# Endpoint regularity of fractional maximal functions

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# The Hardy-Littlewood maximal function theorem

For $f : \mathbb{R}^n \to \mathbb{R}$ the centered Hardy-Littlewood maximal function is defined by

$$\mathrm{M}^{\mathrm{c}} f(x) = \sup_{r>0} f_{B(x,r)} \quad \text{with} \qquad f_{B(x,r)} = \frac{1}{\mathcal{L}(B(x,r))} \int_{B(x,r)} |f|.$$

**Theorem (Hardy-Littlewood maximal function theorem)**

$$\|\mathrm{M}^{\mathrm{c}} f\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|f\|_{L^p(\mathbb{R}^n)}$$

*if and only if* $p > 1$, *and*

$$\|\mathrm{M}^{\mathrm{c}} f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim_{n} \|f\|_{L^1(\mathbb{R}^n)}.$$

# The fractional maximal function theorem

For $0 \leq \alpha \leq n$ and $f : \mathbb{R}^n \to \mathbb{R}$ the centered fractional maximal function is defined by

$$\mathrm{M}^{\mathrm{c}}_{\alpha} f(x) = \sup_{r>0} r^{\alpha} f_{B(x,r)} = \frac{r^{\alpha-n}}{\mathcal{L}(B(0,1))} \int_{B(x,r)} |f|.$$

**Theorem (Fractional maximal function theorem)**

*For $1 \leq p \leq n/\alpha$ denote $p_{\alpha} = \frac{p}{1-\alpha p/n}$. Then*

$$\|\mathrm{M}^{\mathrm{c}} f\|_{L^{p_{\alpha}}(\mathbb{R}^n)} \lesssim_{n,\alpha,p} \|f\|_{L^p(\mathbb{R}^n)}$$

*if and only if $p > 1$, and*

$$\|\mathrm{M}^{\mathrm{c}} f\|_{L^{1_{\alpha},\infty}(\mathbb{R}^n)} \lesssim_{n,\alpha} \|f\|_{L^1(\mathbb{R}^n)}.$$

# The endpoint regularity question

**Theorem (Kinnunen 1997, Kinnunen and Saksman 2003)**

*For $p > 1$ we have*

$$\|\nabla \mathrm{M}_\alpha^{\mathrm{c}} f\|_{L^{p_\alpha}(\mathbb{R}^n)} \lesssim_{n,\alpha,p} \|\nabla f\|_{L^p(\mathbb{R}^n)}$$

**Proof:** For $e \in \mathbb{R}^n$ by the sublinearity of $\mathrm{M}^{\mathrm{c}}$

$$\begin{aligned} \partial_e \mathrm{M}_\alpha^{\mathrm{c}} f(x) &\sim \frac{\mathrm{M}_\alpha^{\mathrm{c}} f(x+he) - \mathrm{M}_\alpha^{\mathrm{c}} f(x)}{h} \\ &\leq \frac{\mathrm{M}_\alpha^{\mathrm{c}}(f(\cdot + he) - f)(x)}{h} \\ &= \mathrm{M}_\alpha^{\mathrm{c}}\Big(\frac{f(\cdot + he) - f}{h}\Big)(x) \sim \mathrm{M}_\alpha^{\mathrm{c}}(\partial_e f)(x). \end{aligned}$$

Then by the fractional maximal function theorem for $p > 1$

$$\|\nabla \mathrm{M}_\alpha^{\mathrm{c}} f\|_{L^{p_\alpha}(\mathbb{R}^n)} \leq \|\mathrm{M}_\alpha^{\mathrm{c}}(|\nabla f|)\|_{L^{p_\alpha}(\mathbb{R}^n)} \lesssim_{n,p} \|\nabla f\|_{L^p(\mathbb{R}^n)}.$$

# The endpoint regularity question

**Question (Hajłasz and Onninen 2004, Carneiro and Madrid 2016)**

*Is it true that*

$$\|\nabla \mathrm{M}^{\mathrm{c}}_{\alpha} f\|_{L^{1}\alpha(\mathbb{R}^n)} \lesssim_n \|\nabla f\|_{L^1(\mathbb{R}^n)}?$$

Uncentered Hardy-Littlewood maximal function

$$\widetilde{\mathrm{M}}_{\alpha} f(x) = \sup_{B \ni x} r(B)^{\alpha} f_B.$$

Endpoint question by Hałjasz and Onninen is interesting for $\widetilde{\mathrm{M}}_{\alpha}$ and other maximal operators.

