Higher Dimensional Techniques for the Regularity of Maximal Functions

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For $f : \mathbb{R}^n \to \mathbb{R}$ the centered Hardy-Littlewood maximal function is defined by

$$\mathrm{M^c} f(x) = \sup_{r>0} f_{B(x,r)} \quad \text{with} \qquad f_{B(x,r)} = \frac{1}{\mathcal{L}(B(x,r))} \int_{B(x,r)} |f|.$$

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$$\|\mathrm{M^c} f\|_{L^p(\mathbb{R}^n)} \leq C_{n,p} \|f\|_{L^p(\mathbb{R}^n)} \qquad \text{if and only if } p > 1$$

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Juha Kinnunen (1997):

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Question (Hajłasz and Onninen 2004)

Is it true that

$$\|\nabla \mathrm{M^c} f\|_{L^1(\mathbb{R}^n)} \leq C_n \|\nabla f\|_{L^1(\mathbb{R}^n)}?$$

For $e \in \mathbb{R}^n$ by the sublinearity of M^{c} Kinnunen proved

$$|\nabla \mathrm{M}^{\mathrm{c}} f(x)| \leq \mathrm{M}^{\mathrm{c}}|\nabla f|(x).$$

Thus by the Hardy-Littlewood maximal function theorem for $p > 1$

$$\|\nabla \mathrm{M}^{\mathrm{c}} f\|_{L^p(\mathbb{R}^n)} \lesssim \|\mathrm{M}^{\mathrm{c}}(|\nabla f|)\|_{L^p(\mathbb{R}^n)} \lesssim \|\nabla f\|_{L^p(\mathbb{R}^n)}$$

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In 2002 Tanaka proved

$$\|\nabla \widetilde{\mathrm{M}} f\|_1 \leq 2\|\nabla f\|_1$$

for the uncentered maximal function of a function $f : \mathbb{R} \to \mathbb{R}$. The proof depends strongly on one-dimensional geometry.

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The corresponding Hardy-Littlewood theorem is is

$$\|\mathrm{M}_\alpha f\|_{L^{\frac{pn}{n-\alpha p}}(\mathbb{R}^n)} \leq C_{n,\alpha,p} \|f\|_{L^p(\mathbb{R}^n)}$$

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if and only if $p > 1$. The corresponding regularity bound is

$$\|\nabla \mathrm{M}_{\alpha} f\|_{L^{\frac{pn}{n-\alpha p}}(\mathbb{R}^n)} \leq C_{n,\alpha,p} \|\nabla f\|_{L^p(\mathbb{R}^n)}.$$

Introduction: Progress

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There are more related bounds, bounds on other maximal operators,. . . .
For example: Continuity of the operator given by $f \mapsto \nabla \mathrm{M} f$ on $W^{1,1}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$. This is a stronger property than boundedness.

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- fractional maximal operator
- cube maximal operator

Proof: Reformulation and decomposition

Coarea formula

$$\|\nabla f\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial\{x \in \mathbb{R}^n : f(x) > \lambda\})\,\mathrm{d}\lambda$$

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Superlevel sets

$$\{x \in \mathbb{R}^n : \mathrm{M}f(x) > \lambda\} = \bigcup\{B : f_B > \lambda\}$$

for *uncentered* maximal operators.

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Decomposition of the boundary

Denote

$$\mathcal{B}_\lambda^{<} = \{B : f_B > \lambda,\ \mathcal{L}(B \cap \{f > \lambda\}) < 2^{-n-1}\mathcal{L}(B)\}$$

and $\mathcal{B}_\lambda^{\geq}$ accordingly.

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We have

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Since $\mathrm{M}f(x) \geq f(x)$ for a.e. $x \in \mathbb{R}^d$ we have

$$\partial\{\mathrm{M}f > \lambda\} \subset (\partial\{\mathrm{M}f > \lambda\} \setminus \overline{\{f > \lambda\}}) \cup \partial\{f > \lambda\}.$$

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Decomposition

$$\int_{\mathbb{R}^d} |\nabla \mathrm{M}f| \leq \int_0^\infty \mathcal{H}^{d-1}\Big(\partial \bigcup \mathcal{B}_\lambda^{\geq} \setminus \overline{\{f > \lambda\}}\Big)\,\mathrm{d}\lambda + \int_0^\infty \mathcal{H}^{d-1}\Big(\partial \bigcup \mathcal{B}_\lambda^{<}\Big)\,\mathrm{d}\lambda$$

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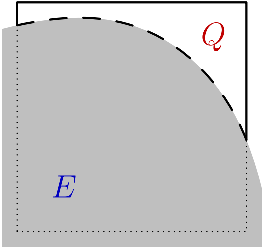
The plan is to estimate each summand separately by $\int_0^\lambda \mathcal{H}^{d-1}(\partial\{f > \lambda\})\,\mathrm{d}\lambda$.

