Alberti representations, rectifiability, PDEs and multilinear Kakeya

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Conference on geometric measure theory and metric geometry

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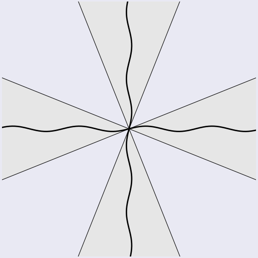
Alberti representations

Definition (Alberti representation)

An *Alberti representation* of a finite measure μ on $\mathbb{R}^d$ is a finite measure η on the space of all Lipschitz curves $\Gamma(\mathbb{R}^d)$ on $\mathbb{R}^d$ such that

$$\mu \ll \int_{\Gamma(\mathbb{R}^n)} \mathcal{H}^1\restriction_\gamma \mathrm{d}\eta(\gamma) = A \mapsto \int_{\Gamma(\mathbb{R}^n)} \mathcal{H}^1(A \cap \gamma)\,\mathrm{d}\eta(\gamma).$$

Alberti representations $\eta_1, \ldots, \eta_n$ are *independent* if for $(\eta_1, \ldots, \eta_n)$-almost any tuple of curves $(\gamma_1, \ldots, \gamma_n) \in \Gamma(\mathbb{R}^d)^n$, the γ_i travel within linearly independent cones.



(everything modulo countable decompositions)

Euclidean case

Example

1. $\mathbb{R}^n$ ($\mathcal{H}^n\restriction_{\mathbb{R}^n\times\{0\}^{d-n}}$) has n independent Alberti representations.
2. By Rademacher's theorem a Lipschitz image of $\mathbb{R}^n$ has n independent Alberti representations.
3. An n-rectifiable set (a countable union of Lipschitz images of $\mathbb{R}^n$) has n independent Alberti representations.

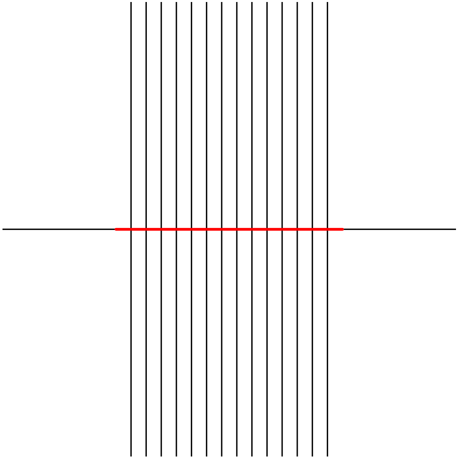
Converse:

Theorem

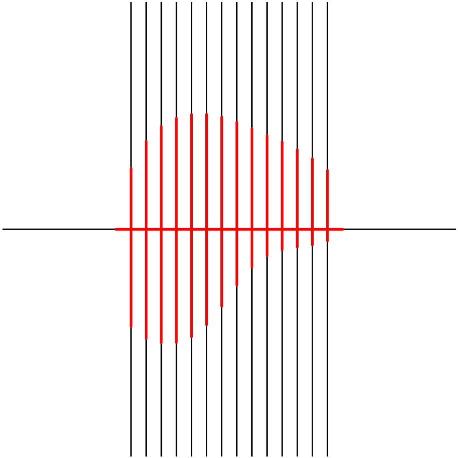
A set with n independent Alberti representations is n-rectifiable.

Axes-parallel case

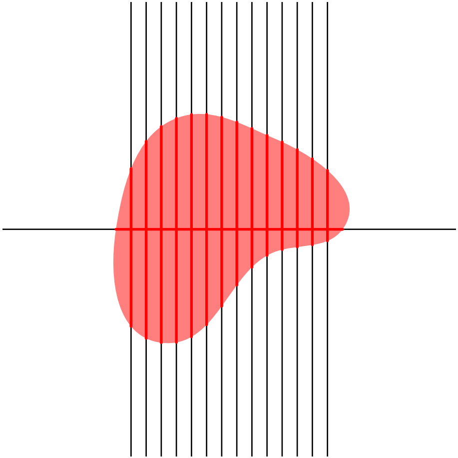
Axes-parallel case



Axes-parallel case



Axes-parallel case



Euclidean case

Theorem (DePhilippis-Rindler, Ann. of Math. (2016), divergence case)