# In one dimension

**Theorem (Tanaka 2002, Aldaz and Pérez Lázaro 2007, Carneiro and Madrid 2016)**

*For* $f : \mathbb{R} \to \mathbb{R}$ *we have*

$$\|\nabla \widetilde{\mathrm{M}}_\alpha f\|_{1_\alpha} \lesssim_\alpha \|\nabla f\|_1$$

In one dimension we have

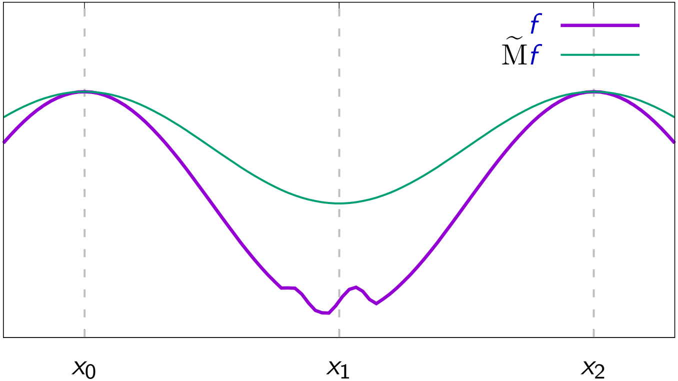
1. $$\|\nabla f\|_1 = \sup_{x_1 < x_2 < \dots} \sum_i |f(x_{i+1}) - f(x_i)|$$

2. $$\mathrm{M}_0 f(x) = \max\{\mathrm{M}_{0,+} f(x), \mathrm{M}_{0,-} f(x)\}$$

   where

   $$\mathrm{M}_{0,\pm} f(x) = \sup_{r>0} \frac{1}{r} \int_{x \pm r} f$$

# In one dimension, $\alpha = 0$




$$\begin{aligned} \|\nabla \widetilde{\mathrm{M}}_0 f\|_{L^1([x_0,x_2])} &= |\widetilde{\mathrm{M}}_0 f(x_1) - \widetilde{\mathrm{M}}_0 f(x_0)| + |\widetilde{\mathrm{M}}_0 f(x_2) - \widetilde{\mathrm{M}}_0 f(x_1)| \\ &\leq |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| \\ &\leq \|\nabla f\|_{L^1([x_0,x_2])} \end{aligned}$$

# Large $\alpha$

**Theorem (Kinnunen and Saksman 2003)**

*For* $\alpha \geq 1$
$$|\nabla \mathrm{M}^{\mathrm{c}}_{\alpha} f(x)| \lesssim_n |\mathrm{M}^{\mathrm{c}}_{\alpha-1} f(x)|.$$

**Corollary (Carneiro and Madrid 2016)**

For $\alpha \geq 1$ we have $1_\alpha = \frac{n}{n-\alpha} = \left(\frac{n}{n-1}\right)_{\alpha-1}$ and $\frac{n}{n-1} > 1$ and therefore

$$\begin{aligned}\|\nabla \mathrm{M}^{\mathrm{c}}_{\alpha} f\|_{L^{1_\alpha}(\mathbb{R}^n)} \lesssim_n \|\mathrm{M}^{\mathrm{c}}_{\alpha-1} f\|_{L^{1_\alpha}(\mathbb{R}^n)} \lesssim_n \|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \\ \lesssim_n \|\nabla f\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

# Continuity

Stronger property than boundedness:

**Operator continuity of $\mathrm{M}_\alpha$**

$f$ close to $g$ $\quad\Rightarrow\quad$ $\mathrm{M}_\alpha f$ close to $\mathrm{M}_\alpha g$ ?

