Higher Dimensional Techniques for the Regularity of Maximal Functions

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Outline

History

Background

Background

For $f : \mathbb{R}^d \to \mathbb{R}$ the centered Hardy-Littlewood maximal function is defined by

$$\mathrm{M}^{\mathrm{c}} f(x) = \sup_{r>0} f_{B(x,r)} \quad \text{with} \quad f_{B(x,r)} = \frac{1}{\mathcal{L}(B(x,r))} \int_{B(x,r)} |f|.$$

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The Hardy-Littlewood maximal function theorem:

$$\|\mathrm{M}^{\mathrm{c}} f\|_{L^p(\mathbb{R}^d)} \leq C_{d,p} \|f\|_{L^p(\mathbb{R}^d)} \qquad \text{if and only if } p > 1$$

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Juha Kinnunen (1997):

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Question (Hajłasz and Onninen 2004)

Is it true that

$$\|\nabla \mathrm{M^c} f\|_{L^1(\mathbb{R}^d)} \leq C_d \|\nabla f\|_{L^1(\mathbb{R}^d)}?$$

Proof

For $e \in \mathbb{R}^d$ by the sublinearity of M^{c}

$$\partial_e \mathrm{M}^{\mathrm{c}} f(x) \sim \frac{\mathrm{M}^{\mathrm{c}} f(x+he) - \mathrm{M}^{\mathrm{c}} f(x)}{h}$$

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For $e \in \mathbb{R}^d$ by the sublinearity of $\mathrm{M^c}$

$$\partial_e \mathrm{M^c} f(x) \sim \frac{\mathrm{M^c} f(x+he) - \mathrm{M^c} f(x)}{h} \leq \frac{\mathrm{M^c}(f(\cdot + he) - f)(x)}{h}$$

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$$= \mathrm{M}^{\mathrm{c}}\Big(\frac{f(\cdot + he) - f)}{h}\Big)(x)$$

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By the Hardy-Littlewood maximal function theorem for $p > 1$

$$\|\nabla \mathrm{M^c} f\|_{L^p(\mathbb{R}^d)} \lesssim \|\mathrm{M^c}(|\nabla f|)\|_{L^p(\mathbb{R}^d)} \lesssim \|\nabla f\|_{L^p(\mathbb{R}^d)}$$

Uncentered maximal operator

For $f : \mathbb{R}^d \to \mathbb{R}$ the uncentered Hardy-Littlewood maximal function is defined by

$$\widetilde{\mathrm{M}} f(x) = \sup_{B \ni x} f_B .$$

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The result by Kinnunen also holds for $\widetilde{\mathrm{M}}$ and various other maximal operators, and the question by Hałjasz and Onninen is being investigated.

Onedimensional case

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In 2002 Tanaka proved

$$\text{var}\, \widetilde{\mathrm{M}} f \leq \text{var}\, f$$

for $f : \mathbb{R} \to \mathbb{R}$, but with a factor 2 on the right hand side. In 2007 Aldaz and Pérez Lázaro reduced that factor to the optimal value 1.

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$$\operatorname{var} f = \sup_{n\in\mathbb{N},\ x_1<\ldots<x_n} \sum_{i=1}^{n-1} |f(x_{n+1}) - f(x_n)|.$$

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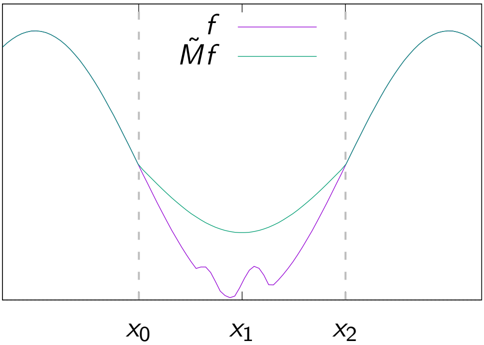
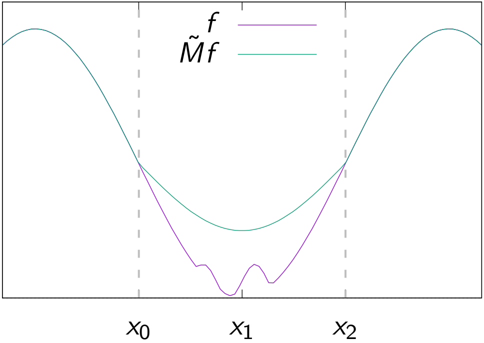
$$\operatorname{var} f = \sup_{n \in \mathbb{N},\ x_1 < \ldots < x_n} \sum_{i=1}^{n-1} |f(x_{n+1}) - f(x_n)|.$$

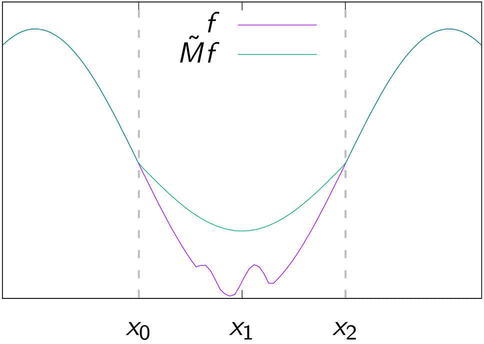
Main ingredient: $\widetilde{\mathrm{M}} f$ is convex on connected components of $\{x \in \mathbb{R} : \widetilde{\mathrm{M}} f(x) > f(x)\}$.



