Alberti representations, rectifiability, PDEs and multilinear Kakeya

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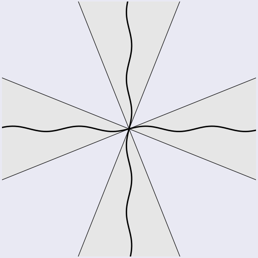
Alberti representations

Definition (Alberti representation)

An *Alberti representation* of a finite measure μ on $\mathbb{R}^d$ is a finite measure η on the space of all Lipschitz curves $\Gamma(\mathbb{R}^d)$ on $\mathbb{R}^d$ such that

$$\mu \ll \int_{\Gamma(\mathbb{R}^n)} \mathcal{H}^1\restriction_{\gamma} \mathrm{d}\eta(\gamma) = A \mapsto \int_{\Gamma(\mathbb{R}^n)} \mathcal{H}^1(A \cap \gamma)\, \mathrm{d}\eta(\gamma).$$

Alberti representations $\eta_1, \dots, \eta_n$ are *independent* if for $(\eta_1, \dots, \eta_n)$-almost any tuple of curves $(\gamma_1, \dots, \gamma_n) \in \Gamma(\mathbb{R}^d)^n$, the γ_i travel within linearly independent cones.



(everything modulo countable decompositions)

Euclidean case

Example

1. $\mathbb{R}^n$ ($\mathcal{H}^n\restriction_{\mathbb{R}^n\times\{0\}^{d-n}}$) has n independent Alberti representations.
2. For $f:\mathbb{R}^n\to\mathbb{R}^d$ with near constant gradient of full rank, $f(\mathbb{R}^n)$ is n-rectifiable.
3. By Rademacher's theorem a Lipschitz image of $\mathbb{R}^n$ has n independent Alberti representations.
4. An n-rectifiable set (a countable union of Lipschitz images of $\mathbb{R}^n$) has n independent Alberti representations.

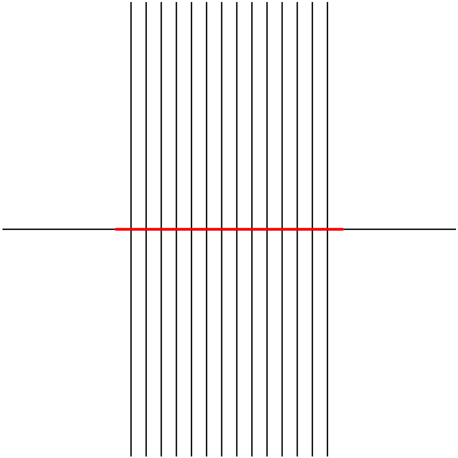
Converse:

Theorem

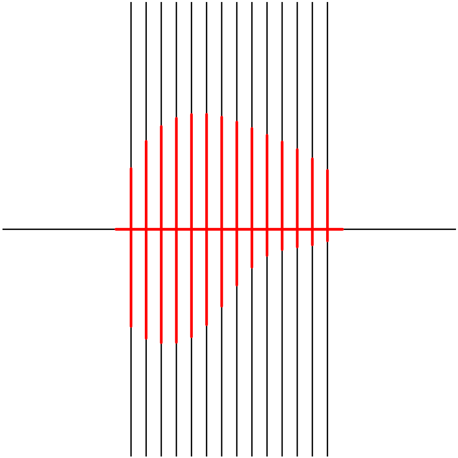
A set with n independent Alberti representations is n-rectifiable.

