Recent results on the regularity of maximal functions

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Background

For $f : \mathbb{R}^n \to \mathbb{R}$ the centered Hardy-Littlewood maximal function is defined by

$$\mathrm{M}^{\mathrm{c}} f(x) = \sup_{r>0} f_{B(x,r)} \quad \text{with} \qquad f_{B(x,r)} = \frac{1}{\mathcal{L}(B(x,r))} \int_{B(x,r)} |f|.$$

Theorem (Hardy-Littlewood maximal function theorem)

$$\|\mathrm{M}^{\mathrm{c}} f\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|f\|_{L^p(\mathbb{R}^n)}$$

if and only if $p > 1$.

$$\|\mathrm{M}^{\mathrm{c}} f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim_{n} \|f\|_{L^1(\mathbb{R}^n)}$$

Background

Theorem (Juha Kinnunen (1997))

For $p > 1$ we have

$$\|\nabla \mathrm{M}^{\mathrm{c}} f\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|\nabla f\|_{L^p(\mathbb{R}^n)}$$

Proof: For $e \in \mathbb{R}^n$ by the sublinearity of M^{c}

$$\begin{aligned}\partial_e \mathrm{M}^{\mathrm{c}} f(x) &\sim \frac{\mathrm{M}^{\mathrm{c}} f(x+he) - \mathrm{M}^{\mathrm{c}} f(x)}{h} \\ &\leq \frac{\mathrm{M}^{\mathrm{c}}(f(\cdot + he) - f)(x)}{h} \\ &= \mathrm{M}^{\mathrm{c}}\Big(\frac{f(\cdot + he) - f}{h}\Big)(x) \sim \mathrm{M}^{\mathrm{c}}(\partial_e f)(x)\end{aligned}$$

By the Hardy-Littlewood maximal function theorem for $p > 1$

$$\|\nabla \mathrm{M}^{\mathrm{c}} f\|_{L^p(\mathbb{R}^n)} \leq \|\mathrm{M}^{\mathrm{c}}(|\nabla f|)\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|\nabla f\|_{L^p(\mathbb{R}^n)}$$

Background

Question (Hajłasz and Onninen 2004)

Is it true that

$$\|\nabla \mathrm{M}^{\mathrm{c}} f\|_{L^1(\mathbb{R}^n)} \lesssim_n \|\nabla f\|_{L^1(\mathbb{R}^n)}?$$

Uncentered Hardy-Littlewood maximal function

$$\widetilde{\mathrm{M}} f(x) = \sup_{B \ni x} f_B.$$

Endpoint question by Hałjasz and Onninen is interesting for $\widetilde{\mathrm{M}}$ and other maximal operators.

In one dimension

Theorem (Tanaka 2002, Aldaz and Pérez Lázaro 2007)

For $f : \mathbb{R} \to \mathbb{R}$ *we have*

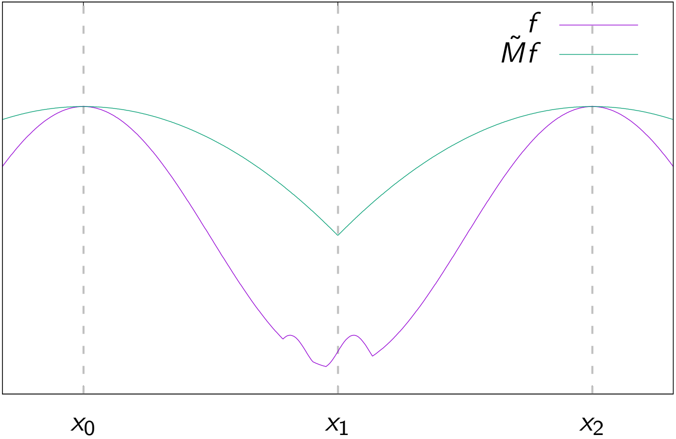
$$\|\nabla \widetilde{\mathrm{M}} f\|_1 \leq \|\nabla f\|_1$$

Proof:

- In one dimension

$$\|\nabla f\|_1 = \sup_{x_1 < x_2 < \dots} \sum_i |f(x_{i+1}) - f(x_i)| = \operatorname{var} f.$$

- For almost all $x \in \mathbb{R}^d$: $\widetilde{\mathrm{M}} f(x) \geq f(x)$
- and $\widetilde{\mathrm{M}} f(x) = f(x)$ at a strict local maximum of $\mathrm{M} f$.



