Total:

 $/40$

Your name:

- Your total number of points will be the sum of points from those **five** out of the six questions for which you will have received the highest number of points.

  That means you can try to solve all questions, and then only your best five out of six solutions will count, or you can leave one question completely unanswered.
- Please write your name on each paper.
- You may completely ignore the *Remark*s below.

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1. Let $\Omega = \{1, 2, 3\}$ and let μ be a measure on Ω with $/8$

$$\mu(\{1\}) = 2, \qquad \mu(\{1,2\}) = 3, \qquad \mu(\{2,3\}) = 3.$$

Let $f : \Omega \to \mathbb{R}$ with $f(x) = x$ and compute $\int_\Omega f \, \mathrm{d}\mu$.

Solution: We have $\mu(\{2\}) = \mu(\{1,2\}) - \mu(\{1\}) = 1$ and $\mu(\{3\}) = \mu(\{2,3\}) - \mu(\{2\}) = 2$. Thus,

$$\int_\Omega f \, \mathrm{d}\mu = 2 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 = 10.$$

2. Let $(\Omega, \mathcal{M}, \mu)$ be a measure space and let $f_1, f_2, \ldots : \Omega \to [0, \infty]$ be measurable. Show, that $f : \mathbb{R} \to [0, \infty]$ given by $f(x) = \sum_{n=1}^\infty f_n(x)$ is measurable and $/8$

$$\int f \, \mathrm{d}\mu = \sum_{n=1}^\infty \int f_n \, \mathrm{d}\mu.$$

Solution: By definition

$$f(x) = \lim_{n\to\infty} \sum_{k=1}^n f_k(x).$$

Thus, f is a pointwise limit of sums of measurable functions, which makes f measurable, too. In fact, f is a monotone limit from below. Therefore, by the linearity of the Lebesgue integral we have

$$\int f \, \mathrm{d}\mu = \lim_{n\to\infty} \int \sum_{k=1}^n f_k \, \mathrm{d}\mu = \lim_{n\to\infty} \int \sum_{k=1}^n f_k \, \mathrm{d}\mu = \int \sum_{k=1}^\infty f_k \, \mathrm{d}\mu.$$

3. Let μ, ν be Radon measures on $\mathbb{R}$. Assume that for every dyadic interval $I \subset \mathbb{R}$ we have $\mu(I) = \nu(I)$. Show, that for every Borel $E \subset \mathbb{R}$ we have $\mu(E) = \nu(E)$. $/8$

Solution: Since E is Borel, it is μ and ν measurable. Thus, by outer regularity we have

$$\mu(E) = \inf\{\mu(U) : \mathbb{R} \supset U \text{ open}\}$$

and the same for ν. That means there exist sequences $U_1, U_2, ... \subset \Omega$ and $V_1, V_2, ... \subset \Omega$ open such that $E \subset U_n, V_n$ and $\mu(U_n) \to \mu(E)$ and $\nu(V_n) \to \nu(E)$. Since $U_n \cap V_n$ is open and a subset of both U_n and V_n and $E \subset U_n \cap V_n$ we have

$$\mu(E) \leq \mu(U_n \cap V_n) \leq \mu(U_n) \to \mu(E)$$

and thus also $\mu(U_n \cap V_n) \to \mu(E)$. Similarly, $\nu(U_n \cap V_n) \to \nu(E)$. Now, we know that since $U_n \cap V_n$ is open it can be written as a disjoint union of dyadic intervals $\mathcal{A}_n$. Thus, by assumption

$$\mu(U_n \cap V_n) = \sum_{I \in \mathcal{A}_n} \mu(I) = \sum_{I \in \mathcal{A}_n} \nu(I) = \nu(U_n \cap V_n).$$

Therefore, $\mu(E) = \nu(E)$.

