Exercise 2
Due: Thursday, 2025-09-25

Total: /24

1. Show that for any $E \subset \mathbb{R}^d$ we have /2

$$\mathcal{L}_*(E) = \inf\Big\{ \sum_{Q\in\mathcal{Q}} |Q| : \mathcal{Q} \text{ is a countable set of open cubes } Q \text{ with } E \subset \bigcup \mathcal{Q} \Big\}$$

Solution: Let $\mathcal{Q}$ be a set of open cubes with $E \subset \bigcup \mathcal{Q}$. Then $\mathcal{P} = \{\overline{Q} : Q \in \mathcal{Q}\}$ is a set of closed cubes with $E \subset \bigcup \mathcal{P}$ and

$$\sum_{Q\in\mathcal{Q}} |Q| = \sum_{Q\in\mathcal{P}} |P|.$$

Hence the set over which the infimum above is taken contains the set over which the infimum from the definition of Lebesgue outer measure is taken, and thus

$$\mathcal{L}_*(E) \le \inf\Big\{ \sum_{Q\in\mathcal{Q}} |Q| : \mathcal{Q} \text{ is a countable set of open cubes } Q \text{ with } E \subset \bigcup \mathcal{Q} \Big\}.$$

For the reverse inequality, let $\mathcal{Q} = \{Q_1, Q_2, ...\}$ be a set of closed cubes with $E \subset \bigcup \mathcal{Q}$. Let $\varepsilon > 0$. For each n there exists an open cube $P_n \supset Q_n$ such that $|P_n| \le |Q_n| + 2^{-n}\varepsilon$ and set $\mathcal{P} = \{P_1, P_2, ...\}$. Then

$$E \subset \bigcup \mathcal{Q} \subset \bigcup \mathcal{P}, \qquad \sum_{P\in\mathcal{P}} |P| \le \varepsilon + \sum_{Q\in\mathcal{Q}} |Q|.$$

Therefore,

$$\begin{aligned} &\inf\Big\{ \sum_{Q\in\mathcal{Q}} |Q| : \mathcal{Q} \text{ is a countable set of open cubes } Q \text{ with } E \subset \bigcup \mathcal{Q} \Big\} \\ &\quad \le \inf\Big\{ \sum_{Q\in\mathcal{Q}} |Q| : \mathcal{Q} \text{ is a countable set of closed cubes } Q \text{ with } E \subset \bigcup \mathcal{Q} \Big\} = \mathcal{L}_*(E), \end{aligned}$$

finishing the proof.

2. Show, that a collection of sets $\mathcal{M} \subset 2^\Omega$ is a σ-algebra if and only if for each $E \in \mathcal{M}$ we have $\Omega \setminus E \in \mathcal{M}$, and for each $E_1, E_2, ... \in \mathcal{M}$ we have /2

$$\bigcap_{n=1}^\infty E_n \in \mathcal{M}.$$

Solution: If $\mathcal{M}$ is a σ-algebra then for each $E \in \mathcal{M}$ we have $\Omega \setminus E \in \mathcal{M}$. For $E_1, E_2, ... \in \mathcal{M}$ we have $\Omega \setminus E_1, \Omega \setminus E_2, ... \in \mathcal{M}$ and thus

$$\bigcap_{n=1}^\infty E_n = \Omega \setminus \bigcup_{n=1}^\infty (\Omega \setminus E_n) \in \mathcal{M}.$$

The reverse implication follows similarly, using

$$\bigcup_{n=1}^\infty E_n = \Omega \setminus \bigcap_{n=1}^\infty (\Omega \setminus E_n).$$

3. Prove Lemma 1.2.7: Let $(\Omega, \mathcal{M}, \mu)$ be a measure space and let $E_1, E_2, ... \in \mathcal{M}$.

   (a) If $E_1 \subset E_2 \subset ...$ then /4

$$\lim_{n\to\infty} \mu(E_n) = \mu\Big(\bigcup_{n=1}^{\infty} E_n\Big),$$

   where both sides may be infinite.

   Hint: Consider the sequence given by $A_1 = E_1$ and $A_n = E_n \setminus E_{n-1}$ and recall the definition of the infinite sum

$$\sum_{k=1}^{\infty} \mu(A_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(A_k).$$

   (b) If $\mu(E_1) < \infty$ and $E_1 \supset E_2 \supset ...$ then /2

$$\lim_{n\to\infty} \mu(E_n) = \mu\Big(\bigcap_{n=1}^{\infty} E_n\Big).$$

   Hint: Use part (a).

