Alberti representations, rectifiability, PDEs and multilinear Kakeya

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Alberti representations

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Definition (Alberti representation)

An *Alberti representation* of a finite measure μ on $\mathbb{R}^d$ is a finite measure η on the space of all Lipschitz curves $\Gamma(\mathbb{R}^d)$ on $\mathbb{R}^d$ such that

$$\mu \ll \int_{\Gamma(\mathbb{R}^n)} \mathcal{H}^1\restriction_\gamma \mathrm{d}\eta(\gamma) = A \mapsto \int_{\Gamma(\mathbb{R}^n)} \mathcal{H}^1(A \cap \gamma)\,\mathrm{d}\eta(\gamma).$$

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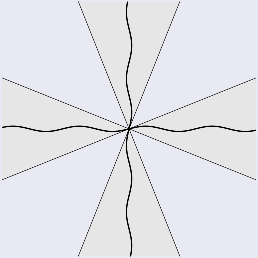
Alberti representations $\eta_1, ..., \eta_n$ are *independent* if for $(\eta_1, ..., \eta_n)$-almost any tuple of curves $(\gamma_1, ..., \gamma_n) \in \Gamma(\mathbb{R}^d)^n$, the γ_i travel within linearly independent cones.



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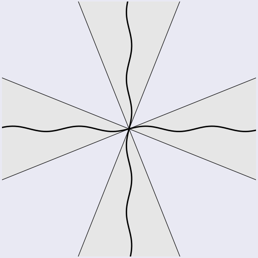

(everything modulo countable decompositions)

Euclidean case

Example

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Converse:

Theorem

A set with n independent Alberti representations is n-rectifiable.

Axes-parallel case

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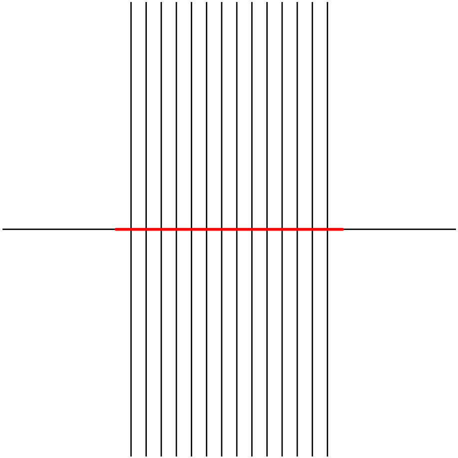

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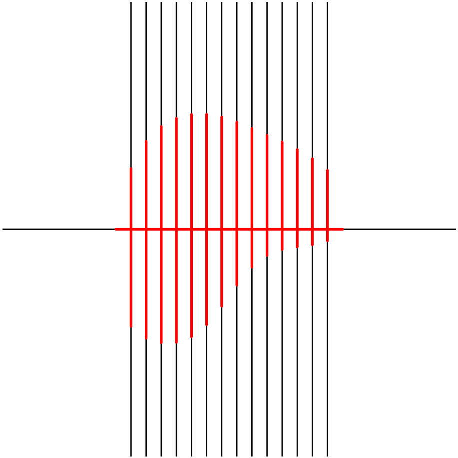

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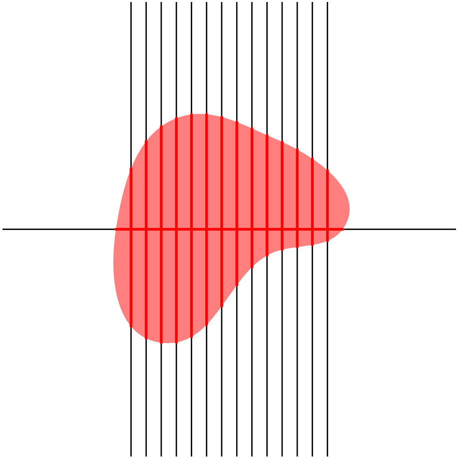

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Theorem (DePhilippis-Rindler, Ann. of Math. (2016), divergence case)

Let $\mathbf{T}$ be an $\mathbb{R}^{n\times n}$-valued finite measure on $\mathbb{R}^n$ such that $\operatorname{div}\mathbf{T}$ is a finite measure. Then the restriction of $\mathbf{T}$ to those points, where its polar $\mathbf{T}/|\mathbf{T}|$ is an invertible matrix, is absolutely continuous with respect to Lebesgue measure.

