Higher Dimensional Techniques for the Regularity of Maximal Functions

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Warwick junior analysis and probability seminar

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For $f : \mathbb{R}^n \to \mathbb{R}$ the centered Hardy-Littlewood maximal function is defined by

$$\mathrm{M}^{\mathrm{c}} f(x) = \sup_{r>0} f_{B(x,r)} \quad \text{with} \quad f_{B(x,r)} = \frac{1}{\mathcal{L}(B(x,r))} \int_{B(x,r)} |f|.$$

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Theorem (Hardy-Littlewood maximal function theorem)

$$\|\mathrm{M}^{\mathrm{c}} f\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|f\|_{L^p(\mathbb{R}^n)}$$

if and only if $p > 1$.

$$\|\mathrm{M}^{\mathrm{c}} f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim_n \|f\|_{L^1(\mathbb{R}^n)}$$

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Theorem (Vitali covering lemma)

Let $\mathcal{B}$ be a (bounded) set of balls. Then it has a subset $\mathcal{S} \subset \mathcal{B}$ of disjoint balls with

$$\bigcup \mathcal{B} \subset \bigcup_{B \in \mathcal{S}} 5B.$$

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$1 < p < \infty$ by interpolation ✓.

Theorem (Juha Kinnunen (1997))

For $p > 1$ we have

$$\|\nabla \mathrm{M}^{\mathrm{c}} f\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|\nabla f\|_{L^p(\mathbb{R}^n)}$$

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Proof: For $e \in \mathbb{R}^n$ by the sublinearity of M^{c}

$$\begin{aligned}
\partial_e \mathrm{M}^{\mathrm{c}} f(x) &\sim \frac{\mathrm{M}^{\mathrm{c}} f(x+he) - \mathrm{M}^{\mathrm{c}} f(x)}{h} \\
&\leq \frac{\mathrm{M}^{\mathrm{c}}(f(\cdot + he) - f)(x)}{h} \\
&= \mathrm{M}^{\mathrm{c}}\Big(\frac{f(\cdot + he) - f}{h}\Big)(x) \sim \mathrm{M}^{\mathrm{c}}(\partial_e f)(x)
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By the Hardy-Littlewood maximal function theorem for $p > 1$

$$\|\nabla \mathrm{M}^{\mathrm{c}} f\|_{L^p(\mathbb{R}^n)} \leq \|\mathrm{M}^{\mathrm{c}}(|\nabla f|)\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|\nabla f\|_{L^p(\mathbb{R}^n)}$$

Question (Hajłasz and Onninen 2004)

Is it true that

$$\|\nabla \mathrm{M}^{\mathrm{c}} f\|_{L^1(\mathbb{R}^n)} \lesssim_n \|\nabla f\|_{L^1(\mathbb{R}^n)}?$$

Introduction: Background

Question (Hajłasz and Onninen 2004)

Is it true that

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Uncentered Hardy-Littlewood maximal function

$$\widetilde{\mathrm{M}} f(x) = \sup_{B \ni x} f_B.$$

Endpoint question by Hałjasz and Onninen is interesting for $\widetilde{\mathrm{M}}$ and other maximal operators.

Theorem (Tanaka 2002, Aldaz and Pérez Lázaro 2007)

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1. For almost all $x \in \mathbb{R}^d$: $\widetilde{\mathrm{M}} f(x) \geq f(x)$

Introduction: In one dimension

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$$\|\nabla f\|_1 = \sup_{x_1 < x_2 < \dots} \sum_i |f(x_{i+1}) - f(x_i)| = \operatorname{var} f.$$

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3. and $\widetilde{\mathrm{M}} f(x) = f(x)$ at a strict local maximum of $\mathrm{M} f$.