Let $\mathbf{T}$ be an $\mathbb{R}^{n\times n}$-valued finite measure on $\mathbb{R}^n$ such that $\operatorname{div}\mathbf{T}$ is a finite measure. Then the restriction of $\mathbf{T}$ to those points, where its polar $\mathbf{T}/|\mathbf{T}|$ is an invertible matrix, is absolutely continuous with respect to Lebesgue measure.

This applies for example to a tuple of n independent Alberti representations since

$$\operatorname{div}\Big(\int \dot\gamma\mathcal{H}^1\restriction_\gamma \,\mathrm{d}\eta(\gamma)\Big) = \int \operatorname{div}(\dot\gamma\mathcal{H}^1\restriction_\gamma)\,\mathrm{d}\eta(\gamma) = 0.$$

Euclidean case

Theorem (Besicovitch projection theorem)

A set $E \subset \mathbb{R}^d$ is purely n-unrectifiable if and only if $\mathcal{H}^{d-n}$-almost every projection of E to an n-plane has $\mathcal{H}^n$-measure 0.

Proof, that n independent Alberti representations imply rectifiability.

Assume $\mathcal{H}^n \restriction_E$ has independent Alberti representations $\eta_1, ..., \eta_n$. Then there exist n linearly independent directions $e_1, ..., e_n$ such that η_i is supported on curve running roughly in direction e_i. Let S be a small open neighborhood of the directions orthogonal to $e_1, ..., e_n$. Then $\mathcal{H}^{d-n}(S) > 0$. Let $e \in S$. Then $\pi_{e\#}\eta_1, ..., \pi_{e\#}\eta_n$ are independent Alberti representations of $\pi_{e\#}\mathcal{H}^n\restriction_E$. By DePhilippis-Rindler this implies $\pi_{e\#}\mathcal{H}^n\restriction_E$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^n$. Since $\mathcal{H}^n(E) > 0$ this means it has support with positive Lebesgue measure. This means $\mathcal{H}^n(\pi_e E) > 0$. Thus E is not purely n-unrectifiable. □

Metric case

Definition (Alberti representation)

An *Alberti representation* of a finite measure μ on X is a finite measure η on the space of all Lipschitz curves $\Gamma(X)$ on X such that

$$\mu \ll \int_{\Gamma(X)} \mathcal{H}^1 \restriction_\gamma \mathrm{d}\eta(\gamma) = A \mapsto \int_{\Gamma(X)} \mathcal{H}^1(A \cap \gamma) \, \mathrm{d}\eta(\gamma).$$

Alberti representations $\eta_1, \dots, \eta_n$ are *independent* if there exists a Lipschitz map $\varphi : X \to \mathbb{R}^n$ such that for $(\eta_1, \dots, \eta_n)$-almost any tuple of curves $(\gamma_1, \dots, \gamma_n) \in \Gamma(X)^n$, their compositions $\varphi \circ \gamma_i$ travel within linearly independent cones.

(everything modulo countable decompositions)

Example

1. $\mathcal{H}^n$ on $\mathbb{R}^n$ has n independent Alberti representations.
2. A Lipschitz image of $\mathbb{R}^n$ has n independent Alberti representations because Lipschitz maps can be decomposed into bi-Lipschitz maps, thus inverted, and their inversions can be extended.
3. An n-rectifiable set has n independent Alberti representations.

Theorem (Bate-Li, 2014)

A set with n independent Alberti representations has positive lower density almost everywhere is n-rectifiable.

Theorem (Bate-W., 2025-26)

A set with n independent Alberti representations has positive lower density almost everywhere.

