Regularity of maximal functions and a Vitali covering lemma for the boundary

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The Hardy-Littlewood maximal operator

For $f : \mathbb{R}^n \to \mathbb{R}$ the centered Hardy-Littlewood maximal function is defined by

$$\mathrm{M}^{\mathrm{c}} f(x) = \sup_{r>0} f_{B(x,r)} \quad \text{with} \quad f_{B(x,r)} = \frac{1}{\mathcal{L}(B(x,r))} \int_{B(x,r)} |f|.$$

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Lemma (Vitali covering lemma)

Let $\mathcal{B}$ *be a bounded set of balls. Then it has a subset* $\tilde{\mathcal{B}} \subset \mathcal{B}$ *of pairwise disjoint balls such that*

$$\bigcup \mathcal{B} \subset \bigcup \{5B : B \in \tilde{\mathcal{B}}\}.$$

Gradient version of the HL theorem

Theorem (Kinnunen 1997)

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Mostly solved in one dimension, few results in higher dimensions. Question is interesting for uncentered Hardy-Littlewood maximal function

$$\mathrm{M}f(x) = \sup_{B \ni x} f_B$$

and other maximal operators.

Coarea formula and Vitali covering for the boundary

Coarea formula

$$\|\nabla f\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}} \mathcal{H}^{n-1}(\partial\{x \in \mathbb{R}^n : f(x) > \lambda\})\,\mathrm{d}\lambda$$

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Covering lemma for the boundary

Lemma (Vitali covering lemma)

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$$\mathcal{L}\Big(\bigcup \mathcal{B}\Big) \lesssim_n \mathcal{L}\Big(\bigcup \tilde{\mathcal{B}}\Big).$$

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Theorem (Covering lemma for boundary, W. 2025)

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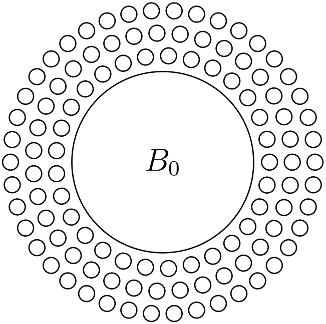
$$\mathcal{H}^{n-1}\Big(\partial \bigcup \mathcal{B}\Big) \lesssim_n \mathcal{H}^{n-1}\Big(\partial \bigcup \tilde{\mathcal{B}}\Big).$$

About the proof

The Vitali selection strategy doesn't work for boundary:



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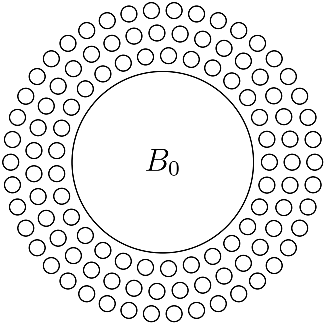

As it turns out, the Besicovitch covering theorem strategy combined with another geometric proposition does, however.

Hybrids

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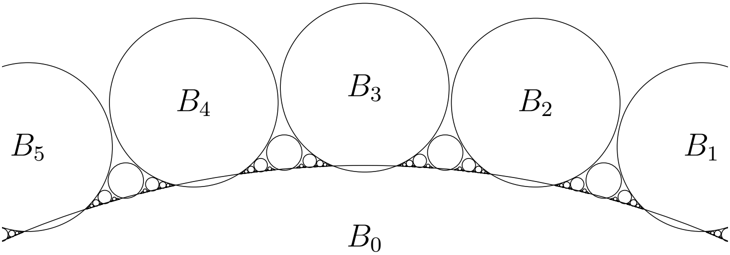

Hybrids

Theorem (W. 2025)

For each $\mathcal{B}$ and any $\varepsilon > 0$ exists a subset $\tilde{\mathcal{B}} \subset \mathcal{B}$ such that for any distinct $B_1, B_2 \in \tilde{\mathcal{B}}$ we have

$$\mathcal{L}(B_1 \cap B_2) \leq \varepsilon \min\{\mathcal{L}(B_1), \mathcal{L}(B_2)\}$$

and with

$$\mathcal{L}\Big(\bigcup \mathcal{B}\Big) \lesssim \mathcal{L}\Big(\bigcup \tilde{\mathcal{B}}\Big), \quad \mathcal{H}^{n-1}\Big(\partial \bigcup \mathcal{B}\Big) \lesssim_n \varepsilon^{-\frac{n-1}{n+1}} \mathcal{H}^{n-1}\Big(\partial \bigcup \tilde{\mathcal{B}}\Big).$$

The rate $\varepsilon^{-\frac{n-1}{n+1}}$ is sharp.