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$$\sum_i |f(x_{i+1}) - f(x_i)| = \sum_i \Big| \int_{x_i}^{x_{i+1}} f' \Big| \leq \sum_i \int_{x_i}^{x_{i+1}} |f'| \leq \|f'\|_{L^1(\mathbb{R})}.$$

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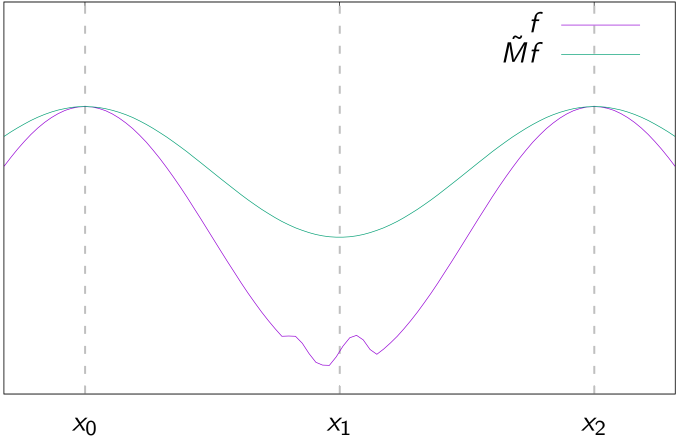
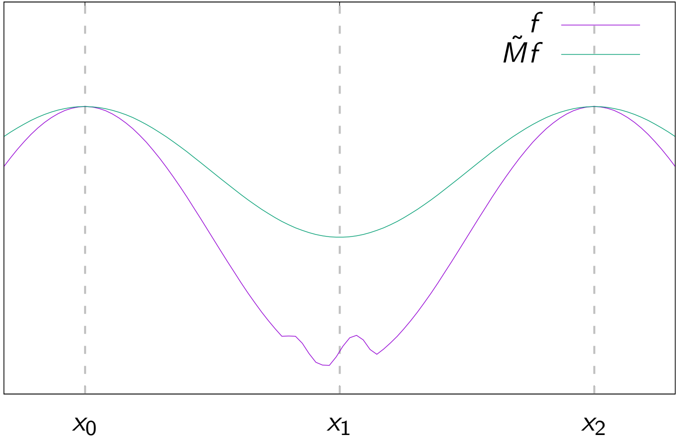
$$\begin{aligned}\|f'\|_{L^1(\mathbb{R})} &= \sum_i \int_{x_i}^{x_{i+1}} |f'| = \sum_i (-1)^i \int_{x_i}^{x_{i+1}} f' \\ &= \sum_i (-1)^i \big(f(x_{i+1}) - f(x_i)\big) = \sum_i |f(x_{i+1}) - f(x_i)|\end{aligned}$$



f
$\tilde{M}f$
x_0
x_1
x_2



$$\begin{aligned}\mathrm{var}_{[x_0,x_2]}\,\widetilde{\mathrm{M}}f &= |\widetilde{\mathrm{M}}f(x_1)-\widetilde{\mathrm{M}}f(x_0)| + |\widetilde{\mathrm{M}}f(x_2)-\widetilde{\mathrm{M}}f(x_1)| \\ &\le |f(x_1)-f(x_0)| + |f(x_2)-f(x_1)| \\ &\le \mathrm{var}_{[x_0,x_2]}\,f\end{aligned}$$

Introduction: Past progress

$n = 1$	[Tanaka 2002, Aldaz +Pérez Lázaro 2007]
block decreasing f	[Aldaz+Pérez Lázaro 2009]
centered M^{c}, $n = 1$	[Kurka 2015]
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more related bounds, bounds on other maximal operators, such as fractional, local,. . . ,

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$$\|f_n - f\|_{W^{1,p}(\mathbb{R}^n)} \to 0 \quad \Longrightarrow \quad \|\nabla \mathrm{M}f_n - \nabla \mathrm{M}f\|_{L^p(\mathbb{R}^n)} \to 0.$$

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For $p = 1$ continuity is known in the same cases as the gradient bound.