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This applies for example to a tuple of n independent Alberti representations since

$$\operatorname{div}\Big(\int \dot\gamma \mathcal{H}^1\restriction_\gamma \mathrm{d}\eta(\gamma)\Big) = \int \operatorname{div}(\dot\gamma\mathcal{H}^1\restriction_\gamma)\,\mathrm{d}\eta(\gamma) = 0.$$

Euclidean space

Theorem (Besicovitch projection theorem)

A set $E \subset \mathbb{R}^d$ is purely n-unrectifiable if and only if $\mathcal{H}^{d-n}$-almost every projection of E to an n-plane has $\mathcal{H}^n$-measure 0.

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The Besicovitch projection theorem together with DePhilippis-Rindler can be used to prove that every $E \subset \mathbb{R}^d$ with n Alberti representations is n-rectifiable.

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What about metric space where we do not have such projection theorem?

Metric space

Theorem (Bate-Li, 2014)

A set with n independent Alberti representations which has positive lower density almost everywhere is n-rectifiable.

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Tool: quantitative regularity

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Theorem (Bate-W., 2025, extracted from DePhilippis-Rindler, with significant contributions from Tuomas Orponen)

Let $\mathbf{T}$ *be an* $\mathbb{R}^{n\times n}$*-valued and* ν *be a nonnegative finite measure on* $B(0,1) \subset \mathbb{R}^n$*. Then for any* $1 \le p < \frac{n}{n-1}$ *we can decompose* $\nu = g + b$ *with*

$$\|g\|_p \lesssim_p \|\nu\|_1 + \|\operatorname{div}\mathbf{T}\|_1,$$

$$\|b\|_1 \lesssim_p (\|\nu\|_1 + \|\operatorname{div}\mathbf{T}\|_1)^{\frac{1}{p}} \|\mathrm{Id}\nu - \mathbf{T}\|_1^{\frac{1}{p'}}.$$

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Corollary

Under the above assumptions, if $\|\operatorname{div}\mathbf{T}\|_1 \lesssim \|\nu\|_1$ and $\|\mathrm{Id}\nu - \mathbf{T}\|_1 \ll \|\nu\|_1$ then ν satisfies a reverse Hölder inequality up to a small L^1-error. In particular, $\operatorname{supp}(\nu) \gtrsim 1$.

Proof sketch of lower density in metric space

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1. Take a point $x \in E \subset X$.
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Also how to filter exactly requires some care, for example the curves in the metric space are not full curves, and thus do not have finite divergence.
And in order to control the divergence after cutting off we actually need to do a smooth cutoff instead of just by $B(x,r)$.

Multilinear Kakeya

Theorem (Guth, 2010, Acta Mathematica)

For $i = 1, ..., n$ and j let T_i^j be a straight tube in $\mathbb{R}^n$ that approximately points in direction e_i and denote by r_i^j its radius. Then

$$\Big\|\Big(\prod_{i=1}^{n}\sum_{j} a_i^j 1_{T_i^j}\Big)^{\frac{1}{n}}\Big\|_{\frac{n}{n-1}} \lesssim \Big(\prod_{i=1}^{n}\sum_{j} a_i^j (r_i^j)^{n-1}\Big)^{\frac{1}{n}}.$$

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with η_i supported on straight lines that approximately point in direction e_i after rescaling.

Strange multilinear Kakeya

Corollary

Under the above assumptions, if $\| \operatorname{div} \mathbf{T} \|_1 \lesssim \|\mathbf{T}\|_1$ *and* $\|\mathrm{Id}|\mathbf{T}| - \mathbf{T}\|_1 \ll \|\mathbf{T}\|_1$ *then* $|\mathbf{T}|$ *satisfies a reverse Hölder inequality up to a small* L^1*-error.*

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The constraint

$$\|\mathrm{Id}|\mathbf{T}| - \mathbf{T}\|_1 \ll \|\mathbf{T}\|_1 \tag{1}$$

implies that in most points $x \in B(0,1)$ the columns $\mathbf{T}_i$ of $\mathbf{T}$ have similar absolute value $|\mathbf{T}_1(x)| \sim ... \sim |\mathbf{T}_n(x)|$, in particular their arithmetic mean is comparable to their geometric mean. That means our PDE result can be seen as a perturbed version of the multilinear Kakeya inequality under the constraint (1).

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Does our regularity result hold also in $p = \frac{n}{n-1}$?

Thank you.