$$
\begin{aligned}
\mathrm{var}_{[x_0,x_2]}\, \widetilde{\mathrm{M}} f &= |\widetilde{\mathrm{M}} f(x_1) - \widetilde{\mathrm{M}} f(x_0)| + |\widetilde{\mathrm{M}} f(x_2) - \widetilde{\mathrm{M}} f(x_1)| \\
&\le |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| \\
&\le \mathrm{var}_{[x_0,x_2]}\, f
\end{aligned}
$$

In one dimension: centered

Theorem (Kurka 2015)

For $f : \mathbb{R} \to \mathbb{R}$ *we have*

$$\|\nabla \mathrm{M}^{\mathrm{c}} f\|_1 \leq C \|\nabla f\|_1.$$

$C = 1$? Yes, for $E \subset \mathbb{R}$ and $f = 1_E$ (Bilz and W. 2022).

Higher dimensions

Theorem (Luiro 2018)

For $f : \mathbb{R}^n \to \mathbb{R}$ *radial we have*

$$\|\nabla \widetilde{\mathrm{M}} f\|_1 \leq C \|\nabla f\|_1.$$

Theorem (Aldaz+Pérez Lázaro 2009)

For $f : \mathbb{R}^n \to \mathbb{R}$ *block-decreasing we have*

$$\|\nabla \widetilde{\mathrm{M}} f\|_1 \leq C \|\nabla f\|_1.$$

The fractional maximal function

For $0 < \alpha < n$ the centered fractional Hardy-Littlewood maximal function is

$$\mathrm{M}_\alpha^{\mathrm{c}} f(x) = \sup_{r>0} r^\alpha f_{B(x,r)}.$$

Corresponding Hardy-Littlewood theorem

$$\|\mathrm{M}_\alpha f\|_{L^{p_\alpha}(\mathbb{R}^n)} \lesssim_{n,\alpha,p} \|f\|_{L^p(\mathbb{R}^n)}$$

with $p_\alpha = \frac{pn}{n-\alpha p} > p$ if and only if $p > 1$. Corresponding regularity bound

$$\|\nabla \mathrm{M}_\alpha f\|_{L^{p_\alpha}(\mathbb{R}^n)} \lesssim_{n,\alpha,p} \|\nabla f\|_{L^p(\mathbb{R}^n)},$$

proven for $p > 1$.

The fractional maximal function

Theorem (Kinnunen and Saksman 2003)

For $\alpha \geq 1$

$$|\nabla \mathrm{M}^{\mathrm{c}}_{\alpha} f(x)| \lesssim_n |\mathrm{M}^{\mathrm{c}}_{\alpha-1} f(x)|.$$

Corollary (Carneiro and Madrid 2016)

For $\alpha \geq 1$ we have $1_\alpha = \frac{n}{n-\alpha} = \left(\frac{n}{n-1}\right)_{\alpha-1}$ and $\frac{n}{n-1} > 1$ and therefore

$$\|\nabla \mathrm{M}^{\mathrm{c}}_{\alpha} f\|_{L^{1_\alpha}(\mathbb{R}^n)} \lesssim_n \|\mathrm{M}^{\mathrm{c}}_{\alpha-1} f\|_{L^{1_\alpha}(\mathbb{R}^n)} \lesssim_n \|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)}$$
$$\lesssim_n \|\nabla f\|_{L^1(\mathbb{R}^n)}.$$

Endpoint bound is known for all $\alpha > 0$ for $n = 1$, radial f, lacunary and same for M^{c} due to [Beltran, Madrid, Luiro, Ramos, Saari 2016-2019].

Higher dimensions

Theorem (W. 2022)

For $E \subset \mathbb{R}^n$ we have

$$\|\nabla \widetilde{\mathrm{M}}(1_E)\|_1 \leq C \|\nabla 1_E\|_1.$$

Theorem (W. 2023)

For $f : \mathbb{R}^n \to \mathbb{R}$ we have

$$\|\nabla \mathrm{M}^{\mathrm{d}} f\|_1 \leq C \|\nabla f\|_1$$

for the dyadic maximal function

$$\mathrm{M}^{\mathrm{d}} f(x) = \sup_{\text{dyadic cube } Q,\ Q \ni x} f_Q.$$

Higher dimensions

Theorem (W. 2024)

Combining tools from both leads to the same bound for cube maximal operator given by

$$\mathrm{M}^{\mathrm{d}} f(x) = \sup_{\textit{cube } Q,\ Q \ni x} f_Q .$$

Proof works for more general sets with a tiling property, but not for balls and certainly not for centered M^{c}.