We prove the endpoint regularity bound for the maximal function for

1. uncentered maximal function of characteristic f
2. dyadic maximal operator
3. fractional maximal operator (uncentered & centered + continuity)
4. cube maximal operator

$$\|\nabla f\|_{L^1(\mathbb{R}^1)} = \sum_i |f(x_{i+1}) - f(x_i)|$$

Introduction: Coarea formula

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&= \sum_i \int_{\mathbb{R}} \#[x_i, x_{i+1}] \cap \partial\{x \in \mathbb{R}^n : f(x) > \lambda\} \, \mathrm{d}\lambda \\
&= \int_{\mathbb{R}} \mathcal{H}^0(\partial\{x \in \mathbb{R} : f(x) > \lambda\}) \, \mathrm{d}\lambda
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Coarea formula

$$\|\nabla f\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}} \mathcal{H}^{n-1}(\partial\{x \in \mathbb{R}^n : f(x) > \lambda\})\,\mathrm{d}\lambda$$

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Compare with layer cake formula/Cavalieri's principle

$$\|f\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}} \mathcal{L}(\{x \in \mathbb{R}^n : f(x) > \lambda\})\,\mathrm{d}\lambda$$

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Superlevel sets

$$\{x \in \mathbb{R}^n : \mathrm{M}f(x) > \lambda\} = \bigcup \{B : f_B > \lambda\}$$

for *uncentered* maximal operators.

Introduction: Reformulation and decomposition

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Decomposition of the boundary

Denote

$$\mathcal{B}_\lambda^{<} = \{B : f_B > \lambda,\ \mathcal{L}(B \cap \{f > \lambda\}) < 2^{-n-1}\mathcal{L}(B)\}$$

and $\mathcal{B}_\lambda^{\geq}$ accordingly.

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$$\min\{\mathcal{L}(Q \cap E), \mathcal{L}(Q \setminus E)\}^{n-1} \lesssim_n \mathcal{H}^{n-1}(Q \cap \partial E)^n.$$

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3. **Besicovitch covering for boundary**

Introduction: Tools

Decomposition of the boundary

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2. **Vitali covering** and similar: general balls $\to$ separated balls
3. **Besicovitch covering for boundary**
4. **superlevelset estimate**: $f < 0$ on most of $B \Rightarrow$ most mass of f lies far above f_B

Proof: Reformulation and decomposition

We have

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$$\{\mathrm{M}f > \lambda\} = \bigcup \mathcal{B}_\lambda^< \cup \bigcup \mathcal{B}_\lambda^\geq.$$

Since $\{f > \lambda\} \subset \{\mathrm{M}f > \lambda\}$ we have

$$\partial\{\mathrm{M}f > \lambda\} \subset (\partial\{\mathrm{M}f > \lambda\} \setminus \overline{\{f > \lambda\}}) \cup \partial\{f > \lambda\}.$$

Proof: Reformulation and decomposition

We have

$$\{\mathrm{M}f > \lambda\} = \bigcup \mathcal{B}_\lambda^< \cup \bigcup \mathcal{B}_\lambda^\geq .$$

Since $\{f > \lambda\} \subset \{\mathrm{M}f > \lambda\}$ we have

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We conclude

Decomposition

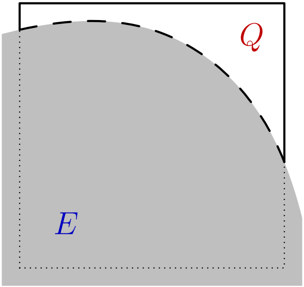
$$\int_{\mathbb{R}^d} |\nabla \mathrm{M}f| \leq \int_0^\infty \mathcal{H}^{n-1}\Big(\partial \bigcup \mathcal{B}_\lambda^\geq \setminus \overline{\{f > \lambda\}}\Big)\,\mathrm{d}\lambda + \int_0^\infty \mathcal{H}^{n-1}\Big(\partial \bigcup \mathcal{B}_\lambda^<\Big)\,\mathrm{d}\lambda + \int_{\mathbb{R}^d} |\nabla f|$$