By sublinearity

$$\mathrm{M}_\alpha f(x) - \mathrm{M}_\alpha g(x) \leq \mathrm{M}_\alpha (f-g)(x) + \mathrm{M}_\alpha g(x) - \mathrm{M}_\alpha g(x)$$

and thus

$$\|\mathrm{M}_\alpha f - \mathrm{M}_\alpha g\|_{L^{p_\alpha}(\mathbb{R}^n)} \leq \|\mathrm{M}_\alpha (f-g)\|_{L^{p_\alpha}(\mathbb{R}^n)} \lesssim_{n,p} \|f-g\|_{L^p(\mathbb{R}^n)}.$$

# Continuity

However,

$$|\nabla \mathrm{M}_\alpha f(x) - \nabla \mathrm{M}_\alpha g(x)| \not\lesssim |\nabla \mathrm{M}_\alpha (f-g)(x)|.$$

Nevertheless, [Luiro 2004] proved for $p > 1$ that

$$\|f_n - f\|_{W^{1,p}(\mathbb{R}^n)} \to 0 \quad \Longrightarrow \quad \|\nabla \mathrm{M}_0 f_n - \nabla \mathrm{M}_0 f\|_{L^p(\mathbb{R}^n)} \to 0.$$

Operator continuity is now known in almost the same cases as boundedness due to many results by [Beltran, Carneiro, González-Riquelme, Madrid, Pierce,... 2013–].

# Previous results

The bound

$$\|\nabla \mathrm{M}_\alpha f\|_{L^{1\alpha}(\mathbb{R}^n)} \lesssim \|\nabla f\|_{L^1(\mathbb{R}^n)}$$

is known to hold in the cases

| | |
|---|---|
| $n = 1$ | [Aldaz, Carneiro, Kurka, Madrid, Pérez Lázaro, Tanaka 2002–2016] |
| $\alpha \geq 1$ | [Beltran, Carneiro, Madrid, Kinnunen, Ramos, Saari, Saksman,. . . ] |
| radial $f$ | [Beltran, Madrid, Luiro 2017-2019] |
| lacunary $\alpha > 0$ | [Beltran + Ramos + Saari 2018] |

Most fractional results are known both for the centered and uncentered maximal function.

# New results

$$\|\nabla \mathrm{M}_\alpha f\|_{L^{1\alpha}(\mathbb{R}^n)} \lesssim \|\nabla f\|_{L^1(\mathbb{R}^n)}$$

| | |
|---|---|
| $0 < \alpha \leq n$ | [W. 2020] |
| continuity | [Beltran, Gonzalez-Riquelme, Madrid, W. 2021] |

Works for both centered and uncentered

Fractional: complete!
New results for $\alpha = 0$, uncentered maximal function:

| | |
|---|---|
| characteristic $f$ | [W. 2020] |
| dyadic maximal function | [W. 2020] |
| cube maximal function | [W. 2021] |

# Proof strategy

$1 \leq \alpha$ [Kinnunen + Saksman, Carneiro + Madrid]

$$\|\nabla \mathrm{M}_\alpha f\|_{\frac{n}{n-\alpha}} \lesssim_n \|\mathrm{M}_{\alpha-1} f\|_{\frac{n}{n-\alpha}} \lesssim_n \|f\|_{\frac{n}{n-1}} \lesssim_n \|\nabla f\|_1.$$

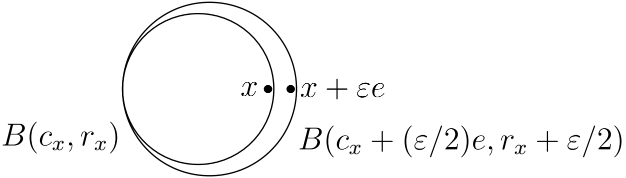
$0 < \alpha$

$$\|\nabla \mathrm{M}_\alpha f\|_{\frac{n}{n-\alpha}} \lesssim_n \|\mathrm{M}_{\alpha,-1} f\|_{\frac{n}{n-\alpha}} \lesssim_{n,\alpha} \|\nabla f\|_1,$$

$\mathrm{M}_{\alpha,-1}$ replacement for $\mathrm{M}_{\alpha-1}$ if $0 < \alpha < 1$.