Proof: High density case $\mathcal{B}_\lambda^{\geq}$

Proposition

For Q, E with $\mathcal{L}(Q \cap E) \geq 2^{-n-1}\mathcal{L}(Q)$ we have

$$\mathcal{H}^{d-1}(\partial Q \setminus \overline{E}) \lesssim \mathcal{H}^{d-1}(Q \cap \partial E)$$



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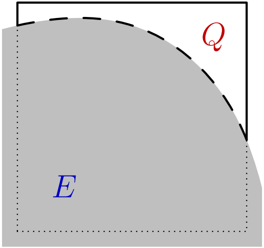

relative isoperimetric inequality

If $\mathcal{L}(Q \cap E) \leq \mathcal{L}(Q)/2$ then

$$\mathcal{L}(Q \cap E)^{n-1} \lesssim \mathcal{H}^{d-1}(Q \cap \partial E)^n$$

Corollary

For a set $\mathcal{Q}$ of dyadic cubes Q with $\mathcal{L}(Q \cap E) \geq 2^{-n-1}\mathcal{L}(Q)$ we have

$$\mathcal{H}^{d-1}\Big(\partial \bigcup \mathcal{Q} \setminus \overline{E}\Big) \lesssim \mathcal{H}^{d-1}\Big(\bigcup \mathcal{Q} \cap \partial E\Big).$$

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For a set $\mathcal{B}$ of balls B with $\mathcal{L}(B \cap E) \geq 2^{-n-1}\mathcal{L}(B)$ we have

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Proof: Low density case $\mathcal{B}_\lambda^<$

$$\int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial \bigcup \mathcal{Q}_\lambda^<)\,\mathrm{d}\lambda \le \sum_{Q \text{ dyadic}} (f_Q - \lambda_Q)\mathcal{H}^{d-1}(\partial Q)$$

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Proposition

$$(f_Q - \lambda_Q)\mathcal{L}(Q) \lesssim \int_{\mathbb{R}} \sum_{P \subsetneq Q : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\})\,\mathrm{d}\lambda$$

where P is maximal above $\bar{\lambda}_P$ and

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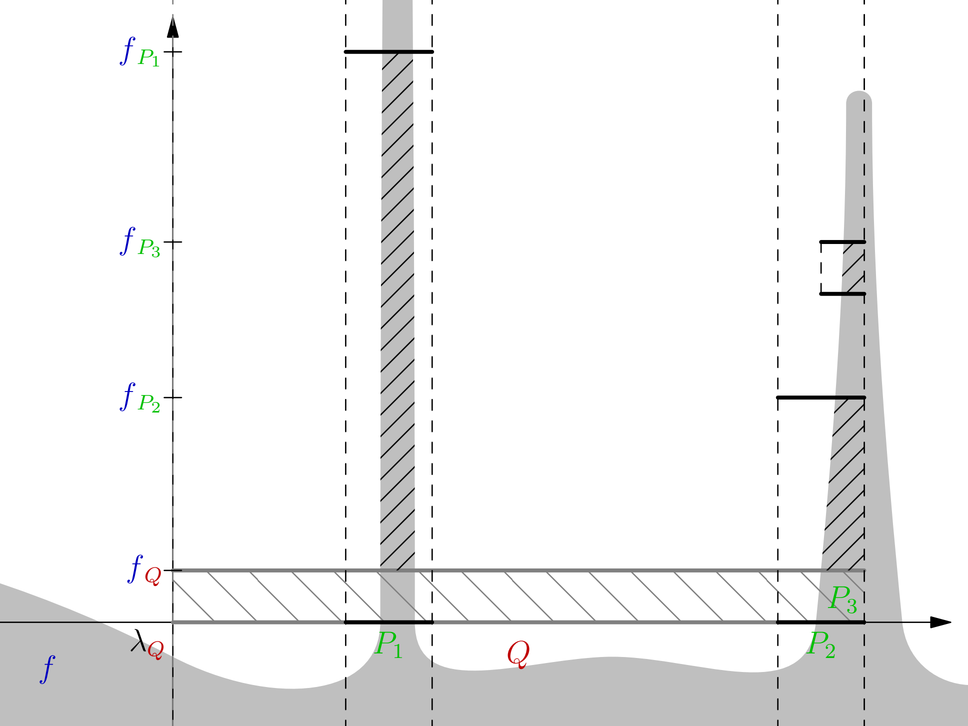

f_{P_1}
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λ_Q
f
P_1
Q
P_2
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Then we change the order of summation, use the convergence of a geometric sum and apply the relative isoperimetric inequality to P. We recover $\|\nabla f\|_1$ on the right hand side.

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Proposition (Vitali for perimeter)

For any (finite) set of cubes $\mathcal{Q}$ there is a subset $\mathcal{S} \subset \mathcal{Q}$ of disjoint cubes such that

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The Vitali covering for the perimeter also works for balls, however we do not have the earlier bound on $(f_Q - \lambda_Q)\mathcal{L}(Q)$ for balls.

Proof: Low density case $\mathcal{B}_\lambda^<$, fractional

$1 \leq \alpha$ [Kinnunen + Saksman, Carneiro + Madrid]

$$\|\nabla \mathrm{M}_\alpha f\|_{\frac{n}{n-\alpha}} \lesssim \|\mathrm{M}_{\alpha-1} f\|_{\frac{n}{n-\alpha}} \lesssim \|f\|_{\frac{n}{n-1}} \lesssim \|\nabla f\|_1.$$

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where $\mathrm{M}_{\alpha,-1}$ is a replacement operator for $\mathrm{M}_{\alpha-1}$ if $0 < \alpha < 1$. Can be bounded using low density arguments from the dyadic proof and extra flexibility coming from $\alpha > 0$.

Thank you