Theorem (Bate-W., 2025-26)

A set with n independent Alberti representations is n-rectifiable.

Tool: quantitative regularity

Theorem (Bate-W., 2025-26, extracted from DePhilippis-Rindler, with significant contributions from Tuomas Orponen)

Let $\mathbf{T}$ *be an* $\mathbb{R}^{n\times n}$*-valued and* ν *be a nonnegative finite measure on* $B(0,1)\subset\mathbb{R}^n$*. Then for any* $1\le p<\frac{n}{n-1}$ *we can decompose* $\nu=g+b$ *with*

$$\|g\|_p \lesssim_p \|\nu\|_1 + \|\operatorname{div}\mathbf{T}\|_1,$$

$$\|b\|_1 \lesssim_p (\|\nu\|_1 + \|\operatorname{div}\mathbf{T}\|_1)^{\frac{1}{p}} \|\mathrm{Id}\nu - \mathbf{T}\|_1^{\frac{1}{p'}}.$$

Corollary

Under the above assumptions, if $\|\operatorname{div}\mathbf{T}\|_1 \lesssim \|\nu\|_1$ *and* $\|\mathrm{Id}\nu - \mathbf{T}\|_1 \ll \|\nu\|_1$ *then* ν *satisfies a reverse Hölder inequality up to a small* L^1*-error. In particular,* $\mathrm{supp}(\nu)\gtrsim 1$*.*

Proof idea of lower density in metric space

1. Take a point $x \in E \subset X$.
2. Zoom in until (hopefully) the assumptions to apply quantitative regularity result are satisfied.
3. Apply quantitative regularity result with $\nu = \varphi_{\#}(\mathcal{H}^n\restriction_{E\cap B(x,r)})$ and $\mathbf{T}$ being the $\varphi_{\#}$-pushforward of the Alberti representations. This yields lower density on $\mathbb{R}^n$ and thus on X.

Obstacles:

- Item 2 does not really work because we do not have the Lebesgue density theorem on a metric space.
- Item 3 does not really work because the cutoff might cause huge divergence.
  And we anyways only have curve fragments which can already have huge divergence to start with.

Proof sketch of lower density in metric space

1. Filter so that $\mathcal{H}^n\restriction_E$ becomes L^1-close to its n Alberti representations.
2. Extend curve fragments to full curves around density points on curves, according to the considered scale. Introduced error remains locally small for $\mathcal{H}^n\restriction_E$-a.e. $x \in X$ at small scales.
3. By positive upper density we find lower density bound at arbitrarily low scale. That means at this scale the above errors are also relatively small.
4. If $\mathcal{H}^n\restriction_E$ is not doubling at that scale then we have lower density bound also at the next higher scale. ✓
5. If $\mathcal{H}^n\restriction_E$ is doubling at that scale then for the Lipschitz tent $\psi_{B(x,r)}(y) = 2 - \mathrm{d}(y,x)/r$ the cutoff $\nu = \varphi_\#(\psi_{B(x,r)}\mathcal{H}^n\restriction_E)$ is ensured to have bounded divergence. Thus, we may apply quantitative regularity result and obtain lower density bound also at the next higher scale. ✓

Kakeya conjecture

Conjecture (Kakeya conjecture)

Let $E \subset \mathbb{R}^n$ such that E contains a unit line segment in every direction. Then E has Hausdorff dimension n.

Roughly: On how little space can we concentrate using translations the set of all unit line segments from $B(0, 1/2)$?
Known: On a set of Lebesgue measure zero.