1. apply Kinnunen-Saksman
2. exploit $\alpha > 0$ to separate balls, allowing for Vitali covering arguments
3. apply
    - relative isoperimetric inequality and
    - result used for the cube maximal operator for $\alpha = 0$.

# Kinnunen-Saksman argument




$$
\begin{aligned}
&\frac{|\mathrm{M}_\alpha f(x)-\mathrm{M}_\alpha f(x+\varepsilon e)|}{\varepsilon}\\
&\quad\le \frac{1}{\varepsilon}\Big(r_x^{\alpha-n}\int_{B(c_x,r_x)} f-(r_x+\varepsilon/2)^{\alpha-n}\int_{B(c_x+(\varepsilon/2)e,r_x+\varepsilon/2)} f\Big)\\
&\quad\le \frac{r_x^{\alpha-n}-(r_x+\varepsilon/2)^{\alpha-n}}{\varepsilon}\int_{B(c_x,r_x)} f\\
&\quad\to \frac{n-\alpha}{2}r_x^{\alpha-1-n}\int_{B(c_x,r_x)} f=:\frac{n-\alpha}{2}\mathrm{M}_{\alpha,-1}f(x)\\
&\quad\le \frac{n-\alpha}{2}\mathrm{M}_{\alpha-1}f(x).
\end{aligned}
$$

# Separation of balls

With

$$\mathcal{B}_\alpha = \{B : \forall C \supset B \ \ \mathrm{r}(C)^\alpha f_C \le \mathrm{r}(B)^\alpha f_B\}$$

we have

$$\mathrm{M}_\alpha f(x) = \sup_{x \in B \in \mathcal{B}_\alpha} \mathrm{r}(B)^\alpha f_B$$

and define

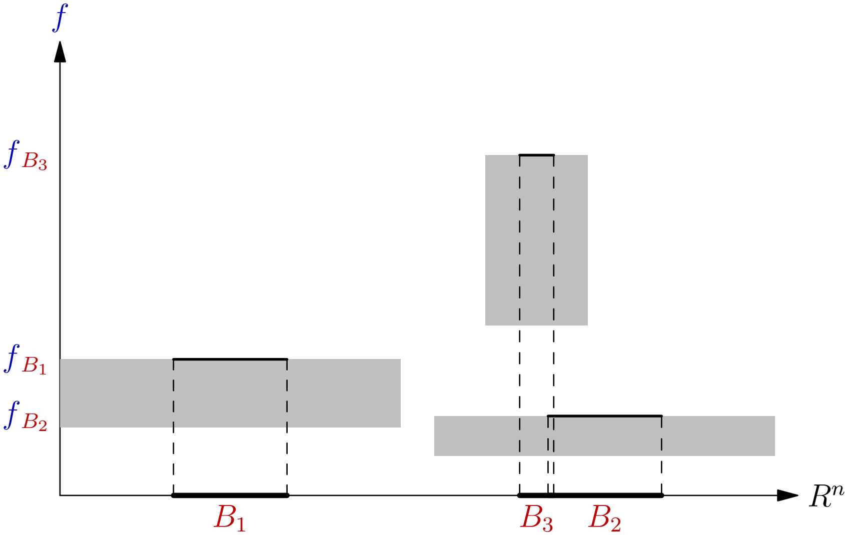
$$\mathrm{M}_{\alpha,-1} f(x) = \sup_{x \in B \in \mathcal{B}_\alpha} \mathrm{r}(B)^{\alpha-1} f_B$$

**Lemma**

For any $B, C \in \mathcal{B}_\alpha$ we have

1. $\mathrm{r}(B) \sim \mathrm{r}(C)$ and $f_B \sim f_C$ or
2. $(2B) \times (f_B/2, f_B)$ and $(2C) \times (f_C/2, f_C)$ are disjoint.

