Alberti representations, rectifiability, PDEs and multilinear Kakeya

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Conference on geometric measure theory and metric geometry

27.03.2026

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Definition (Alberti representation)

An *Alberti representation* of a finite measure μ on $\mathbb{R}^d$ is a finite measure η on the space of all Lipschitz curves $\Gamma(\mathbb{R}^d)$ on $\mathbb{R}^d$ such that

$$\mu \ll \int_{\Gamma(\mathbb{R}^n)} \mathcal{H}^1\restriction_\gamma \mathrm{d}\eta(\gamma) = A \mapsto \int_{\Gamma(\mathbb{R}^n)} \mathcal{H}^1(A \cap \gamma)\, \mathrm{d}\eta(\gamma).$$

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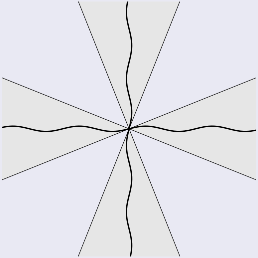
Alberti representations $\eta_1, \ldots, \eta_n$ are *independent* if for $(\eta_1, \ldots, \eta_n)$-almost any tuple of curves $(\gamma_1, \ldots, \gamma_n) \in \Gamma(\mathbb{R}^d)^n$, the γ_i travel within linearly independent cones.



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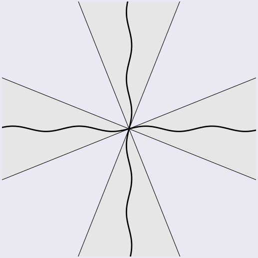

(everything modulo countable decompositions)

Euclidean case

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Converse:

Theorem

A set with n independent Alberti representations is n-rectifiable.

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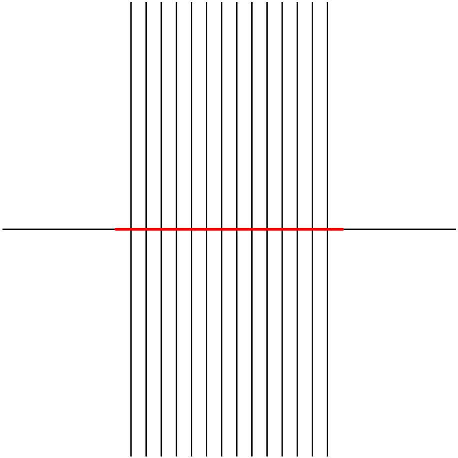

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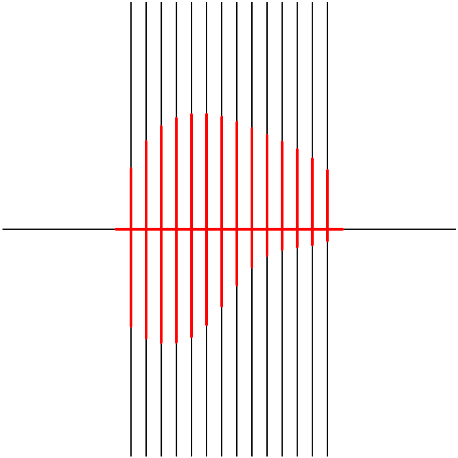

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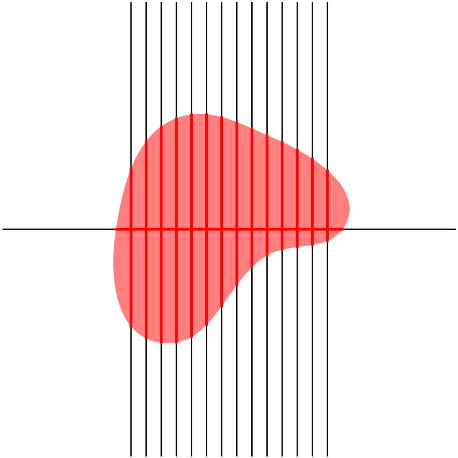

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Theorem (DePhilippis-Rindler, Ann. of Math. (2016), divergence case)