f
$\tilde{M}f$
x_0
x_1
x_2



$$\operatorname{var} \widetilde{\mathrm{M}} f = \operatorname{var}_{[0,x_0]} \widetilde{\mathrm{M}} f + \operatorname{var}_{[x_2,1]} \widetilde{\mathrm{M}} f + |\widetilde{\mathrm{M}} f(x_0) - \widetilde{\mathrm{M}} f(x_1)| + |\widetilde{\mathrm{M}} f(x_2) - \widetilde{\mathrm{M}} f(x_1)|$$



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&\leq \operatorname{var}_{[0,x_0]} f + \operatorname{var}_{[x_2,1]} f \\
&\quad + |f(x_0) - f(x_1)| + |f(x_2) - f(x_1)|
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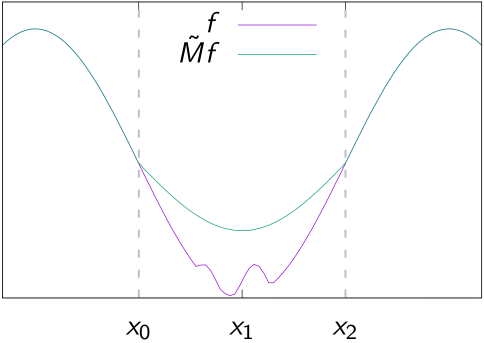

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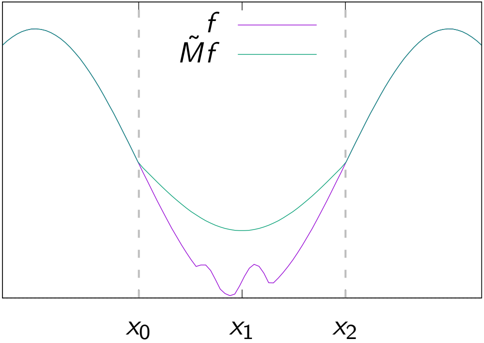

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&\quad + |f(x_0) - f(x_1)| + |f(x_2) - f(x_1)| \\
&\leq \operatorname{var}_{[0,x_0]} f + \operatorname{var}_{[x_2,1]} f + \operatorname{var}_{[x_0,x_2]} f = \operatorname{var} f
\end{aligned}
$$

Onedimensional case

For the centered maximal function $\mathrm{M}^{\mathrm{c}} f$ the convexity property does not hold. Nevertheless,

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Kurka proved $\operatorname{var} \mathrm{M}^{\mathrm{c}} f \leq C \operatorname{var} f$ for $f : \mathbb{R} \to \mathbb{R}$ in a very involved paper in 2015.

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He did case distinctions with respect to the shape of triples $x_0 < x_1 < x_2$ with $\mathrm{M}^{\mathrm{c}} f(x_0) < \mathrm{M}^{\mathrm{c}} f(x_1) > \mathrm{M}^{\mathrm{c}} f(x_2)$ and a decomposition in scales.

Onedimensional case

For radial functions $f : \mathbb{R}^d \to \mathbb{R}$ with $f(x) = f(|x|)$ we have

$$\|\nabla f\|_{L^1(\mathbb{R}^d)} = \int_0^\infty |\nabla f(r)| r^{d-1} \, \mathrm{d}r$$

and also $\widetilde{\mathrm{M}} f$ is radial.

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In 2018 Luiro used this one-dimensional representation to prove $\|\nabla \widetilde{\mathrm{M}} f\|_{L^1(\mathbb{R}^d)} \leq C_d \|\nabla f\|_{L^1(\mathbb{R}^d)}$ for radial functions $f : \mathbb{R}^d \to \mathbb{R}$.

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block-decreasing

In 2009 Aldaz and Pérez Lázaro proved $\|\nabla \widetilde{\mathrm{M}} f\|_{L^1(\mathbb{R}^d)} \leq C_d \|\nabla f\|_{L^1(\mathbb{R}^d)}$ for block-decreasing $f : \mathbb{R}^d \to \mathbb{R}$,

which are to some extent similar to radially decreasing functions.

Other maximal operators and related questions

- fractional maximal operators
- convolution operators
- local maximal operators
- discrete maximal operators
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related: Continuity of the operator given by $f \mapsto \nabla \mathrm{M} f$ on $W^{1,1}(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$. This is a stronger property than boundedness.

Core Techniques

Reduction and decomposition

reformulations

definition

$$\operatorname{var} f = \sup\left\{ \int f \operatorname{div} \varphi : \varphi \in C^1_c(\mathbb{R}^d; \mathbb{R}^d),\ |\varphi| \le 1 \right\}$$

reformulations

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coarea formula

$$\operatorname{var} f = \int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial\{x \in \mathbb{R}^d : f(x) > \lambda\})\, \mathrm{d}\lambda$$

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superlevel sets

$$\{x \in \mathbb{R}^d : \mathrm{M}f(x) > \lambda\} = \bigcup \{B : f_B > \lambda\}$$

for *uncentered* maximal operators.

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for *uncentered* maximal operators.