# Separation of balls



# Separation of balls

**Lemma**

For any $B, C \in \mathcal{B}_\alpha$ we have

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2. $(2B) \times (f_B/2, f_B)$ and $(2C) \times (f_C/2, f_C)$ are disjoint.

Proof: If $B, C \in \mathcal{B}_\alpha$ with $B \subset 2C$ then

$$\mathrm{r}(B)^\alpha f_B \geq \mathrm{r}(2C)^\alpha f_{2C} \geq 2^{\alpha-n}\,\mathrm{r}(C)^\alpha f_C.$$

Thus, if $\mathrm{r}(B) \ll \mathrm{r}(C)$ then $f_B \gg f_C$. Moreover, if $\mathrm{r}(B) \sim \mathrm{r}(C)$ then $C \subset 2B$ and thus $f_B \sim f_C$.

Using the lemma, by Vitali covering type argument take a subset $\tilde{\mathcal{B}}_\alpha \subset \mathcal{B}_\alpha$ of which all balls satisfy the disjointness 2, and such that for all $B \in \mathcal{B}_\alpha \exists C \in \tilde{\mathcal{B}}_\alpha$ with $\mathrm{r}(C) \sim \mathrm{r}(B)$ and $f_B \sim f_C$. Then

$$\begin{aligned}
\int |\nabla \mathrm{M}_\alpha f|^{\frac{d}{d-\alpha}} &\lesssim_\alpha \int (\mathrm{M}_{\alpha,-1} f)^{\frac{d}{d-\alpha}} \\
&= \int_0^\infty \lambda^{\frac{\alpha}{d-\alpha}} \mathcal{L}(\{\mathrm{M}_{\alpha,-1} f > \lambda\}) \,\mathrm{d}\lambda \\
&= \int_0^\infty \lambda^{\frac{\alpha}{d-\alpha}} \mathcal{L}\Big(\bigcup \{\overline{B} : B \in \mathcal{B}_\alpha, r(B)^{\alpha-1} f_B > \lambda\}\Big) \,\mathrm{d}\lambda \\
&\sim_\alpha \int_0^\infty \lambda^{\frac{\alpha}{d-\alpha}} \mathcal{L}\Big(\bigcup \{\overline{B} : B \in \tilde{\mathcal{B}}_\alpha, r(B)^{\alpha-1} f_B > \lambda\}\Big) \,\mathrm{d}\lambda \\
&\leq \int_0^\infty \lambda^{\frac{\alpha}{d-\alpha}} \sum_{B \in \tilde{\mathcal{B}}_\alpha, c r(B)^{\alpha-1} f_B > \lambda} \mathcal{L}(B) \,\mathrm{d}\lambda
\end{aligned}$$

$$\begin{aligned}\int_0^\infty \lambda^{\frac{\alpha}{d-\alpha}} \sum_{B\in\tilde{\mathcal{B}}_\alpha, cr(B)^{\alpha-1} f_B > \lambda} \mathcal{L}(B)\,\mathrm{d}\lambda &= \sum_{B\in\tilde{\mathcal{B}}_\alpha} \int_0^{cr(B)^{\alpha-1} f_B} \lambda^{\frac{\alpha}{d-\alpha}}\,\mathrm{d}\lambda \\ &\sim_\alpha \sum_{B\in\tilde{\mathcal{B}}_\alpha} (f_B \mathcal{H}^{n-1}(\partial B))^{\frac{d}{d-\alpha}} \\ &\leq \left( \sum_{B\in\tilde{\mathcal{B}}_\alpha} f_B \mathcal{H}^{n-1}(\partial B) \right)^{\frac{d}{d-\alpha}} \\ &\lesssim_\alpha \left( \sum_{Q\in\tilde{\mathcal{Q}}_\alpha} f_Q \mathcal{H}^{n-1}(\partial Q) \right)^{\frac{d}{d-\alpha}}.\end{aligned}$$