Proof: High density case $\mathcal{B}_\lambda^{\geq}$

Proposition

For Q, E with $\mathcal{L}(Q \cap E) \geq 2^{-n-1}\mathcal{L}(Q)$ we have

$$\mathcal{H}^{n-1}(\partial Q \setminus \overline{E}) \lesssim_n \mathcal{H}^{n-1}(Q \cap \partial E)$$



dyadic maximal operator

$$\mathrm{M}^{\mathrm{d}} f(x) = \sup_{\text{dyadic } Q,\ Q \ni x} f_Q.$$

dyadic maximal operator

$$\mathrm{M}^{\mathrm{d}} f(x) = \sup_{\text{dyadic } Q,\ Q \ni x} f_Q.$$

$$\begin{aligned}
\mathcal{H}^{n-1}(\partial \bigcup \mathcal{Q}_\lambda^{\geq} \setminus \overline{\{f > \lambda\}}) &\leq \sum_{Q \in \mathcal{Q}_\lambda^{\geq}} \mathcal{H}^{n-1}(\partial Q \setminus \overline{\{f > \lambda\}}) \\
&\lesssim_n \sum_{Q \in \mathcal{Q}_\lambda^{\geq}} \mathcal{H}^{n-1}(Q \cap \partial\{f > \lambda\}) \\
&\leq \mathcal{H}^{n-1}(\partial\{f > \lambda\})
\end{aligned}$$

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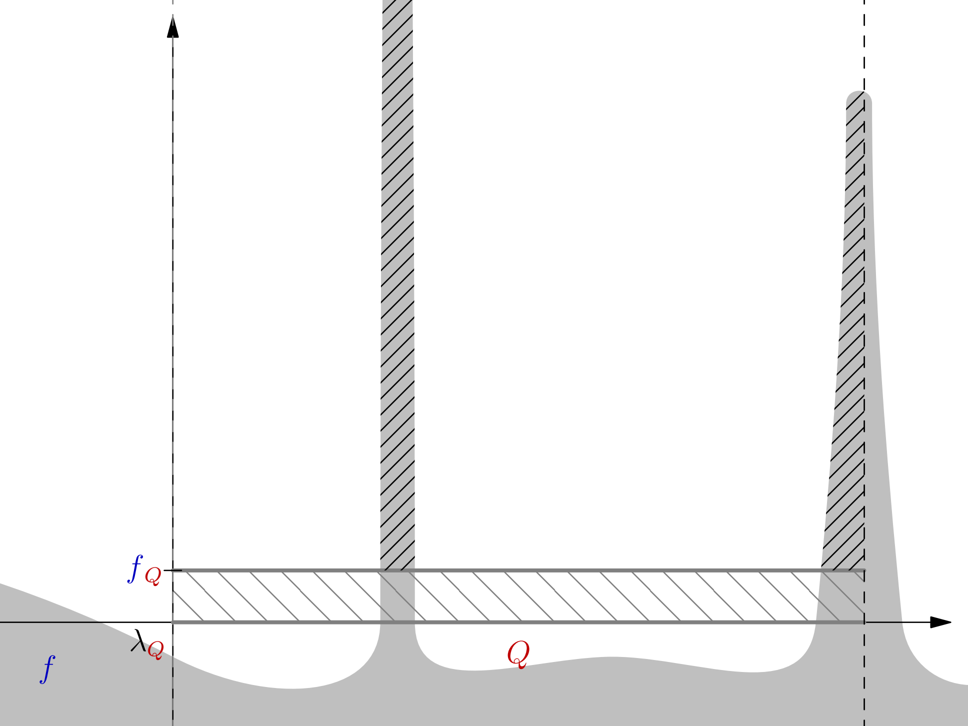
$$\begin{aligned}\mathcal{H}^{n-1}(\partial \bigcup \mathcal{Q}_\lambda^{\geq} \setminus \overline{\{f > \lambda\}}) &\leq \sum_{Q \in \mathcal{Q}_\lambda^{\geq}} \mathcal{H}^{n-1}(\partial Q \setminus \overline{\{f > \lambda\}}) \\ &\lesssim_n \sum_{Q \in \mathcal{Q}_\lambda^{\geq}} \mathcal{H}^{n-1}(Q \cap \partial\{f > \lambda\}) \\ &\leq \mathcal{H}^{n-1}(\partial\{f > \lambda\})\end{aligned}$$