Denote

$$\mathcal{B}_\lambda^{<} = \{B : f_B > \lambda,\ \mathcal{L}(B \cap \{f > \lambda\}) < \mathcal{L}(B)/2\}$$

and $\mathcal{B}_\lambda^{\geq}$ accordingly. We split the boundary

$$\partial \bigcup \{B : f_B > \lambda\} \subset \partial \bigcup \mathcal{B}_\lambda^{<} \cup \partial \bigcup \mathcal{B}_\lambda^{\geq}. \tag{1}$$

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Since $\mathrm{M}f \geq f$ a.e. we have $\{f > \lambda\} \subset \{\mathrm{M}f > \lambda\}$ up to measure zero, and thus

$$\partial \bigcup \{B : f_B > \lambda\} \subset \left(\partial \bigcup \{B : f_B > \lambda\}\right) \setminus \overline{\{f > \lambda\}} \cup \partial \{f > \lambda\}. \tag{2}$$

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Plug (1) into (2) and that into the coarea formula

$$\operatorname{var} \mathrm{M}f = \int_0^\infty \mathcal{H}^{d-1}\Big(\partial \bigcup \{B : f_B > \lambda\}\Big) \,\mathrm{d}\lambda.$$

Decomposition of the boundary

decomposition

$$\operatorname{var} \mathrm{M}f \leq \int_0^\infty \mathcal{H}^{d-1}\Big(\partial \bigcup \mathcal{B}_\lambda^<\Big)\,\mathrm{d}\lambda + \int_0^\infty \mathcal{H}^{d-1}\Big(\Big(\partial \bigcup \mathcal{B}_\lambda^\geq\Big) \setminus \overline{\{f>\lambda\}}\Big)\,\mathrm{d}\lambda + \operatorname{var} f$$

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High density case

Relative isoperimetric inequality

A is a *John domain* if there is a $K > 0$ and point $x \in A$ such that for any $y \in A$ there is a path γ from x to y with

$$\mathrm{dist}(\gamma(t), A^{\complement}) \geq K^{-1}|\gamma(t) - y|.$$

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Relative isoperimetric inequality

Let A be a John domain and $\mathcal{L}(A \cap E) \leq \mathcal{L}(A)/2$. Then

$$\mathcal{L}(A \cap E)^{\frac{d-1}{d}} \lesssim \mathcal{H}^{d-1}(A \cap \partial E)$$

High density case

Corollary: For a ball or cube B with $\mathcal{L}(B)/4 \leq \mathcal{L}(B \cap E) \leq \mathcal{L}(B)/2$ we have

$$\mathcal{H}^{d-1}(\partial B) \lesssim \mathcal{L}(B)^{\frac{d-1}{d}} \lesssim \mathcal{L}(B \cap E)^{\frac{d-1}{d}} \lesssim \mathcal{H}^{d-1}(B \cap \partial E).$$

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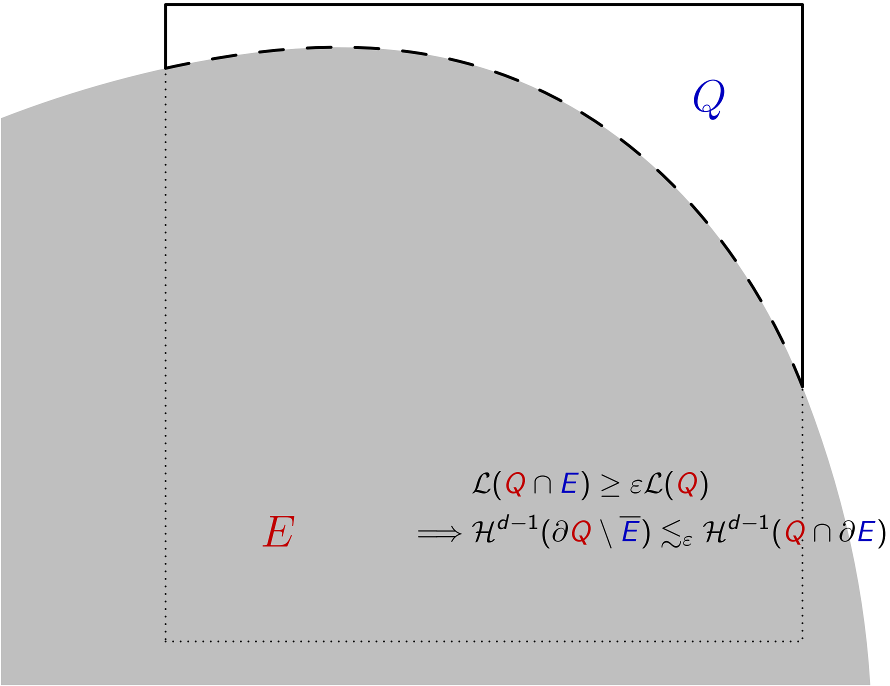
Proposition (High density)

For $\mathcal{L}(B \cap E) \geq \mathcal{L}(B)/2$ we have

$$\mathcal{H}^{d-1}(\partial B \setminus \overline{E}) \lesssim \mathcal{H}^{d-1}(B \cap \partial E).$$



Q
$\mathcal{L}(Q \cap E) \geq \varepsilon \mathcal{L}(Q)$
E
$\Longrightarrow \mathcal{H}^{d-1}(\partial Q \setminus \overline{E}) \lesssim_\varepsilon \mathcal{H}^{d-1}(Q \cap \partial E)$

Proof of high density proposition

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For every $x \in \partial B \setminus \overline{E}$ there is an $\varepsilon > 0$ with

$$\mathcal{L}(B(x,\varepsilon) \cap E) = 0,$$
$$\mathcal{L}(B \cap B(x, \mathrm{diam}(B)) \cap E) \geq \mathcal{L}(B)/2 = 2^{-d-1}\mathcal{L}(B(x, \mathrm{diam}(B)))$$

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Thus $\exists r \in [\varepsilon, \mathrm{diam}(B)]$

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Let $\mathcal{B}$ be the collection of all such balls $B(x,r)$ and apply the Vitali covering. Let $\mathcal{S}$ be the resulting disjoint subset.