# Two cases

**Lemma (coarea formula)**

$$\|\nabla f\|_1 = \int_0^\infty \mathcal{H}^{n-1}(\partial\{f > \lambda\})\,\mathrm{d}\lambda$$

**Lemma (relative isoperimetric inequality)**

Let $E \subset B$ with $\mathcal{L}(E \cap B) \leq \mathcal{L}(B)/2$. Then

$$\mathcal{L}(B \cap E)^{n-1} \lesssim \mathcal{H}^{n-1}(B \cap \partial E)^n.$$

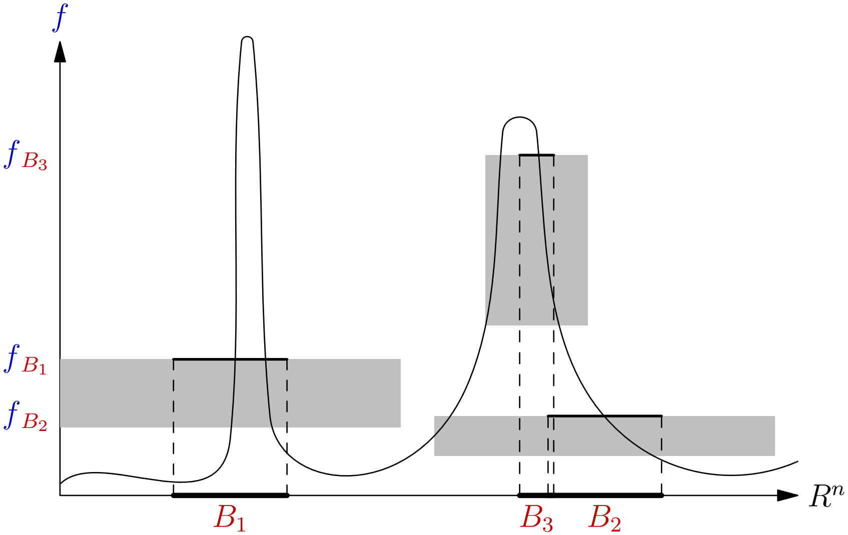
In particular, if $\mathcal{L}(B) \lesssim \mathcal{L}(E \cap B) \leq \mathcal{L}(B)/2$ then

$$\mathcal{H}^{n-1}(\partial B) \sim \mathcal{L}(B)^{\frac{n-1}{n}} \sim \mathcal{L}(B \cap E)^{\frac{n-1}{n}} \lesssim \mathcal{H}^{n-1}(B \cap \partial E).$$

Not too much of $2B \times (f_B/2, f_B)$ can be occupied by $\{(x,t) : t \leq f(x)\}$, because otherwise $\mathrm{r}(2B)^\alpha f_{2B} \geq \mathrm{r}(B)^\alpha f_B$.

Divide the balls in $\tilde{\mathcal{B}}_\alpha$ (cubes in $\tilde{\mathcal{Q}}_A$) according to two cases.

# Two cases



# High density case

If much of $2B \times (f_B/2, f_B)$ is occupied by $\{(x,t) : t \le f(x)\}$ then for a typical $f_B/2 \le \lambda \le f_B$ we have

$$\mathcal{L}(2B) \lesssim \mathcal{L}(2B \cap \{f > \lambda\}) \le \mathcal{L}(2B)/2$$

and thus by the relative isoperimetric inequality we can conclude

$$f_B \mathcal{H}^{n-1}(\partial B) \lesssim \int_{f_B/2}^{f_B} \mathcal{H}^{n-1}(\partial B)\,\mathrm{d}\lambda \lesssim \int_{f_B/2}^{f_B} \mathcal{H}^{n-1}(2B \cap \partial\{f > \lambda\})\,\mathrm{d}\lambda.$$