Proposition

For a set $\mathcal{B}$ of balls B with $\mathcal{L}(B \cap E) \geq 2^{-n-1}\mathcal{L}(B)$ we have

$$\mathcal{H}^{n-1}\Big(\partial \bigcup \mathcal{B} \setminus \overline{E}\Big) \lesssim_n \mathcal{H}^{n-1}\Big(\bigcup \mathcal{B} \cap \partial E\Big).$$



f_Q
λ_Q
f
Q

Proof: Low density case $\mathcal{B}_\lambda^<$, dyadic

$$\int_{\mathbb{R}} \mathcal{H}^{n-1}(\partial \bigcup \mathcal{Q}_\lambda^<)\,\mathrm{d}\lambda \le \sum_{Q \text{ dyadic}} (f_Q - \lambda_Q)\mathcal{H}^{n-1}(\partial Q)$$

with

$$\mathcal{L}(Q \cap \{f > \lambda_Q\}) = 2^{-n-1}\mathcal{L}(Q)$$

Proposition

$$(f_Q - \lambda_Q)\mathcal{L}(Q) \lesssim_n \int_{f_Q}^{\infty} \mathcal{L}(Q \cap \{f > \lambda\}) \qquad \mathrm{d}\lambda$$

Proof: Low density case $\mathcal{B}_\lambda^<$, dyadic

$$\int_{\mathbb{R}} \mathcal{H}^{n-1}(\partial \bigcup \mathcal{Q}_\lambda^<) \, \mathrm{d}\lambda \le \sum_{Q \text{ dyadic}} (f_Q - \lambda_Q) \mathcal{H}^{n-1}(\partial Q)$$

with

$$\mathcal{L}(Q \cap \{f > \lambda_Q\}) = 2^{-n-1} \mathcal{L}(Q)$$

Proposition

$$(f_Q - \lambda_Q)\mathcal{L}(Q) \lesssim_n \int_{f_Q}^{\infty} \sum_{P \subsetneq Q : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\}) \, \mathrm{d}\lambda$$

Proof: Low density case $\mathcal{B}_\lambda^<$, dyadic

$$\int_{\mathbb{R}} \mathcal{H}^{n-1}(\partial \bigcup \mathcal{Q}_\lambda^<)\,\mathrm{d}\lambda \leq \sum_{Q \text{ dyadic}} (f_Q - \lambda_Q)\mathcal{H}^{n-1}(\partial Q)$$

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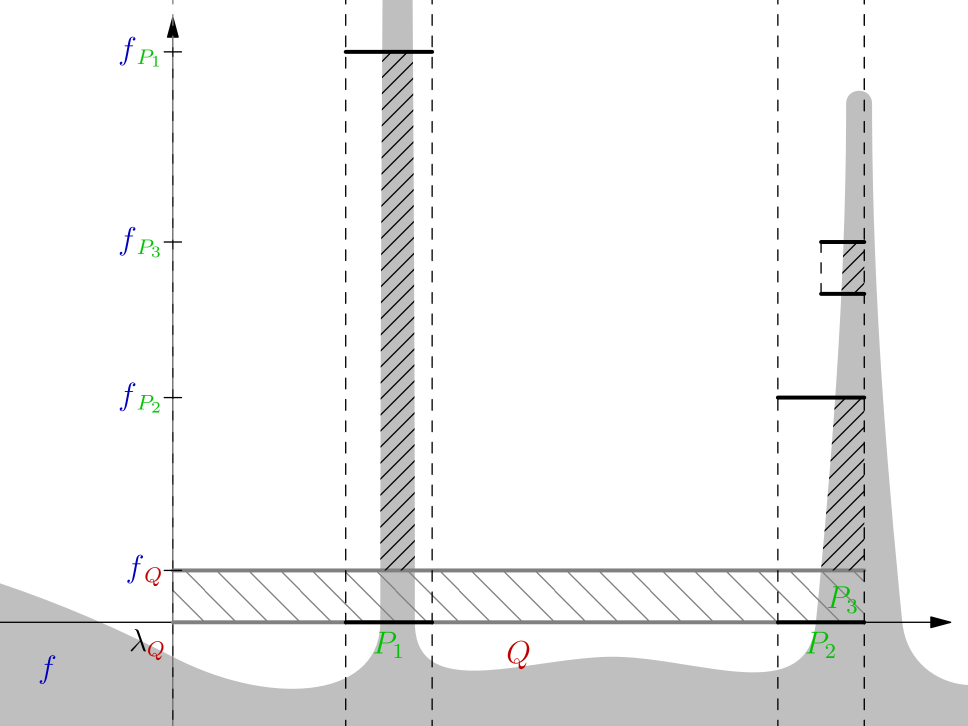
Proposition