Relative isoperimetric inequality

For each $B(x,r) \in \mathcal{S}$ the set $A = B \cap B(x,r)$ is a John domain and thus satisfies the

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$$\min\{\mathcal{L}(A \cap E), \mathcal{L}(A \setminus E)\}^{\frac{d-1}{d}} \lesssim \mathcal{H}^{d-1}(\partial E \cap A)$$

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(Proof of first inequality can be made precise.)

$\mathcal{S}$ Vitali covering of $\partial B \setminus \overline{E}$. We can conclude

$$\begin{aligned}
\mathcal{H}^{d-1}\left(\partial B \setminus \overline{E}\right) &= \mathcal{H}^{d-1}\left(\bigcup \mathcal{B} \cap \partial B \setminus \overline{E}\right) \le \mathcal{H}^{d-1}\left(\bigcup \mathcal{B} \cap \partial B\right) \\
&= \mathcal{H}^{d-1}\left(\bigcup 5\mathcal{S} \cap \partial B\right) \le \sum_{S \in \mathcal{S}} \mathcal{H}^{d-1}(5S \cap \partial B) \\
&\lesssim \sum_{S \in \mathcal{S}} \mathcal{H}^{d-1}(\partial 5S) \lesssim \sum_{S \in \mathcal{S}} \mathcal{H}^{d-1}(\partial S) \\
&\lesssim \sum_{S \in \mathcal{S}} \mathcal{H}^{d-1}(\partial E \cap B \cap S) \le \mathcal{H}^{d-1}(\partial E \cap B)
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\end{aligned}$$

(Proof of fifth step can be made precise.)

High density case

Proposition (High density, general version)

Let $\mathcal{B}$ be a set of balls B with $\mathcal{L}(B \cap E) \geq \varepsilon \mathcal{L}(B)$. Then

$$\mathcal{H}^{d-1}\left(\partial \bigcup \mathcal{B} \setminus \overline{E}\right) \lesssim_\varepsilon \mathcal{H}^{d-1}\left(\bigcup \mathcal{B} \cap \partial E\right).$$

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$$\begin{aligned}
\int_0^\infty \mathcal{H}^{d-1}\left(\left(\partial \bigcup \mathcal{B}_\lambda^{\geq}\right) \setminus \overline{\{f > \lambda\}}\right) \mathrm{d}\lambda
&\lesssim \int_0^\infty \mathcal{H}^{d-1}\left(\bigcup \mathcal{B}_\lambda^{\geq} \cap \partial\{f > \lambda\}\right) \mathrm{d}\lambda \\
&\leq \operatorname{var} f.
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Proof works almost the same as with $\mathcal{B} = \{B\}$ if all balls in $\mathcal{B}$ have the same scale. But we need one extra covering tool from the next section.

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Proof works almost the same as with $\mathcal{B} = \{B\}$ if all balls in $\mathcal{B}$ have the same scale. But we need one extra covering tool from the next section. Then we prove a modified version for each scale separately and add up all scales.

Low density case

Low density case

Have to bound

$$\int_0^\infty \mathcal{H}^{d-1}\Big(\partial \bigcup \mathcal{B}_\lambda^<\Big)\,\mathrm{d}\lambda \lesssim \mathsf{var}\, f,$$

where

$$\mathcal{B}_\lambda^< = \{B : f_B > \lambda,\ \mathcal{L}(B \cap \{f > \lambda\}) < \mathcal{L}(B)/2\}.$$

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I can't :(

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$$\mathrm{M}^{\mathrm{d}} f(x) = \sup_{Q \ni x,\ Q \text{ dyadic}} f_Q .$$

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Definition

Q is maximal for $\lambda < f_Q$ if for all $P \supsetneq Q$ we have $f_P \leq \lambda$.

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Definition

Q is maximal for $\lambda < f_Q$ if for all $P \supsetneq Q$ we have $f_P \leq \lambda$. Given Q, let λ_Q be the smallest such λ.

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Q is maximal for $\lambda < f_Q$ if for all $P \supsetneq Q$ we have $f_P \leq \lambda$. Given Q, let λ_Q be the smallest such λ.

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial \bigcup \mathcal{Q}_\lambda^<) \, \mathrm{d}\lambda &\leq \int_{\mathbb{R}} \sum_{Q \in \mathcal{Q}_\lambda^<} \mathcal{H}^{d-1}(\partial Q) \, \mathrm{d}\lambda \\ &= \int_{\mathbb{R}} \sum_{Q : \tilde{\lambda}_Q < \lambda < f_Q} \mathcal{H}^{d-1}(\partial Q) \, \mathrm{d}\lambda \\ &= \sum_Q (f_Q - \tilde{\lambda}_Q) \mathcal{H}^{d-1}(\partial Q)\end{aligned}$$

where

$$\tilde{\lambda}_Q = \sup\Big\{\lambda_Q, \sup\{\lambda : \mathcal{L}(Q \cap \{f > \tilde{\lambda}_Q\}) \geq 2^{-d-2} \cdot \mathcal{L}(Q)\}\Big\}$$

