1. Show that for any $E \subset \mathbb{R}^d$ we have /2

$$\mathcal{L}_*(E) = \inf\Big\{\sum_{Q\in\mathcal{Q}} |Q| : \mathcal{Q} \text{ is a countable set of open cubes } Q \text{ with } E \subset \bigcup \mathcal{Q}\Big\}$$

2. Show, that a collection of sets $\mathcal{M} \subset 2^\Omega$ is a σ-algebra if and only if for each $E \in \mathcal{M}$ we have $\Omega \setminus E \in \mathcal{M}$, and for each $E_1, E_2, \ldots \in \mathcal{M}$ we have /2

$$\bigcap_{n=1}^\infty E_n \in \mathcal{M}.$$

3. Prove Lemma 1.2.6: Let $(\Omega, \mathcal{M}, \mu)$ be a measure space and let $E_1, E_2, \ldots \in \mathcal{M}$.
   (a) If $E_1 \subset E_2 \subset \ldots$ then /4

   $$\lim_{n\to\infty} \mu(E_n) = \mu\Big(\bigcup_{n=1}^\infty E_n\Big),$$

   where both sides may be infinite.
   Hint: Consider the sequence given by $A_1 = E_1$ and $A_n = E_n \setminus E_{n-1}$ and recall the definition of the infinite sum

   $$\sum_{k=1}^\infty \mu(A_k) = \lim_{n\to\infty} \sum_{k=1}^n \mu(A_k).$$

   (b) If $\mu(E_1) < \infty$ and $E_1 \supset E_2 \supset \ldots$ then /2

   $$\lim_{n\to\infty} \mu(E_n) = \mu\Big(\bigcap_{n=1}^\infty E_n\Big).$$

   Hint: Use part (a).
   (c) Find an example of sets $\mathbb{R}^d \supset E_1 \supset E_2 \supset \ldots$ with $\mathcal{L}(E_1) = \infty$ for which /2

   $$\lim_{n\to\infty} \mathcal{L}(E_n) \neq \mathcal{L}(\bigcap_{n=1}^\infty E_n).$$

4. We want to prove Fact 1.2.16 in one dimension. Let $d = 1$ so that

$$\mathcal{D}_n = \{[2^n k, 2^n(k+1)) : k \in \mathbb{Z}\}, \qquad \mathcal{D} = \bigcup_{n\in\mathbb{Z}} \mathcal{D}_n.$$

   (a) Show, that for any $Q, P \in \mathcal{D}$ we have $P \subset Q$ or $Q \subset P$ or $P \cap Q = \emptyset$. /2
   (b) Let $n \in \mathbb{Z}$, /1

   $$\mathcal{Q} \subset \bigcup_{k\in\mathbb{Z},\ k\le n} \mathcal{D}_k,$$

   and

   $$\tilde{\mathcal{Q}} := \{Q \in \mathcal{Q} : \forall P \in \mathcal{Q}\ \neg Q \subsetneq P\},$$

   the subset of maximal cubes.
   Show, that for any $Q, P \in \tilde{\mathcal{Q}}$ with $P \neq Q$ we have $Q \cap P = \emptyset$.
   (c) For $\tilde{\mathcal{Q}}$ from above, show that and for every $Q \in \mathcal{Q}$ exists a $P \in \tilde{\mathcal{Q}}$ with $Q \subset P$. /3
   (d) Show, that for any $x \in \mathbb{R}$ and $r > 0$ exists a $Q \in \mathcal{D}$ with $x \in Q$ and $Q \subset (x-r, x+r)$. /1
   (e) Conclude Fact 1.2.16 for $d = 1$. Precisely, show, that for every open set $U \subset \mathbb{R}$ with $\mathcal{L}(U) < \infty$ exists a set $\mathcal{Q} \subset \mathcal{D}$ of disjoint dyadic intervals with $U = \bigcup \mathcal{Q}$. /3
   (f) Show, that the restriction $k \le n$ in part (b) is necessary. /2
   More precisely, provide an example of a collection $\mathcal{Q} \subset \mathbb{D}$ such that for any subcollection $\tilde{\mathcal{Q}} \subset \mathcal{Q}$, consisting of disjoint dyadic cubes, there exists a $Q \in \mathcal{Q}$ such that there is no $P \in \tilde{\mathcal{Q}}$ with $Q \subset P$.