   (c) Find an example of sets $\mathbb{R}^d \supset E_1 \supset E_2 \supset ...$ with $\mathcal{L}(E_1) = \infty$ for which /2

$$\lim_{n\to\infty} \mathcal{L}(E_n) \neq \mathcal{L}(\bigcap_{n=1}^{\infty} E_n).$$

Solution:

(a) By definition of infinite sums we have

$$\lim_{n\to\infty} \sum_{k=1}^{n} \mu(A_k) = \sum_{k=1}^{\infty} \mu(A_k).$$

The sets $A_1, A_2, ...$ are disjoint and for each $n \in \mathbb{N}$ we have $E_n = A_1 \cup ... \cup A_n$ and

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} E_n,$$

which implies

$$\mu(E_n) = \sum_{k=1}^{n} \mu(A_k), \qquad \mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{k=1}^{\infty} \mu(A_k).$$

Combining with the definition of the infinite sum concludes the proof.

(b) Set $A_n = E_1 \setminus E_n$. Then $A_1, A_2, ...$ is an icreasing sequence as in part (a) and

$$\mu(A_n) = \mu(E_1) - \mu(E_n),$$
$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \mu\Big(E_1 \setminus \bigcap_{n=1}^{\infty} E_n\Big) = \mu(E_1) - \mu\Big(\bigcap_{n=1}^{\infty} E_n\Big)$$

After substracting $\mu(E_1) < \infty$ from both sides we can conclude the result from part (a).

(c) For $d = 1$ set

$$E_n = \bigcup_{k \in \mathbb{Z}} (k, k + 2^{-n}).$$

Then $E_1 \supset E_2 \supset \dots$ and

$$\bigcap_{n=0}^{\infty} E_n = \emptyset$$

which has zero Lebesgue measure. However, for each n we have $\mathcal{L}(E_n) = \infty$.

4. We want to prove Fact 1.2.17 in one dimension. Let $d = 1$ so that

$$\mathcal{D}_n = \{[2^n k, 2^n(k+1)) : k \in \mathbb{Z}\}, \qquad \mathcal{D} = \bigcup_{n \in \mathbb{Z}} \mathcal{D}_n.$$

(a) Show, that for any $Q, P \in \mathcal{D}$ we have $P \subset Q$ or $Q \subset P$ or $P \cap Q = \emptyset$. /2

(b) Let $n \in \mathbb{Z}$, /1

$$\mathcal{Q} \subset \bigcup_{k \in \mathbb{Z},\ k \leq n} \mathcal{D}_k,$$

and

$$\tilde{\mathcal{Q}} := \{Q \in \mathcal{Q} : \forall P \in \mathcal{Q}\ \neg Q \subsetneq P\},$$

the subset of maximal cubes.

Show, that for any $Q, P \in \tilde{\mathcal{Q}}$ with $P \neq Q$ we have $Q \cap P = \emptyset$.

(c) For $\tilde{\mathcal{Q}}$ from above, show that and for every $Q \in \mathcal{Q}$ exists a $P \in \tilde{\mathcal{Q}}$ with $Q \subset P$. /3

(d) Show, that for any $x \in \mathbb{R}$ and $r > 0$ exists a $Q \in \mathcal{D}$ with $x \in Q$ and $Q \subset (x - r, x + r)$. /1

(e) Conclude Fact 1.2.17 for $d = 1$. Precisely, show, that for every open set $U \subset \mathbb{R}$ with $\mathcal{L}(U) < \infty$ exists a set $\mathcal{Q} \subset \mathcal{D}$ of disjoint dyadic intervals with $U = \bigcup \mathcal{Q}$. /3

(f) Show, that the restriction $k \leq n$ in part (b) is necessary. /2

More precisely, provide an example of a collection $\mathcal{Q} \subset \mathbb{D}$ such that for any subcollection $\tilde{\mathcal{Q}} \subset \mathcal{Q}$, consisting of disjoint dyadic cubes, there exists a $Q \in \mathcal{Q}$ such that there is no $P \in \tilde{\mathcal{Q}}$ with $Q \subset P$.