Multilinear Kakeya

Theorem (Guth, 2010, Acta Mathematica)

For $i = 1, \ldots, n$ and j let T_i^j be a straight tube in $\mathbb{R}^n$ that approximately points in direction e_i and denote by r_i^j its radius. Then

$$\Big\| \Big(\prod_{i=1}^n \sum_j a_i^j 1_{T_i^j} \Big)^{\frac{1}{n}} \Big\|_{\frac{n}{n-1}} \lesssim \Big(\prod_{i=1}^n \sum_j a_i^j (r_i^j)^{n-1} \Big)^{\frac{1}{n}}.$$

The previous inequality is scaling invariant and thus equivalent to

$$\Big\| \Big(\prod_{i=1}^n \int \mathcal{H}^1 \restriction_\gamma \mathrm{d}\eta_i(\gamma) \Big)^{\frac{1}{n}} \Big\|_{L^{\frac{n}{n-1}}(B(0,1))} \lesssim \Big\| \sum_{i=1}^n \int \mathcal{H}^1 \restriction_\gamma \mathrm{d}\eta_i(\gamma) \Big\|_{L^1(B(0,1))}$$

with η_i supported on straight lines that approximately point in direction e_i after rescaling.

Tool: quantitative regularity

Theorem (Bate-W., 2025-26, extracted from DePhilippis-Rindler, with significant contributions from Tuomas Orponen)

Let $\mathbf{T}$ *be an* $\mathbb{R}^{n\times n}$*-valued and* ν *be a nonnegative finite measure on* $B(0,1) \subset \mathbb{R}^n$*. Then for any* $1 \leq p < \frac{n}{n-1}$ *we can decompose* $\nu = g + b$ *with*

$$\|g\|_p \lesssim_p \|\nu\|_1 + \|\operatorname{div}\mathbf{T}\|_1,$$

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Corollary

Under the above assumptions, if $\|\operatorname{div}\mathbf{T}\|_1 \lesssim \|\nu\|_1$ *and* $\|\mathrm{Id}\nu - \mathbf{T}\|_1 \ll \|\nu\|_1$ *then* ν *satisfies a reverse Hölder inequality up to a small* L^1*-error. In particular,* $\mathrm{supp}(\nu) \gtrsim 1$*.*

Strange multilinear Kakeya

Corollary

Under the above assumptions, if $\|\operatorname{div} \mathbf{T}\|_1 \lesssim \|\mathbf{T}\|_1$ *and* $\|\mathrm{Id}|\mathbf{T}| - \mathbf{T}\|_1 \ll \|\mathbf{T}\|_1$ *then* $|\mathbf{T}|$ *satisfies a reverse Hölder inequality up to a small* L^1*-error.*

The constraint

$$\|\mathrm{Id}|\mathbf{T}| - \mathbf{T}\|_1 \ll \|\mathbf{T}\|_1 \tag{1}$$

implies that in most points $x \in B(0,1)$ the columns $\mathbf{T}_i$ of $\mathbf{T}$ have similar absolute value $|\mathbf{T}_1(x)| \sim ... \sim |\mathbf{T}_n(x)|$, in particular their arithmetic mean is comparable to their geometric mean. That means quantitative DePhilippis-Rindler with straight lines can be seen as the multilinear Kakeya inequality under the constraint (1).

Lipschitz multilinear Kakeya

- Quantitative DePhilippis-Rindler is true not only for straight lines but also for Lipschitz curves.
- Quantitative DePhilippis-Rindler is only known for $1 \leq p < \frac{n}{n-1}$, while multilinear Kakeya is true also for $p = \frac{n}{n-1}$.
- Lipschitz multilinear Kakeya on the other hand is announced by Csörnyej and Jones to fail in a small interval around $\frac{n}{n-1}$, and to hold between $p = 1$ and some value near $\frac{n}{n-1}$.
- The exception is $n = 2$ (and $n = 1$) where LMK is straightforward to prove at $p = \frac{n}{n-1}$.
- A version of LMK with a lower bound on the diameter of the tubes however holds for all $1 \leq p \leq \frac{n}{n-1}$ due to Guth 2014.
- There are certain versions of LMK for C^1 and C^2 curves, some at and some above the endpoint $p = \frac{n}{n-1}$, see Carbery-Hänninen-Valdimarsson 2018 and Tao 2020.

Does quantitative DePhilippis-Rindler hold also for $p = \frac{n}{n-1}$?

Thank you.