Since the $2B \times (f_B/2, f_B)$ are pairwise disjoint we sum over all $B \in \tilde{\mathcal{B}}_\alpha$ in the **high density case** and recover by the coarea formula

$$\sum_{B \in \tilde{\mathcal{B}}_\alpha} \int_{f_B/2}^{f_B} \mathcal{H}^{n-1}(2B \cap \partial\{f > \lambda\})\,\mathrm{d}\lambda \le \|\nabla f\|_1 \checkmark$$

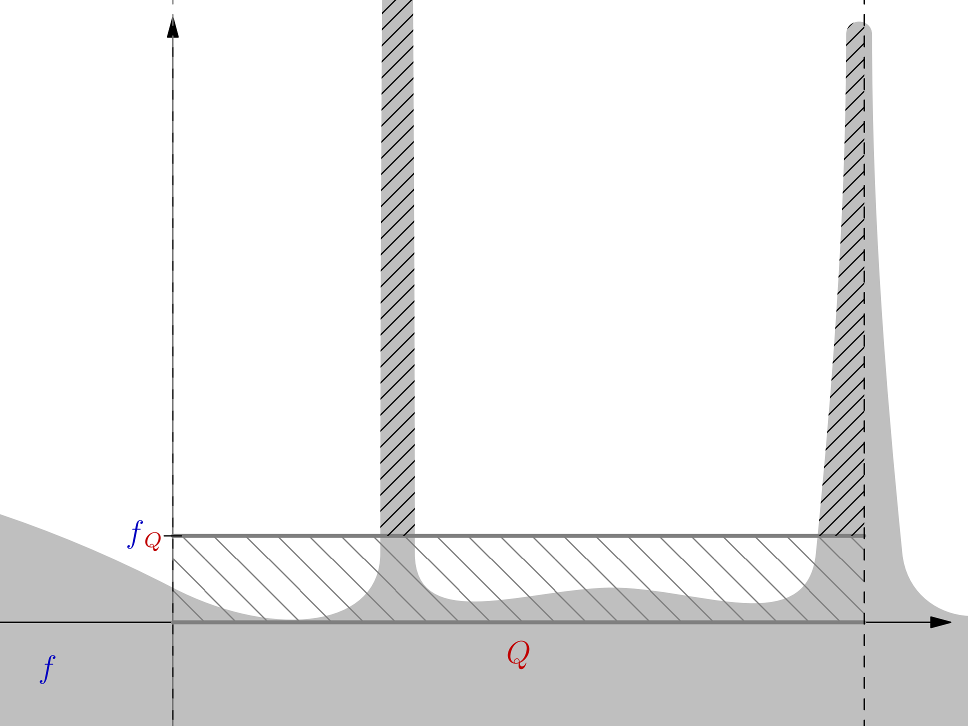
# Low density case

If little of $2B \times (f_B/2, f_B)$ is occupied by $\{(x,t) : t \leq f(x)\}$ the idea is to bound $f_B \mathcal{H}^{n-1}(\partial B)$ by the part of $\|\nabla f\|_1$ on $2B \times (f_B, \infty)$.

**Proposition**

If $Q$ is a low density cube then

$$f_Q \mathcal{L}(Q) \lesssim_n \int_{f_Q}^{\infty} \mathcal{L}(Q \cap \{f > \lambda\}) \qquad \mathrm{d}\lambda$$