Solution:

(a) Let $Q_1, Q_2 \in \mathcal{D}$. Then there are $n_1, n_2 \in \mathbb{Z}$ with $Q_i \in \mathcal{D}_{n_i}$. By symmetry it suffices to consider $n_1 \leq n_2$. Moreover there exist $k_1, k_2 \in \mathbb{Z}$ such that $Q_i = [2^{n_i} k_i, 2^{n_i}(k_i + 1))$. If $2^{n_1} k_1 \geq 2^{n_2}(k_2 + 1)$ then $Q_1 \cap Q_2 = \emptyset$. If $2^{n_1} k_1 < 2^{n_2} k_2$ then $k_1 < 2^{n_1 - n_2} k_2$, and since the right hand side is an integer, also $2^{n_1}(k_1 + 1) \leq 2^{n_2} k_2$ and thus also $Q_1 \cap Q_2 = \emptyset$. It remains to consider the case $2^{n_2} k_2 \leq 2^{n_1} k_1 < 2^{n_2}(k_2 + 1)$. Then $k_1 < 2^{n_2 - n_1}(k_2 + 1)$. Since the right hand side is an integer, also $2^{n_1}(k_1 + 1) \leq 2^{n_2}(k_2 + 1)$. We can conclude $Q_1 \subset Q_2$.

In fact, we have shown that for $Q_i \in \mathcal{D}_{n_i}$ with $n_1 \leq n_2$ we have $Q_1 \cap Q_2 \in \{\emptyset, Q_2\}$.

(b) Let $P, Q \in \tilde{\mathcal{Q}}$ with $P \neq Q$. Then $P, Q \in \mathcal{Q}$ and by the previous part we have $P \subset Q, Q \subset P$ or $P \cap Q = \emptyset$. If $P \subset Q$ then in fact $P \subsetneq Q$ since $P \neq Q$, but this is not possible by definition of $\tilde{\mathcal{Q}}$. Similary we can exclude $Q \subset P$, and thus only the possibility $P \cap Q$ remains.

(c) Let $Q \in \mathcal{Q} \cap \mathcal{D}_k$. Then there exists a maximal $N \leq n$ for which there exists a $P \in \mathcal{Q} \cap \mathcal{D}_N$ with $Q \cap P \neq \emptyset$. Since $Q \cap Q \neq \emptyset$ we have $k \leq N$. That means $Q \subset P$ by the first part. Now, there exists no $m > k$ and $R \in \mathcal{Q} \cap \mathcal{D}_m$ with $R \cap P \neq \emptyset$, because then we would have $Q \subset P \subset R$, and in particular $Q \cap R \neq \emptyset$. For any $m \leq k$ and $R \in \mathbb{D}_m$, by the previous part we do not have $P \subsetneq R$. We can conclude $P \in \tilde{\mathcal{Q}}$.

(d) Take $n \in \mathbb{Z}$ such that $2^n < r$. Take $k \in \mathbb{Z}$ maximal with $2^n k \leq x$. Then $2^n(k+1) > x$. We can conclude

$$x - r < 2^n(k+1) - 2^n = 2^n k \leq x \leq 2^n(k+1) = 2^n k + 2^n < x + r$$

and thus the claim is true for $Q = [2^n k, 2^n(k+1))$.

In fact, we have show that for any $n \in \mathbb{Z}$ with $2^n < r$ exists such a $Q \in \mathcal{D}_n$ with $x \in Q \subset (x - r, x + r)$.

(e) Let $n \in \mathbb{Z}$. If U is open then for any $x \in U$ exists an $r > 0$ such that $B(x, r) \subset U$. By the previous part exists a $k \leq n$ and a $Q \in \mathcal{D}_k$ with $x \in Q \subset B(x, r) \subset U$. This means that for

$$\mathcal{Q} = \bigcup_{k=-\infty}^{n} \{Q \in \mathcal{D}_k : \exists x \in U,\ \exists r > 0\ x \in Q \subset B(x, r) \subset U\}$$

we have $U = \bigcup \mathcal{Q}$. That means $\tilde{\mathcal{Q}}$ is a set of disjoint cubes and by the third part we have $U = \bigcup \tilde{\mathcal{Q}}$.

(f) Set $\mathcal{Q} = \{[0, 2^n) : n \in \mathbb{Z}\}$. The cubes in $\mathcal{Q}$ are nested, which means that any disjoint subcollection consists of at most one cube. But $\bigcup \mathcal{Q} = [0, \infty)$ is not a single cube or empty.