

1. Let $E \subset \mathbb{R}^d$ be countable. Show, that $\mathcal{L}_*(E) = 0$. /2
2. Let $Q \subset \mathbb{R}^d$ be a cube. Show, that $\mathcal{L}_*(\partial Q) = 0$. **Only consider the case $d = 2$.** /4
3. Find a set Ω and an outer measure μ_* on Ω such that there exist disjoint $A, B \subset \Omega$ with $\mu_*(A \cup B) \neq \mu_*(A) + \mu_*(B)$. /4

Hint: An example already exists with a finite set Ω , i.e. $\#\Omega < \infty$.

4. (a) Let μ_* be an outer measure on Ω and let $A \subset \Omega$. Show, that the map $\mu_*^A : 2^\Omega \rightarrow [0, \infty]$ given by $\mu_*^A(E) = \mu_*(A \cap E)$ is an outer measure. /2
- (b) Let $(\Omega, \mathcal{M}, \mu)$ be a measure space and let $A \in \mathcal{M}$. Show, that the map $\mu^A : \mathcal{M} \rightarrow [0, \infty]$ given by $\mu^A(E) = \mu(A \cap E)$ is a measure. /2
5. Let μ_* be an outer measure on Ω and let $\mathcal{M}$ be a σ -algebra on Ω . Show, that /4

$$\{(A \cup E) \setminus F : A \in \mathcal{M}, E, F \subset \Omega, \mu_*(E) = \mu_*(F) = 0\}$$

is a σ -algebra, too.

This σ -algebra is called the **completion** of $\mathcal{M}$.

6. Let Ω be a set and $\mathcal{M} \subset 2^\Omega$. Show, that /4

$$\bigcap \{\mathcal{A} : \mathcal{A} \subset 2^\Omega \text{ is a } \sigma\text{-algebra, } \mathcal{M} \subset \mathcal{A}\}$$

is a σ -algebra. Conclude, that the above collection of sets is the smallest σ -algebra that contains $\mathcal{M}$.