbb{R}^d$ with $\mathcal{L}_*(B) < \infty$. Let $\varepsilon > 0$. Then there exists an open set $U \supset B$ with $\mathcal{L}_*(U) \leq \mathcal{L}_*(B) + \varepsilon$. We can tile U by a set $\mathcal{Q}$ of disjoint dyadic cubes. By the previous argument we can conclude

$$\begin{aligned} \mathcal{L}_*(B) + \varepsilon &\geq \sum_{Q \in \mathcal{Q}} \mathcal{L}_*(Q) = \sum_{Q \in \mathcal{Q}} \mathcal{L}_*(Q \cap E_n) + \mathcal{L}_*(Q \setminus E_n) \\ &= \sum_{Q \in \mathcal{Q}} \mathcal{L}_*(Q \cap A) + \mathcal{L}_*(Q \cap E_n \setminus A) + \mathcal{L}_*(Q \setminus E_n) \\ &\geq \mathcal{L}_*(U \cap A) + \mathcal{L}_*(U \cap E_n \setminus A) + \mathcal{L}_*(U \setminus E_n) \\ &\geq \mathcal{L}_*(B \cap A) + \mathcal{L}_*(B \setminus A). \end{aligned}$$

We finish the proof by letting $\varepsilon \rightarrow 0$.

Finally, let $A \in \mathcal{M}$. Then $A = \bigcup_{n \in \mathbb{N}} A \cap E_n$. Then by the previous case we have that $A \cap E_n$ satisfies the Caratheodory criterion. Since sets that satisfy the Caratheodory criterion are a σ -algebra we can conclude that also A satisfies the Caratheodory criterion.

2. Prove Lemma 1.3.2: Let $f : \Omega \rightarrow [-\infty, \infty]$. Show, that each of the following are equivalent to f being measurable.

- (a) For every $a \in \mathbb{R}$ the set $\{f > a\}$ is measurable. /1
- (b) For every $a \in \mathbb{R}$ the set $\{f \leq a\}$ is measurable. /1
- (c) The function $-f$ is measurable. /1

If $f : \Omega \rightarrow (-\infty, \infty)$ then measurability is also equivalent to each of the following:

- (d) For every $a, b \in \mathbb{R}$ the set $\{a < f < b\}$ is measurable. Equivalently we can replace $<$ by $\leq$ in either instance, but you don't need to show it. /2
- (e) For every open $U \subset \mathbb{R}$ the set $f^{-1}(U)$ is measurable. /2
- (f) For every closed $C \subset \mathbb{R}$ the set $f^{-1}(C)$ is measurable. /1
- (g) For every Borel $B \subset \mathbb{R}$ the set $f^{-1}(B)$ is measurable. /2

The latter also apply to functions $f : \Omega \rightarrow [-\infty, \infty]$ if in addition we require $f^{-1}(\{-\infty\})$ and $f^{-1}(\{\infty\})$ to be measurable, but you don't need to show it.

Solution: For every $a \in \mathbb{R}$ we have $\{f \geq a\} = \Omega \setminus \{f < a\}$. That means $\{f \geq a\}$ is measurable if and only if $\{f < a\}$ is measurable. This proves (i).

This follows from the fact that measurable sets are a σ algebra, and

$$\{f \geq a\} = \Omega \setminus \{f < a\}, \quad \{f > a\} = \bigcup_{n \in \mathbb{N}} \{f \geq a + 1/n\}, \quad \{f \leq a\} = \Omega \setminus \{f > a\}.$$

3. Show Lemma 1.3.4: Let $f_1, f_2, \dots : \Omega \rightarrow [-\infty, \infty]$ be measurable. Show, that the following functions are measurable

- (a) $x \mapsto \inf_n f_n(x)$ /1
- (b) $x \mapsto \limsup_n f_n(x)$ /2
- (c) $x \mapsto \liminf_n f_n(x)$ /1
- (d) If $(f_n)_n$ converges pointwise, then $x \mapsto \lim_n f_n(x)$ is measurable. /1

4. Show Lemma 1.3.5: Let $f : \Omega \rightarrow (-\infty, \infty)$ be measurable and $g : \mathbb{R} \rightarrow \mathbb{R}$ be continuous. Then $g \circ f$ is measurable. /2

5. Show Lemma 1.3.8: Let μ be a complete measure and let $f, g : \Omega \rightarrow [-\infty, \infty]$ such that f is measurable and $f(x) = g(x)$ for μ -almost every x . Show, that g is μ -measurable. /2