Proposition

$$(f_Q - \tilde{\lambda}_Q)\mathcal{L}(Q) \lesssim \int_{\mathbb{R}} \sum_{P \subsetneq Q : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\})\, \mathrm{d}\lambda$$

where P is maximal above $\bar{\lambda}_P$ and

$$"\mathcal{L}(P \cap \{f > \bar{\lambda}_P\}) = 2^{-1}\mathcal{L}(P)"$$
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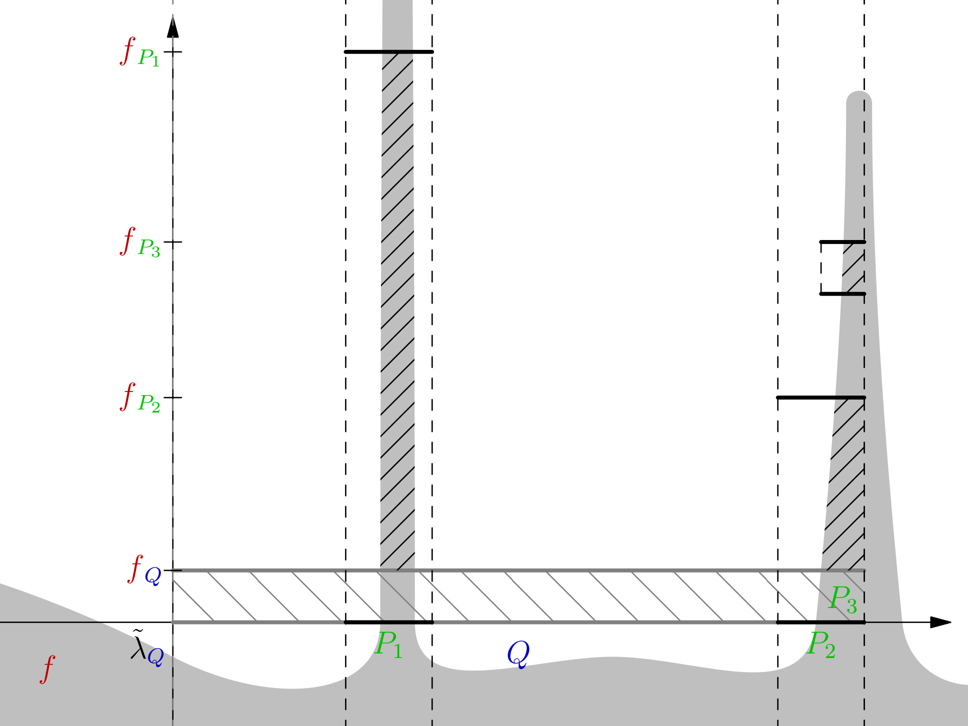
The proof uses a stopping time argument: Start with Q and then iteratively descend into all children P. Stop if $f_P < f_{\mathrm{prt}(P)}$ or $f_P > \tilde{\lambda}_P$. All cubes which don't have a stopping cube as an ancestor will contribute on the right hand side above.



f_{P_1}
f_{P_3}
f_{P_2}
f_Q
$\tilde{\lambda}_Q$
f
P_1
Q
P_2
P_3

$$\sum_{Q}(f_Q - \tilde{\lambda}_Q)\mathcal{H}^{d-1}(\partial Q) \lesssim \int_{\mathbb{R}} \sum_{Q} \mathrm{l}(Q)^{-1} \sum_{P \subsetneq Q : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\})\, \mathrm{d}\lambda$$

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\begin{aligned}
\sum_{Q}(f_Q - \tilde{\lambda}_Q)\mathcal{H}^{d-1}(\partial Q) &\lesssim \int_{\mathbb{R}} \sum_{Q} \mathsf{l}(Q)^{-1} \sum_{P \subsetneq Q : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\})\, \mathrm{d}\lambda \\
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\end{aligned}
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&= \int_{\mathbb{R}} \sum_{P : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\}) \, \mathrm{l}(P)^{-1} \, \mathrm{d}\lambda \\
&\le \int_{\mathbb{R}} \sum_{P : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\})^{\frac{d-1}{d}} \, \mathrm{d}\lambda \\
&\lesssim \int_{\mathbb{R}} \sum_{P : \bar{\lambda}_P < \lambda < f_P} \mathcal{H}^{d-1}(P \cap \partial\{f > \lambda\}) \, \mathrm{d}\lambda \\
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\end{aligned}
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&\leq \int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial\{f > \lambda\}) \,\mathrm{d}\lambda = \operatorname{var} f \qquad \square
\end{aligned}$$

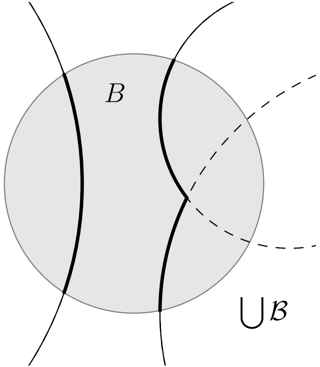
Covering Techniques

Boundary of large balls

Proposition

Let B be a ball and $\mathcal{B}$ be a set of balls C with $\operatorname{diam}(C) \geq K \operatorname{diam}(B)$. Then

$$\mathcal{H}^{d-1}\left(\partial \bigcup \mathcal{B} \cap B\right) \lesssim (1 + K^{-d})\mathcal{H}^{d-1}(\partial B).$$



Proof

Center B in the origin and let $e \in \partial B(0,1)$ be a direction.

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is a Lipschitz graph with constant 1 which thus has perimeter $\lesssim \operatorname{diam}(B)^{d-1} \sim \mathcal{H}^{d-1}(\partial B)$. Take a maximal set of ε-separated directions and the result follows.

Actually this only works if $\operatorname{diam}(C) \geq 2\operatorname{diam}(B)$. For $\operatorname{diam}(C) \geq K\operatorname{diam}(B)$ we cover B by $\sim K^d$ many balls B with $\operatorname{diam}(B) = \operatorname{diam}(B)/2K$, for which we have $\operatorname{diam}(C) \geq 2\operatorname{diam}(B)$ for each $C \in \mathcal{B}$. Then do the argument in each B.