$$(f_Q - \lambda_Q)\mathcal{L}(Q) \lesssim_n \int_{f_Q}^{\infty} \sum_{P \subsetneq Q : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\})\,\mathrm{d}\lambda$$

where P is maximal above $\bar{\lambda}_P$ and

$$\mathcal{L}(P \cap \{f > \bar{\lambda}_P\}) = 2^{-1}\mathcal{L}(P)$$



f_{P_1}
f_{P_3}
f_{P_2}
f_Q
λ_Q
f
P_1
Q
P_2
P_3

Proof: Low density case $\mathcal{B}_\lambda^<$, dyadic

Combining, we obtain

$$\int_{\mathbb{R}} \mathcal{H}^{n-1}(\partial \bigcup \mathcal{Q}_\lambda^<) \, \mathrm{d}\lambda$$

$$\lesssim_n \int_{\mathbb{R}} \sum_{Q \text{ dyadic}} \sum_{P \subsetneq Q : \bar{\lambda}_P < \lambda < f_P} \frac{\mathcal{L}(P \cap \{f > \lambda\})}{l(Q)} \, \mathrm{d}\lambda$$

Proof: Low density case $\mathcal{B}_\lambda^<$, dyadic

Combining, we obtain

$$\int_{\mathbb{R}} \mathcal{H}^{n-1}(\partial \bigcup \mathcal{Q}_\lambda^<) \, \mathrm{d}\lambda \lesssim_n \int_{\mathbb{R}} \sum_{Q \text{ dyadic}} \sum_{P \subsetneq Q : \bar{\lambda}_P < \lambda < f_P} \frac{\mathcal{L}(P \cap \{f > \lambda\})}{l(Q)} \, \mathrm{d}\lambda$$

1. change the order of summation
2. convergence of the geometric sum
3. apply the relative isoperimetric inequality to P.
4. coarea formula to recover $\|\nabla f\|_1$

Proof: Low density case $\mathcal{B}_\lambda^<$, general cubes

cube maximal function

$$\mathrm{M}f(x) = \sup_{\text{cube } Q,\ Q \ni x} f_Q .$$

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We reduce to almost dyadic cubes, using

Proof: Low density case $\mathcal{B}_\lambda^<$, general cubes

cube maximal function

$$\mathrm{M}f(x) = \sup_{\text{cube } Q,\ Q \ni x} f_Q.$$

We reduce to almost dyadic cubes, using

Proposition (Vitali/Besicovitch for perimeter)

For any (finite) set of cubes $\mathcal{Q}$ there is a subset $\mathcal{S} \subset \mathcal{Q}$ of disjoint cubes such that

$$\mathcal{H}^{n-1}\Big(\partial \bigcup \mathcal{Q}\Big) \lesssim_n \sum_{S \in \mathcal{S}} \mathcal{H}^{n-1}(\partial S).$$

Uncentered HL $\widetilde{\mathrm{M}}f$ (balls)?

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- All arguments work

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- All arguments work
- except low density bound $(f_B - \lambda_B)\mathcal{L}(B) \lesssim_n$?

Thank you