Let $\mathbf{T}$ *be an* $\mathbb{R}^{n\times n}$*-valued finite measure on* $\mathbb{R}^n$ *such that* $\operatorname{div}\mathbf{T}$ *is a finite measure. Then the restriction of* $\mathbf{T}$ *to those points, where its polar* $\mathbf{T}/|\mathbf{T}|$ *is an invertible matrix, is absolutely continuous with respect to Lebesgue measure.*

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This applies for example to a tuple of n independent Alberti representations since

$$\operatorname{div}\Big(\int \dot\gamma \mathcal{H}^1\restriction_\gamma \,\mathrm{d}\eta(\gamma)\Big) = \int \operatorname{div}(\dot\gamma\mathcal{H}^1\restriction_\gamma)\,\mathrm{d}\eta(\gamma) = 0.$$

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Theorem (Besicovitch projection theorem)

A set $E \subset \mathbb{R}^d$ is purely n-unrectifiable if and only if $\mathcal{H}^{d-n}$-almost every projection of E to an n-plane has $\mathcal{H}^n$-measure 0.

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Proof, that n independent Alberti representations imply rectifiability.

Assume $\mathcal{H}^n\restriction_E$ has independent Alberti representations $\eta_1, \ldots, \eta_n$. Then there exist n linearly independent directions $e_1, \ldots, e_n$ such that η_i is supported on curve running roughly in direction e_i. Let S be a small open neighborhood of the directions orthogonal to $e_1, \ldots, e_n$. Then $\mathcal{H}^{d-n}(S) > 0$. Let $e \in S$. Then $\pi_{e\#}\eta_1, \ldots, \pi_{e\#}\eta_n$ are independent Alberti representations of $\pi_{e\#}\mathcal{H}^n\restriction_E$. By DePhilippis-Rindler this implies $\pi_{e\#}\mathcal{H}^n\restriction_E$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^n$. Since $\mathcal{H}^n(E) > 0$ this means it has support with positive Lebesgue measure. This means $\mathcal{H}^n(\pi_e E) > 0$. Thus E is not purely n-unrectifiable. □

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Corollary

Under the above assumptions, if $\|\operatorname{div}\mathbf{T}\|_1 \lesssim \|\nu\|_1$ *and* $\|\mathrm{Id}\nu - \mathbf{T}\|_1 \ll \|\nu\|_1$ *then* ν *satisfies a reverse Hölder inequality up to a small* L^1*-error. In particular,* $\operatorname{supp}(\nu) \gtrsim 1$*.*

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  And we anyways only have curve fragments which can already have huge divergence to start with.

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Known: On a set of Lebesgue measure zero.

Multilinear Kakeya

Theorem (Guth, 2010, Acta Mathematica)

For $i = 1, ..., n$ and j let T_i^j be a straight tube in $\mathbb{R}^n$ that approximately points in direction e_i and denote by r_i^j its radius. Then

$$\Big\|\Big(\prod_{i=1}^{n}\sum_{j} a_i^j 1_{T_i^j}\Big)^{\frac{1}{n}}\Big\|_{\frac{n}{n-1}} \lesssim \Big(\prod_{i=1}^{n}\sum_{j} a_i^j (r_i^j)^{n-1}\Big)^{\frac{1}{n}}.$$

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Corollary

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Under the above assumptions, if $\|\operatorname{div}\mathbf{T}\|_1 \lesssim \|\mathbf{T}\|_1$ *and* $\|\mathrm{Id}|\mathbf{T}| - \mathbf{T}\|_1 \ll \|\mathbf{T}\|_1$ *then* $|\mathbf{T}|$ *satisfies a reverse Hölder inequality up to a small* L^1*-error.*

The constraint

$$\|\mathrm{Id}|\mathbf{T}| - \mathbf{T}\|_1 \ll \|\mathbf{T}\|_1 \tag{1}$$

implies that in most points $x \in B(0,1)$ the columns $\mathbf{T}_i$ of $\mathbf{T}$ have similar absolute value $|\mathbf{T}_1(x)| \sim ... \sim |\mathbf{T}_n(x)|$, in particular their arithmetic mean is comparable to their geometric mean. That means quantitative DePhilippis-Rindler with straight lines can be seen as the multilinear Kakeya inequality under the constraint (1).

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Does quantitative DePhilippis-Rindler hold also for $p = \frac{n}{n-1}$?

Thank you.