High density, general version

Proposition (High density, general version)

Let $\mathcal{B}$ be a set of balls B with $\mathcal{L}(B \cap E) \geq \varepsilon \mathcal{L}(B)$. Then

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Proposition (High density, single scale version)

Let $\mathcal{B}$ be a set of balls B with $\operatorname{diam}(B) \geq 1$ and $\mathcal{L}(B \cap E) \geq \varepsilon \mathcal{L}(B)$ and let $\mathcal{S}$ be a set of disjoint balls S centered on $\partial \bigcup \mathcal{B} \setminus \overline{E}$ with $\operatorname{diam}(S) \leq 1$ and $\varepsilon \mathcal{L}(S) \leq \mathcal{L}(S \cap \bigcup \mathcal{B} \cap E) \leq (1-\varepsilon)\mathcal{L}(S)$. Then

$$\mathcal{H}^{d-1}\left(\partial \bigcup \mathcal{B} \cap \bigcup 5\mathcal{S} \setminus \overline{E}\right) \lesssim_\varepsilon$$
$$\mathcal{H}^{d-1}\left(\bigcup_{S \in \mathcal{S}} \{x \in S : \operatorname{dist}(x, \bigcup \mathcal{B}^\complement) > \varepsilon \operatorname{diam}(S)\} \cap \partial E\right).$$

Proof of high density, single scale version

$\mathcal{S}$ Vitali covering of $\partial B \setminus \overline{E}$. We can conclude

$$\begin{aligned}
\mathcal{H}^{d-1}\left(\partial B \setminus \overline{E}\right) &= \mathcal{H}^{d-1}\left(\bigcup \mathcal{B} \cap \partial B \setminus \overline{E}\right) \leq \mathcal{H}^{d-1}\left(\bigcup \mathcal{B} \cap \partial B\right) \\
&= \mathcal{H}^{d-1}\left(\bigcup 5\mathcal{S} \cap \partial B\right) \leq \sum_{S \in \mathcal{S}} \mathcal{H}^{d-1}(5S \cap \partial B) \\
&\lesssim \sum_{S \in \mathcal{S}} \mathcal{H}^{d-1}(\partial 5S) \lesssim \sum_{S \in \mathcal{S}} \mathcal{H}^{d-1}(\partial S) \\
&\lesssim \sum_{S \in \mathcal{S}} \mathcal{H}^{d-1}(\partial E \cap B \cap S) \leq \mathcal{H}^{d-1}(\partial E \cap B)
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&\lesssim \sum_{S \in \mathcal{S}} \mathcal{H}^{d-1}(\partial 5S) \lesssim \sum_{S \in \mathcal{S}} \mathcal{H}^{d-1}(\partial S) \\
&\lesssim \sum_{S \in \mathcal{S}} \mathcal{H}^{d-1}(\partial E \cap B \cap S) \leq \mathcal{H}^{d-1}(\partial E \cap B)
\end{aligned}$$

(Proof of fifth step can be made precise.)

Proof of high density, general version

Do Vitali covering $\mathcal{S}$ of $\partial \bigcup \mathcal{B} \setminus \overline{E}$ but only make the balls in $\mathcal{S}_n = \{S \in \mathcal{S} : 2^n \leq \text{diam}(S) < 2^{n+1}\}$ disjoint.

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$$
\begin{aligned}
&\mathcal{H}^{d-1}\left(\partial \bigcup \mathcal{B} \setminus \overline{E}\right) \\
&\leq \sum_{n\in\mathbb{Z}} \mathcal{H}^{d-1}\left(\partial \bigcup \mathcal{B} \cap \bigcup 5\mathcal{S}_n \setminus \overline{E}\right) \\
&\lesssim \sum_{n\in\mathbb{Z}} \mathcal{H}^{d-1}\left(\bigcup_{S\in\mathcal{S}_n} \{x \in S : \varepsilon 2^n < \operatorname{dist}(x, \bigcup \mathcal{B}^{\complement}) < 2^n\} \cap \partial E\right) \\
&\lesssim |1 - \log \varepsilon| \mathcal{H}^{d-1}\left(\bigcup \mathcal{B} \cap \partial E\right).
\end{aligned}
$$

Dyadic cubes to general cubes

Want to show $\operatorname{var} \mathrm{M} f \lesssim \operatorname{var} f$ for $\mathrm{M} f(x) = \sup_{Q \ni x} f_Q$, where the supremum is taken over all cubes.

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Vitali covering

For any (finite) set of balls $\mathcal{B}$ For any (finite) set of balls $\mathcal{B}$, there is a subset $\mathcal{S} \subset \mathcal{B}$ of disjoint balls with

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Question

For any (finite) set of balls $\mathcal{B}$, is there a subset $\mathcal{S} \subset \mathcal{B}$ of disjoint balls with

$$\mathcal{H}^{d-1}\Big(\partial \bigcup \mathcal{B}\Big) \lesssim \sum_{S \in \mathcal{S}} \mathcal{H}^{d-1}(\partial S)?$$

I think not.

Problem with the Vitali covering-proof:

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Proposition (Vitali (replacement) for perimeter)

For any (finite) set of balls $\mathcal{B}$ there is a subset $\mathcal{S} \subset \mathcal{B}$ of balls such that for any $S_1, S_2 \in \mathcal{S}$ with $S_1 \neq S_2$ we have

$$\mathcal{L}(S_1 \cap S_2) \le \frac{\min\{\mathcal{L}(S_1), \mathcal{L}(S_2)\}}{2}$$

and

$$\mathcal{H}^{d-1}\Big(\partial \bigcup \mathcal{B}\Big) \lesssim \sum_{S \in \mathcal{S}} \mathcal{H}^{d-1}(\partial S).$$

(The factor $1/2$ can be made arbitrarily small.)