"Application" to maximal operator in characteristic function case

Theorem (W. 2020)

For $E \subset \mathbb{R}^n$ we have

$$\mathrm{var}(\mathrm{M}(1_E)) \lesssim_n \mathrm{var}(1_E).$$

For a partial proof recall

Relative isoperimetric inequality

$$\min\{\mathcal{L}(B \cap E), \mathcal{L}(B \setminus E)\}^{n-1} \lesssim_n \mathcal{H}^{n-1}(B \cap \partial E)^n.$$

Partial proof

We have $f = 1_E$ with $E \subset \mathbb{R}^n$. Note, $f_B = \mathcal{L}(E \cap B)/\mathcal{L}(B)$ and for $0 \leq \lambda \leq 1$ define

$$\mathcal{B}_\lambda := \{B : f_B = \lambda\}, \qquad \text{so that} \quad \{\mathrm{M}f \geq \lambda\} = \bigcup \mathcal{B}_\lambda.$$

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Apply covering lemma for the boundary: $\tilde{\mathcal{B}}_\lambda$. Then for $0 < \lambda \leq 1/2$

$$\begin{aligned} \mathcal{H}^{n-1}(\partial\{\mathrm{M}f > \lambda\}) &\lesssim_n \sum_{B \in \tilde{\mathcal{B}}_\lambda} \mathcal{H}^{n-1}(\partial B) \sim_n \sum_{B \in \tilde{\mathcal{B}}_\lambda} \mathcal{L}(B)^{\frac{n-1}{n}} \\ &= \lambda^{-\frac{n-1}{n}} \sum_{B \in \tilde{\mathcal{B}}_\lambda} \mathcal{L}(B \cap E)^{\frac{n-1}{n}} \lesssim_n \lambda^{-\frac{n-1}{n}} \sum_{B \in \tilde{\mathcal{B}}_\lambda} \mathcal{H}^{n-1}(B \cap \partial E) \\ &\leq \lambda^{-\frac{n-1}{n}} \mathcal{H}^{n-1}(\partial E) = \lambda^{-\frac{n-1}{n}} \operatorname{var} f, \end{aligned}$$

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$$\int_0^{\frac{1}{2}} \mathcal{H}^{n-1}(\partial\{\mathrm{M}f > \lambda\}) \, \mathrm{d}\lambda \lesssim_n \operatorname{var} f.$$

But...

For $1/2 \leq \lambda \leq 1$ we use

Proposition

Let $\mathcal{B}_\lambda$ be a set of balls B with $\mathcal{L}(B \cap E) \geq \frac{1}{2}\mathcal{L}(B)$. Then

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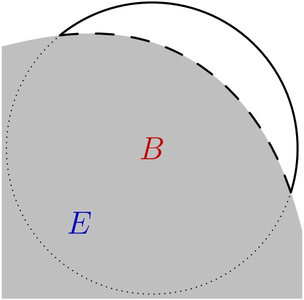

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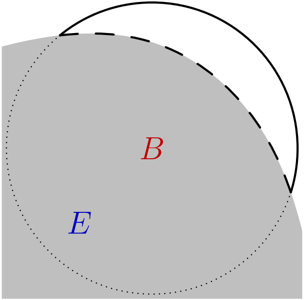

(This also happens to be the previously mentioned proposition we used to prove the covering lemma for the boundary.)

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- But, using the covering lemma for the boundary we can now (more easily) remove the term $|\log \lambda|$ in the above proposition.

Thank you