

1. Prove what remains to be proven of Proposition 1.2.18:

- (a) Show, that for every $E \subset \mathbb{R}^d$ we have

$$\mathcal{L}_*(E) = \inf\{\mathcal{L}(U) : U \subset \mathbb{R}^d \text{ open, } E \subset U\}.$$

Hint: Use Exercise 2, Question 1.

- (b) If E is measurable then for every $\varepsilon > 0$ exists an open set $U \supset E$ with

$$\mathcal{L}_*(U \setminus E) < \varepsilon.$$

Hint: Show this first for E restricted to an annulus $A_n = B(0, n) \setminus B(0, n-1)$, and then use $\mathbb{R}^d = A_1 \cup A_2 \cup \dots$

- (c) Show, that for every $E \subset \mathbb{R}^d$ exists a Borel set $B \supset E$ such that

$$\mathcal{L}_*(E) = \mathcal{L}(B).$$

If $E \subset \mathbb{R}^d$ is measurable then there exists a Borel set $B \supset E$ such that

$$\mathcal{L}_*(B \setminus E) = 0.$$

- (d) Show, that for every measurable $E \subset \mathbb{R}^d$ exists a Borel set $B \subset E$ such that

$$\mathcal{L}(E \setminus B) = 0.$$

- (e) Show, that if E is Lebesgue measurable with $\mathcal{L}(E) < \infty$ then for every $\varepsilon > 0$ exists a finite collection $\mathcal{Q}$ of disjoint dyadic cubes with

$$\mathcal{L}(E \Delta \bigcup \mathcal{Q}) < \varepsilon.$$

Hint: Use Fact 1.2.17.

2. Given a metric space Ω , show, that each of the following σ -algebras are equal to the Borel σ -algebra.

- (a) The smallest σ -algebra that contains all closed sets.

- (b) For $\Omega = \mathbb{R}^d$ the smallest σ -algebra that contains all compact sets.

- (c) For $\Omega = \mathbb{R}^d$ the smallest σ -algebra that contains all balls $B(x, r)$ with $x \in \mathbb{Q}^d$ and $r \in \mathbb{Q}$.

Note, that this means the Borel σ -algebra can be generated by a countable set.

3. Consider the following set functions $\mu : 2^{\mathbb{R}} \rightarrow [0, \infty]$. Which of those is a measure on $2^{\mathbb{R}}$? Which of those is an outer measure on $\mathbb{R}$? For an outer measure, which are the measurable sets? Prove your answers.

- (a)

$$\mu(A) = \begin{cases} 0 & A \text{ is countable} \\ 1 & A \text{ is uncountable} \end{cases}$$

- (b)

$$\mu(A) = \begin{cases} 0 & A \text{ is countable} \\ \infty & A \text{ is uncountable} \end{cases}$$

- (c) for a fixed $x \in \mathbb{R}$,

$$\mu(A) = \delta_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

- (d)

$$\mu(A) = \#A,$$

the number of elements in A .