Axes-parallel case

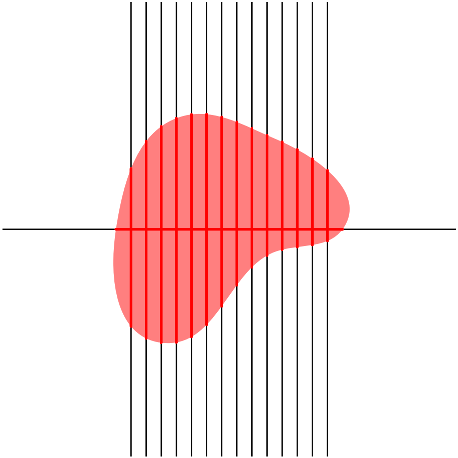
Axes-parallel case



Axes-parallel case



Axes-parallel case



Euclidean case

Theorem (DePhilippis-Rindler, Ann. of Math. (2016), divergence case)

Let $\mathbf{T}$ be an $\mathbb{R}^{n\times n}$-valued finite measure on $\mathbb{R}^n$ such that $\operatorname{div}\mathbf{T}$ is a finite measure. Then the restriction of $\mathbf{T}$ to those points, where its polar $\mathbf{T}/|\mathbf{T}|$ is an invertible matrix, is absolutely continuous with respect to Lebesgue measure.

This applies for example to a tuple of n independent Alberti representations since

$$\operatorname{div}\Big(\int \dot\gamma \mathcal{H}^1\restriction_\gamma \mathrm{d}\eta(\gamma)\Big) = \int \operatorname{div}(\dot\gamma\mathcal{H}^1\restriction_\gamma)\,\mathrm{d}\eta(\gamma) = 0.$$

Euclidean space

Theorem (Besicovitch projection theorem)

A set $E \subset \mathbb{R}^d$ is purely n-unrectifiable if and only if $\mathcal{H}^{d-n}$-almost every projection of E to an n-plane has $\mathcal{H}^n$-measure 0.

The Besicovitch projection theorem together with DePhilippis-Rindler can be used to prove that every $E \subset \mathbb{R}^d$ with n Alberti representations is n-rectifiable.

What about metric space where we do not have such projection theorem?

Metric space

Theorem (Bate-Li, 2014)

A set with n independent Alberti representations which has positive lower density almost everywhere is n-rectifiable.

Theorem (Bate-W., 2025)

A set with n independent Alberti representations has positive lower density almost everywhere.

$\Longrightarrow$

Theorem (Bate-W., 2025)

A set with n independent Alberti representations is n-rectifiable.

Tool: quantitative regularity

Theorem (Bate-W., 2025, extracted from DePhilippis-Rindler, with significant contributions from Tuomas Orponen)

Let $\mathbf{T}$ *be an* $\mathbb{R}^{n\times n}$*-valued and* ν *be a nonnegative finite measure on* $B(0,1) \subset \mathbb{R}^n$*. Then for any* $1 \leq p < \frac{n}{n-1}$ *we can decompose* $\nu = g + b$ *with*

$$\|g\|_p \lesssim_p \|\nu\|_1 + \|\operatorname{div}\mathbf{T}\|_1,$$

$$\|b\|_1 \lesssim_p (\|\nu\|_1 + \|\operatorname{div}\mathbf{T}\|_1)^{\frac{1}{p}} \|\mathrm{Id}\nu - \mathbf{T}\|_1^{\frac{1}{p'}}.$$

Corollary

Under the above assumptions, if $\|\operatorname{div}\mathbf{T}\|_1 \lesssim \|\nu\|_1$ *and* $\|\mathrm{Id}\nu - \mathbf{T}\|_1 \ll \|\nu\|_1$ *then* ν *satisfies a reverse Hölder inequality up to a small* L^1*-error. In particular,* $\operatorname{supp}(\nu) \gtrsim 1$*.*

Proof sketch of lower density in metric space

1. Take a point $x \in E \subset X$.
2. Zoom in and filter so that $\mathcal{H}^n\restriction_{E \cap B(x,r)}$ becomes L^1-close to its n Alberti representations.
3. Apply quantitative regularity result with $\nu = \varphi_\#(\mathcal{H}^n\restriction_{E \cap B(x,r)})$ and $\mathbf{T}$ being the $\varphi_\#$-pushforward of the Alberti representations. This yields lower density on $\mathbb{R}^n$ and thus on X.