4. Let $E \subset \mathbb{R}^2$ be Borel, let $x \in \mathbb{R}$ and denote $E^x = \{y \in \mathbb{R} : (x, y) \in E\} \subset \mathbb{R}$. Show, that E^x is Borel. /8

Hint: Show, that the collection of all sets $E \subset \mathbb{R}^2$ which have the desired property is a σ-algebra which contains all open sets.

Remark: The corresponding statement for Lebesgue measurable sets fails.

Solution: Let $x \in \mathbb{R}$. Then for every open set U the set U^x is open, too. In particular, U^x is Borel. Let $A_1, A_2, ... \subset \mathbb{R}^2$ have the property that A_n^x is Borel. Then

$$\left(\bigcup_{n \in \mathbb{N}} A_n\right)^x = \bigcup_{n \in \mathbb{N}} A_n^x,$$

which is a countable union of Borel sets and thus Borel itself. Moreover,

$$(\mathbb{R}^2 \setminus A_1)^x = \mathbb{R} \setminus (A_1^x),$$

which is the complement of a Borel set and hence Borel itself. We can conclude that the set of all sets $A \subset \mathbb{R}^2$ such that A^x is Borel is a σ-algebra which contains all open sets. Since the Borel σ-algebra is the smallest σ-algebra which contains all Borel sets, it must belong to this σ-algebra. Hence, for every Borel set E also E^x is Borel.

5. For $f, g : \mathbb{R} \to [-\infty, \infty]$ recall the definition of the convolution, /8

$$(f * g)(x) := \int f(x - y) \cdot g(y) \, \mathrm{d}\mathcal{L}(y).$$

Let $E, F \subset \mathbb{R}$ be Lebesgue measurable with $\mathcal{L}(E), \mathcal{L}(F) < \infty$. Show, that the convolution $1_E * 1_F : \mathbb{R} \to \mathbb{R}$ is a continuous function.

Hint: First consider the case that E is an interval. (However, there is also a way to solve this question that doesn't use this hint.)

Remark: In fact $f * g$ is continuous if $f \in L^p(\mathbb{R}^d)$ and $g \in L^{p'}(\mathbb{R}^d)$.

Solution: Fast solution: By Proposition 4.2.7 we have

$$\begin{aligned}|1_E * 1_F(x) - 1_E * 1_F(y)| &= \int (1_E(x-z) - 1_E(y-z))1_F(z)\,\mathrm{d}\mathcal{L}(z)\\ &\leq \|1_{E-x} - 1_{E-y}\|_1 \to 0\end{aligned}$$

as $y \to x$. This convergence is actually uniform in $|x-y|$ so this even proves uniform convergence. (This proof combined with Hölder's inequality also shows that $f * g$ is continuous if $f \in L^p$ and $g \in L^{p'}$.)

Slow solution that uses hint: The plan is to do the same thing but prove it without referring to Proposition 4.2.7. Let $x \in \mathbb{R}$ and let $\varepsilon > 0$. Since $\mathcal{L}(F) < \infty$ we have $1_F \in L^1$, and thus there exists a step function with

$$\|1_F - \sum_{k=1}^n a_k 1_{I_k}\|_1 < \varepsilon.$$

Thus,

$$\begin{aligned}\|1_E * 1_F - (\sum_{k=1}^n a_k 1_{I_k}) * 1_F\|_\infty &= \|(1_E - \sum_{k=1}^n a_k 1_{I_k}) * 1_F\|_\infty\\ &\leq \|1_E - (\sum_{k=1}^n a_k 1_{I_k})\|_1 < \varepsilon.\end{aligned}$$

Now, for any $k \leq n$ we have

$$\begin{aligned}|(1_E * 1_{I_k})(x) - (1_E * 1_{I_k})(y)| &= |\int 1_E(x-z)(1_{I_k}(z) - 1_{I_k}(z+y-x))\,\mathrm{d}\mathcal{L}(z)\\ &\leq \int |1_{I_k}(z) - 1_{I_k}(z+y-x)|\,\mathrm{d}\mathcal{L}(z) = 2|x-y|.\end{aligned}$$