Proof of Vitali for perimeter

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be the set of balls already covered in earlier steps. Set

$$\mathcal{B}_n = \{B \in \mathcal{B} \setminus \mathcal{C}_n : 2^{-n-1} < \operatorname{diam}(B) \leq 2^{-n}\}.$$

Let $\mathcal{S}_n$ be a maximal disjoint subset of $\mathcal{B}_n$ such that for all $S, T \in \mathcal{S}_n$ we have $\mathcal{L}(S \cap T) \leq \min\{\mathcal{L}(S), \mathcal{L}(T)\}/2$.

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$$\mathcal{L}(S \cap T) \leq \min\{\mathcal{L}(S), \mathcal{L}(T)\}/2.$$

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$$\mathcal{L}(B \cap S) \geq \frac{\min\{\mathcal{L}(B), \mathcal{L}(S)\}}{2} \geq \frac{\mathcal{L}(B(0, 2^{-n-2})}{2} \geq 2^{-n-1}\mathcal{L}(B).$$

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Proposition (High density)

Let $\mathcal{B}$ be a set of balls B with $\mathcal{L}(B \cap E) \geq \varepsilon\mathcal{L}(B)$. Then

$$\mathcal{H}^{d-1}\left(\partial \bigcup \mathcal{B} \setminus \overline{E}\right) \lesssim_\varepsilon \mathcal{H}^{d-1}\left(\bigcup \mathcal{B} \cap \partial E\right).$$

$$\mathcal{H}^{d-1}\left(\partial\bigcup\mathcal{B}\right) \leq \mathcal{H}^{d-1}\left(\partial\bigcup\mathcal{B}\setminus\bigcup\{\overline{S} : S \in \mathcal{S}\}\right) + \sum_{S\in\mathcal{S}}\mathcal{H}^{d-1}(\partial S)$$

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&\lesssim \sum_{S\in\mathcal{S}}\mathcal{H}^{d-1}\left(\bigcup\{B\in\mathcal{B} : \mathcal{L}(B\cap S)\geq 2^{-n-1}\mathcal{L}(B)\}\cap\partial S\right)\\
&\leq \sum_{S\in\mathcal{S}}\mathcal{H}^{d-1}(\partial S).
\end{aligned}$$

Recall the strategy for dyadic

Want to estimate

$$\sum_{Q \text{ dyadic}} (f_Q - \tilde{\lambda}_Q)\mathcal{H}^{d-1}(\partial Q) \lesssim \operatorname{var} f.$$

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$$(f_Q - \tilde{\lambda}_Q)\mathcal{H}^{d-1}(\partial Q) \cdot \mathrm{l}(Q) \lesssim \int_{\mathbb{R}} \sum_{P \in \mathcal{D}^{\lambda}(Q)} \mathcal{L}(P \cap \{f > \lambda\})\, \mathrm{d}\lambda,$$

where $\mathcal{D}^{\lambda}(Q)$ is the set of dyadic cubes P with base cube Q such that $\bar{\lambda}_P < \lambda < f_P$.

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Do Fubini. Each dyadic cube P on the RHS will appear with a factor $\mathrm{l}(Q)^{-1}$ for each dyadic parent of P.

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Do Fubini. Each dyadic cube P on the RHS will appear with a factor $\mathrm{l}(Q)^{-1}$ for each dyadic parent of P. Geometric sum will coverge and yield $\mathcal{H}^{d-1}(\partial P)$ and relative isoperimetric inequality will turn it into into $\mathcal{H}^{d-1}(\partial\{f > \lambda\} \cap P)$. By disjonintness the right hand side then is $\operatorname{var} f$.

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The following weaker assumptions are actually enough.

1. There is a small $\varepsilon > 0$ such that for each $\lambda \in \mathbb{R}$ the cubes in $\{(1-\varepsilon)P : P \in \mathcal{D}^\lambda\}$ have bounded overlap.

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2. For each $P \in \mathcal{D}^\lambda$ we only get a contribution from a bounded number of cubes Q with the same scale.

Strategy for general cubes

Split the cubes $\mathcal{Q}_\lambda = \{Q : f_Q > \lambda\}$ into $\mathcal{Q}_\lambda^{>} \cup \mathcal{Q}_\lambda^{>,2} \cup \mathcal{Q}_\lambda^{\leq}$, where

$$\begin{aligned}
\mathcal{Q}_\lambda^{>} &= \{Q \in \mathcal{Q}_\lambda : \mathcal{L}(\{f > \lambda\} \cap Q) > 2^{-1}\mathcal{L}(Q)\} \\
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\mathcal{Q}_\lambda^{\leq} &= \mathcal{Q}_\lambda \setminus \mathcal{Q}_\lambda^{>} \setminus \mathcal{Q}_\lambda^{>,2}.
\end{aligned}$$

Then by the *high density* argument
$\mathcal{H}^{d-1}(\partial \bigcup \mathcal{Q}_\lambda^{>,2}) \lesssim \mathcal{H}^{d-1}(\partial \bigcup \mathcal{Q}_\lambda^{>}) \lesssim \mathcal{H}^{d-1}(\partial\{f > \lambda\})$.
To $\mathcal{Q}_\lambda^{\leq}$ apply the Vitali covering argument for the boundary.