Theorem (W. 2022)

The arguments for the dyadic maximal operator can be used also for the fractional maximal operators $\widetilde{\mathrm{M}}_\alpha, \mathrm{M}^{\mathrm{c}}_\alpha$ *for all* $\alpha > 0$*.*

Proof ingredients

Coarea formula

$$\|\nabla f\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}} \mathcal{H}^{n-1}(\partial\{x \in \mathbb{R}^n : f(x) > \lambda\})\,\mathrm{d}\lambda$$

Superlevel sets

$$\{x \in \mathbb{R}^n : \mathrm{M}f(x) > \lambda\} = \bigcup \{B : f_B > \lambda\}$$

for *uncentered* maximal operators.

Relative isoperimetric inequality

$$\min\big\{\mathcal{L}(Q \cap E), \mathcal{L}(Q \setminus E)\big\}^{n-1} \lesssim_n \mathcal{H}^{n-1}(Q \cap \partial E)^n.$$

Example: dyadic, characteristic, low level

Consider $f = 1_E$ and $\mathrm{M} = \mathrm{M}^{\mathrm{d}}$ and let $0 < \lambda < 2^{-n-1}$. Let Q be a maximal dyadic cube with $f_Q > \lambda$. Then

$$\mathcal{L}(E \cap Q)/\mathcal{L}(Q) = f_Q \in (\lambda, 2^n \lambda] \subset (0, 1/2],$$

$$\mathcal{H}^{n-1}(\partial Q) \sim (\mathcal{L}(E \cap Q)/\lambda)^{\frac{n-1}{n}} \lesssim \lambda^{-\frac{n-1}{n}} \mathcal{H}^{n-1}(Q \cap \partial E),$$

$$\begin{aligned}
\mathcal{H}^{n-1}(\partial\{\mathrm{M}^{\mathrm{d}} f > \lambda\}) &= \mathcal{H}^{n-1}\Big(\partial \bigcup \{\text{maximal } Q : f_Q > \lambda\}\Big) \\
&\le \sum_{\text{maximal } Q : f_Q > \lambda} \mathcal{H}^{n-1}(\partial Q) \\
&\lesssim \lambda^{-\frac{n}{n-1}} \sum_{\text{maximal } Q : f_Q > \lambda} \mathcal{H}^{n-1}(Q \cap \partial E) \\
&\le \lambda^{-\frac{n}{n-1}} \mathcal{H}^{n-1}(\partial E) = \lambda^{-\frac{n}{n-1}} \operatorname{var}(f).
\end{aligned}$$

Proof ingredients

1. **relative isoperimetric inequality**
2. **Vitaly covering** and similar: general balls $\to$ separated balls
3. **Vitaly covering for boundary**
4. **superlevelset estimate**: $f < 0$ on most of $B \Rightarrow$ most mass of f lies far above f_B

used in proof	isoperimetric, Vitali	boundary Vitaly	superlevel
dyadic char. f.	**x**		
char. f.	**x**	**x**	
dyadic	**x**		**x**
fractional	**x**		**x**
cube	**x**	**x**	**x**

Variants

1. Centered maximal operator M_φ that averages against a **smooth kernel** φ satisfies

$$\|\nabla \mathrm{M}_\varphi f\|_1 \leq \|\nabla f\|_1$$

if $\varphi : \mathbb{R} \to (0, \infty)$ is associated to a PDE [Carneiro, Finder, Sousa and Svaiter 2013,2018]

2. **Discrete** $f : \mathbb{Z} \to \mathbb{R}$ mostly mirrors continuous setting but not entirely. Also $f : G \to \mathbb{R}$ on a graph. [Bober, Carneiro, Gonzalez-Riquelme, Hughes, Madrid, Pierce,. . .]
3. on **Hardy-Sobolev space** [Pérez, Picon, Saari, Sousa 2018]

Variants

4. **local** maximal functions on **domains** $\Omega \subset \mathbb{R}^n$ that average only over balls $B \subset \Omega$: Many questions remain open for the local fractional maximal function since it prefers to average over large balls. [Heikkinen, Kinnunen, Korvenpää, Lindqvist, Raamos, Saari, Tuominen, W.,. . .]
5. fractional **smoothing**

$$\|\nabla \mathrm{M}_\alpha f\|_{p_\alpha} \leq C\|f\|_{\frac{pn}{n-p}},$$

known to hold or fail in some cases and open in others.