Thus, for $\delta < \varepsilon \sum_{k=1}^n a_k/2$ and $|x-y| < \delta$ we obtain

$$\Big|1_E * (\sum_{k=1}^n a_k 1_{I_k})(x) - 1_E * (\sum_{k=1}^n a_k 1_{I_k})(z)\Big| \leq \sum_{k=1}^n a_k |1_E * 1_{I_k} * (x) - 1_E * 1_{I_k} * (z)| \leq \varepsilon.$$

Combining the estimates we can conclude

$$\begin{aligned}&|1_E * 1_F(x) - 1_E * 1_F(z)|\\ &\quad\leq \|1_E * 1_F - 1_E * (\sum_{k=1}^n a_k 1_{I_k})\|_\infty + |1_E * (\sum_{k=1}^n a_k 1_{I_k})(x) - 1_E * (\sum_{k=1}^n a_k 1_{I_k})(z)| \leq 2\varepsilon.\end{aligned}$$

This proves continuity, even uniform continuity.

6. Let $f \in L^1(\mathbb{R}^d)$. For $x \in \mathbb{R}^d$ and $r > 0$ abbreviate

$$f_{B(x,r)} := \frac{1}{\mathcal{L}(B(x,r))} \int_{B(x,r)} f\,\mathrm{d}\mathcal{L}.$$

(a) Show, that for any $r > 0$ the map $\mathbb{R}^d \ni x \mapsto f_{B(x,r)}$ is continuous, and that for any $x \in \mathbb{R}^d$ the map $(0,\infty) \ni r \mapsto f_{B(x,r)}$ is continuous. /4

(b) Conclude, that the Hardy-Littlewood maximal function $Mf : \mathbb{R}^d \to (-\infty, \infty]$ given by $Mf(x) = \sup_{r>0} f_{B(x,r)}$ is a Lebesgue measurable function. /4

Remark: We already used in the lecture that the Hardy-Littlewood maximal function is measurable. I forgot to prove it there so we do it here instead. Also, the Hardy-Littlewood maximal function is usually defined in terms of $|f|$, but it makes not much difference.

Solution:

(i) Let $r_n \to r$ and consider $g_n = f 1_{B(x,r_n)}$. Then $g_n \leq |f|$ and $g_n(x) \to f 1_{B(x,r)}$ for every $x \in \mathbb{R}^d \setminus \partial B(x,r)$, so in particular for $\mathcal{L}$-almost every x. By dominated convergence we can conclude

$$\lim_{n\to\infty} \int_{B(x,r_n)} f \, \mathrm{d}\mathcal{L} = \int_{B(x,r)} f \, \mathrm{d}\mathcal{L}.$$

Applying the same to $f = 1_{B(x,2r)}$ we can conclude $\mathcal{L}(B(x,r_n)) \to \mathcal{L}(B(x,r))$. Thus,

$$\lim_{n\to\infty} \frac{1}{B(x,r_n)} \int_{B(x,r_n)} f \, \mathrm{d}\mathcal{L} = \frac{1}{B(x,r)} \int_{B(x,r)} f \, \mathrm{d}\mathcal{L}.$$

Similarly, let $x_n \to x$ and denote $g_n(x) = f_{B(x_n,r)}$. Then again $g_n(x) \to f_{B(x,r)}$ for every $x \in \mathbb{R}^d \setminus \partial B(x,r)$ and thus

$$\lim_{n\to\infty} \frac{1}{B(x_n,r)} \int_{B(x_n,r)} f \, \mathrm{d}\mathcal{L} = \frac{1}{B(x,r)} \int_{B(x,r)} f \, \mathrm{d}\mathcal{L}.$$

(ii) By the first continuity we have $Mf(x) = \sup_{r\in\mathbb{Q}, r>0} f_{B(x,r)}$. This makes Mf the supremum of a countable set of functions which by the second continuity are continuous and thus measurable. Hence also Mf is measurable.