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To $\mathcal{Q}_\lambda^\leq$ apply the Vitali covering argument for the boundary. This can actually be done in a consistent way through all $\lambda \in \mathbb{R}$, so that we obtain a set S such that

$$\int_0^\infty \mathcal{H}^{d-1}\Big(\partial \bigcup \mathcal{Q}_\lambda^\leq\Big)\,\mathrm{d}\lambda \lesssim \sum_{Q \in \mathcal{S}} (f_Q - \tilde{\lambda}_Q)\mathcal{H}^{d-1}(\partial Q).$$

For any $Q_1, Q_2 \in \mathcal{S}$ with $\mathrm{diam}(Q_1) \leq \mathrm{diam}(Q_2)$ we have

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For cubes $Q_1, Q_2 \in \mathcal{S}$ the cubes in $\mathcal{D}^\lambda(Q_1), \mathcal{D}^\lambda(Q_2)$ can have bad overlap. So we run again a Vitali-type argument on $\mathcal{D}^\lambda$ to select a set of almost disjoint representatives $\mathcal{F}^\lambda$.

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What goes wrong: Let $Q_1, Q_2 \in \mathcal{S}$ be intersecting and with $\operatorname{diam}(Q_1) \ll \operatorname{diam}(Q_2)$. Then there might be a $P \in \mathcal{D}^\lambda(Q_2)$ with $\operatorname{diam}(Q_1) \ll \operatorname{diam}(P)$ which covers all cubes in $\mathcal{D}^\lambda(Q_2)$.

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$$(f_{Q_2} - \tilde{\lambda}_{Q_2})\mathcal{H}^{d-1}(\partial Q_2) \cdot \mathrm{l}(Q_2) \lesssim \int_{\mathbb{R}} \sum_{P\in\mathcal{F}^\lambda(Q)} \mathcal{L}(P \cap \{f > \lambda\})\,\mathrm{d}\lambda,$$

because we must have $\operatorname{diam}(P) \lesssim \operatorname{diam}(Q_2)$ for all $P \in \mathcal{F}^\lambda(Q_2)$ for the geometric sum to converge.

Vitali covering creates an actual disjoint cover, but we only need

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Fix: Take $(1-\varepsilon)P$ instead. Then in the above situation $(1-\varepsilon)P$ is disjoint from any cube in $\mathcal{D}^\lambda(Q_2)$ and we can still use them.

Vitali covering creates an actual disjoint cover, but we only need

1. There is a small $\varepsilon > 0$ such that for each $\lambda \in \mathbb{R}$ the cubes in $\{(1-\varepsilon)P : P \in \mathcal{F}^\lambda\}$ have bounded overlap.

Fix: Take $(1-\varepsilon)P$ instead. Then in the above situation $(1-\varepsilon)P$ is disjoint from any cube in $\mathcal{D}^\lambda(Q_2)$ and we can still use them. If we are not in the situation $\operatorname{diam}(Q_1) \ll \operatorname{diam}(Q_2)$ then all cubes have a similar scale and we are safe to do Vitali covering and we just lose some constants.

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Invoke the coarea formula

$$\operatorname{var} \mathrm{M}f = \int_0^\infty \mathcal{H}^{d-1}\Big(\partial \bigcup \mathcal{Q}_\lambda\Big)\,\mathrm{d}\lambda,$$

where $\mathcal{Q}_\lambda = \{Q : f_Q > \lambda\}$.

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from which the coarea formula yields var f.

Do a Vitali type covering to find a set S of dyadic like cubes with

$$\int_0^\infty \mathcal{H}^{d-1}\Big(\partial \bigcup \mathcal{Q}_{\lambda}^{\leq}\Big)\,\mathrm{d}\lambda \lesssim \sum_{Q\in\mathcal{S}} (f_Q - \tilde{\lambda}_Q)\mathcal{H}^{d-1}(\partial Q).$$

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Prove a bound

$$(f_Q - \tilde\lambda_Q)\mathcal{H}^{d-1}(\partial Q)\cdot \mathrm{l}(Q) \lesssim \int_{\mathbb{R}} \sum_{P\in\mathcal{D}^\lambda(Q)} \mathcal{L}(P\cap\{f>\lambda\})\,\mathrm{d}\lambda.$$

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Do a Vitali type covering to find a set S of dyadic like cubes with

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For each $\lambda \in \mathbb{R}$ do a Vitali type covering to extract almost disjoint cubes $\mathcal{F}^\lambda$ from $\bigcup_{Q\in\mathcal{S}} \mathcal{D}^\lambda(Q)$. Change the order of summation, have a geometric sum converge, apply the relative isoperimetric inequality and use almost disjointness

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For each $\lambda\in\mathbb{R}$ do a Vitali type covering to extract almost disjoint cubes $\mathcal{F}^\lambda$ from $\bigcup_{Q\in\mathcal{S}}\mathcal{D}^\lambda(Q)$. Change the order of summation, have a geometric sum converge, apply the relative isoperimetric inequality and use almost disjointness and the coarea formula to recover var f.

This proof works for

- dyadic maximal function [2]

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- uncentered Hardy-Littlewood maximal function if f is characteristic function [1]
- Hardy-Littlewood fractional maximal function, both uncentered and centered [3]

[1] Julian Weigt. "Variation of the uncentered maximal characteristic Function". In: *arXiv e-prints* (Apr. 2020). to appear in: Rev. Mat. Iber., arXiv:2004.10485. arXiv: 2004.10485 [math.CA].

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Thank you