In fact, 2 is not really possible like that because we do not have the Lebesgue density theorem on a metric space. What we do actually to achieve this is an induction on scales argument that uses lower density from the previous lower scale.
Also how to filter exactly requires some care, for example the curves in the metric space are not full curves, and thus do not have finite divergence.
And in order to control the divergence after cutting off we actually need to do a smooth cutoff instead of just by $B(x,r)$.

Multilinear Kakeya

Theorem (Guth, 2010, Acta Mathematica)

For $i = 1, ..., n$ and j let T_i^j be a straight tube in $\mathbb{R}^n$ that approximately points in direction e_i and denote by r_i^j its radius. Then

$$\Big\|\Big(\prod_{i=1}^{n}\sum_{j} a_i^j 1_{T_i^j}\Big)^{\frac{1}{n}}\Big\|_{\frac{n}{n-1}} \lesssim \Big(\prod_{i=1}^{n}\sum_{j} a_i^j (r_i^j)^{n-1}\Big)^{\frac{1}{n}}.$$

The previous inequality is scaling invariant and thus equivalent to

$$\Big\|\Big(\prod_{i=1}^{n}\int \mathcal{H}^1\restriction_{\gamma} \mathrm{d}\eta_i(\gamma)\Big)^{\frac{1}{n}}\Big\|_{L^{\frac{n}{n-1}}(B(0,1))} \lesssim \Big\|\sum_{i=1}^{n}\int \mathcal{H}^1\restriction_{\gamma} \mathrm{d}\eta_i(\gamma)\Big\|_{L^1(B(0,1))}$$

with η_i supported on straight lines that approximately point in direction e_i after rescaling.

Strange multilinear Kakeya

Corollary

Under the above assumptions, if $\|\operatorname{div}\mathbf{T}\|_1 \lesssim \|\mathbf{T}\|_1$ *and* $\|\mathrm{Id}|\mathbf{T}| - \mathbf{T}\|_1 \ll \|\mathbf{T}\|_1$ *then* $|\mathbf{T}|$ *satisfies a reverse Hölder inequality up to a small* L^1*-error.*

The constraint

$$\|\mathrm{Id}|\mathbf{T}| - \mathbf{T}\|_1 \ll \|\mathbf{T}\|_1 \tag{1}$$

implies that in most points $x \in B(0,1)$ the columns $\mathbf{T}_i$ of $\mathbf{T}$ have similar absolute value $|\mathbf{T}_1(x)| \sim ... \sim |\mathbf{T}_n(x)|$, in particular their arithmetic mean is comparable to their geometric mean. That means our PDE result can be seen as a perturbed version of the multilinear Kakeya inequality under the constraint (1).

Lipschitz multilinear Kakeya

- Our regularity result is true not only for straight lines but also for Lipschitz curves.
- Our regularity result is only known for $1 \leq p < \frac{n}{n-1}$, while multilinear Kakeya is true also for $p = \frac{n}{n-1}$.
- Lipschitz multilinear Kakeya on the other hand is announced by Csörnyej and Jones to fail in a small interval around $\frac{n}{n-1}$, and to hold between $p = 1$ and some value near $\frac{n}{n-1}$.
- The exception is $n = 2$ (and $n = 1$) where LMK is straightforward to prove at $p = \frac{n}{n-1}$.
- A version of LMK with a lower bound on the diameter of the tubes however holds for all $1 \leq p \leq \frac{n}{n-1}$ due to Guth 2014.
- There are certain versions of LMK for C^1 and C^2 curves, some at and some above the endpoint $p = \frac{n}{n-1}$, see Carbery-Hänninen-Valdimarsson 2018 and Tao 2020.

Does our regularity result hold also in $p = \frac{n}{n-1}$?

Thank you.