$f_Q$
$Q$
$f$

# Low density case

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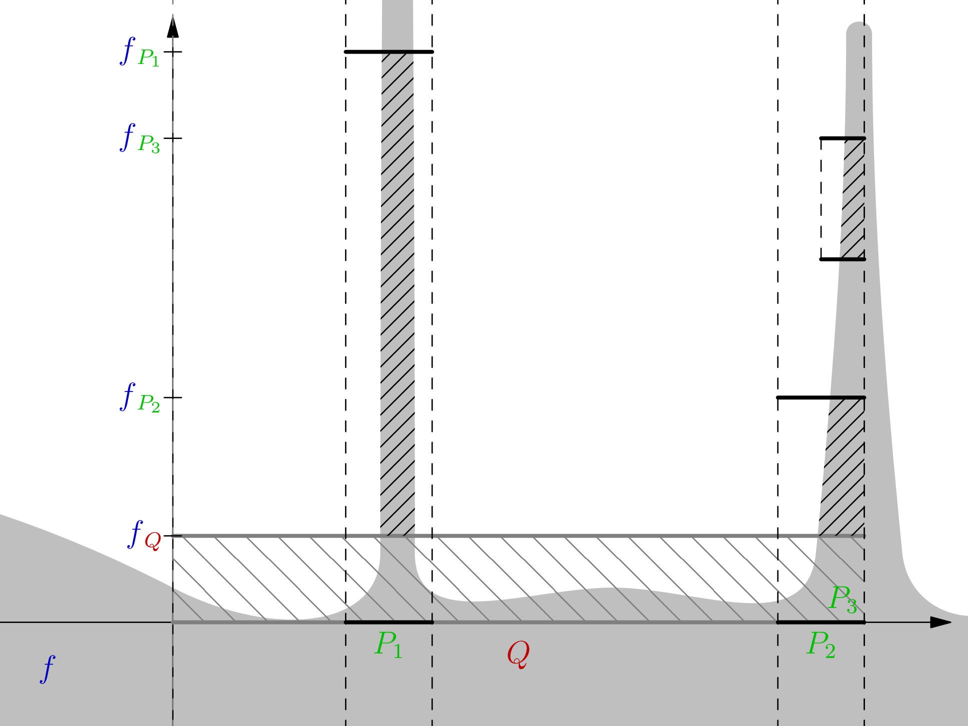
## Proposition

If $Q$ is a low density cube then

$$f_Q \mathcal{L}(Q) \lesssim_n \int_{f_Q}^{\infty} \sum_{P \subsetneq Q : \lambda_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\}) \, \mathrm{d}\lambda$$

where $P$ is maximal above $\lambda_P$ and

$$\mathcal{L}(P \cap \{f > \lambda_P\}) = 2^{-1} \mathcal{L}(P)$$



$f_{P_1}$
$f_{P_3}$
$f_{P_2}$
$f_Q$
$P_3$
$P_1$
$Q$
$P_2$
$f$

# Low density case

Sum over all $Q \in \tilde{\mathcal{Q}}_\alpha$ in the **low density case** and obtain

$$\sum_{Q \in \tilde{\mathcal{Q}}_\alpha} f_Q \mathcal{H}^{n-1}(\partial Q) \lesssim \int_{\mathbb{R}} \sum_{Q \text{ dyadic}} \sum_{P \subsetneq Q : \lambda_P < \lambda < f_P} \frac{\mathcal{L}(P \cap \{f > \lambda\})}{l(Q)} \, \mathrm{d}\lambda$$

change the order of summation, convergence of the geometric sum

$$\lesssim \sum_{P \text{ dyadic}} \int_{\lambda_P}^{f_P} \frac{\mathcal{L}(P \cap \{f > \lambda\})}{l(P)} \, \mathrm{d}\lambda$$

apply the relative isoperimetric inequality
$\mathcal{L}(P \cap \{f > \lambda\})^{\frac{d}{d-1}} \lesssim \mathcal{H}^{n-1}(P \cap \partial\{f > \lambda\})$ and coarea formula

$$\lesssim \|\nabla f\|_1$$

# Continuity

Let $\|f_n - f\|_{W^{1,1}(\mathbb{R}^n)} \to 0$.

1. [Beltran, Carneiro, González-Riquelme, Madrid, Pierce,... 2013–] have already established that certain parts of $\nabla \mathrm{M}_\alpha f_n$ converge to the corresponding part of $\nabla \mathrm{M}_\alpha f$, such as where $\mathrm{M}_\alpha$ averages only over balls with radius bounded from below, and outside a compact set.
2. Use a local version of $\|\nabla \mathrm{M}_\alpha f\|_{1_\alpha} \lesssim \|\nabla f\|_1$ to control the remaining part.

Thank you