6. **local regularity**: For $f \in \mathrm{BV}(\mathbb{R}^n)$ is $\nabla \mathrm{M} f \in L^1(\mathbb{R}^n)$ or only a Radon measure? some cases known, some open [Gonzalez-Riquelme 2022, Lahti 2021]

Continuity

Stronger property than boundedness:

Operator continuity of M

f close to g $\quad\Rightarrow\quad$ $\mathrm{M}f$ close to $\mathrm{M}g$?

By sublinearity

$$\mathrm{M}f(x) - \mathrm{M}g(x) \leq \mathrm{M}(f-g)(x) + \mathrm{M}g(x) - \mathrm{M}g(x)$$

and thus

$$\|\mathrm{M}f - \mathrm{M}g\|_{L^p(\mathbb{R}^n)} \leq \|\mathrm{M}(f-g)\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|f-g\|_{L^p(\mathbb{R}^n)}.$$

Continuity

However,

$$|\nabla \mathrm{M} f(x) - \nabla \mathrm{M} g(x)| \not\lesssim |\nabla \mathrm{M}(f-g)(x)|.$$

Nevertheless, [Luiro 2004] proved for $p > 1$ that

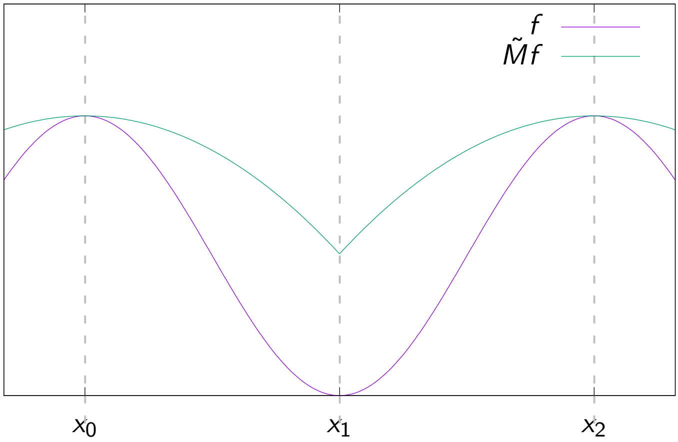
$$\|f_n - f\|_{W^{1,p}(\mathbb{R}^n)} \to 0 \quad \Longrightarrow \quad \|\nabla \mathrm{M} f_n - \nabla \mathrm{M} f\|_{L^p(\mathbb{R}^n)} \to 0.$$

Operator continuity is known in many of the cases of boundedness due to many results by [Beltran, Carneiro, González-Riquelme, Madrid, Pierce,... 2013–], but not all cases.

Higher derivatives

What about $(\widetilde{\mathrm{M}}f)''$?

Typically, $(\widetilde{\mathrm{M}}f)'' \notin L^p(\mathbb{R})$, similarly to how $|f|'' \notin L^p(\mathbb{R})$.



And if we relax to $\operatorname{var}((\widetilde{\mathrm{M}}f)')$?

For $f \in C_0^1(\mathbb{R})$ it is easy to see that $\operatorname{var}(|f|') \leq 2\operatorname{var}(f')$.

Theorem (Temur 2022)

If $f : \mathbb{Z} \to \mathbb{R}$ is of the form $f = 1_E$ we have

$$\|(\widetilde{\mathrm{M}}f)''\|_1 \leq C\|f''\|_1.$$

Theorem (W. 2024)

If $f : \mathbb{R} \to \mathbb{R}$ is radially decreasing and symmetric then

$$\operatorname{var}((\widetilde{\mathrm{M}}f)') \leq C\operatorname{var}(f').$$

Theorem (W. 2024)

There exist radially decreasing $f_k : \mathbb{R} \to \mathbb{R}$ is with

$$\lim_{k\to\infty} \frac{\operatorname{var}((\widetilde{\mathrm{M}}f_k)')}{\operatorname{var}(f_k')} = \infty.$$

M^{c}? Fractional derivatives? Best constants?

Thank you