Weighted and fractional Poincaré Inequalities

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based on work with

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and

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Classical Poincaré

Poincaré inequality

$$\int_Q |f - f_Q| \lesssim_d \mathrm{l}(Q) \int_Q |\nabla f|$$

with $f_Q = \frac{1}{\mathcal{L}(Q)} \int_Q f$.

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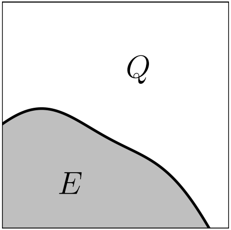
$$\int_Q |f - f_Q| \lesssim_d \mathrm{l}(Q) \int_Q |\nabla f|$$

with $f_Q = \frac{1}{\mathcal{L}(Q)} \int_Q f$. It's equivalent to

relative isoperimetric inequality

$$\min\Big\{\mathcal{L}(Q \cap E), \mathcal{L}(Q \setminus E)\Big\}^{d-1} \lesssim_d \mathcal{H}^{d-1}(Q \cap \partial E)^d$$



Classical Poincaré

For $1 \leq p \leq d$ there exist strengthened versions

Strengthened p-Poincaré

$$\left(\int_Q |f - f_Q|^{p^*}\right)^{\frac{1}{p^*}} \lesssim_d \left(\int_Q |\nabla f|^p\right)^{\frac{1}{p}}$$

where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$.

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where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$.

With $Q \to \mathbb{R}^n$ we obtain Sobolev embedding from p-Poincaré.

$$\|f\|_{p^*} \lesssim_d \|\nabla f\|_p.$$

Fractional Poincaré

Theorem (Bourgain, Brezis, and Mironescu 2002; Maz'ya and Shaposhnikova 2002; Ponce 2004; Milman 2005)

Let $0 \leq \delta < 1$*. Then*

$$\int_Q |f - f_Q| \lesssim_d (1-\delta)\, l(Q)^\delta \int_Q \int_Q \frac{|f(x) - f(y)|}{|x-y|^{d+\delta}} \,\mathrm{d}x\,\mathrm{d}y \lesssim \|\nabla f\|_1$$

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- Without $(1-\delta)$ the first inequality follows from the triangle inequality.
- $$\int_Q \frac{1}{r^{d+\delta-1}} \sim_d \frac{\mathrm{l}(Q)^{1-\delta}}{1-\delta}$$
- L^p version with $\frac{1}{p_\delta^*} = \frac{1}{p} - \frac{\delta}{d}$.

With weights

For $0 \leq \alpha \leq d$ the fractional maximal function is

$$\mathrm{M}_\alpha \mu(x) = \sup_{r>0} r^\alpha \frac{\mu(B(x,r))}{\mathcal{L}(B(x,r))}.$$

Theorem (Franchi, Pérez, and Wheeden 2000)

$$\int_Q |f - f_Q| \,\mathrm{d}\mu \lesssim_d \int_Q |\nabla f(x)| \mathrm{M}_1 \mu(x) \,\mathrm{d}x.$$

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- Constant blows up for $q \searrow 1$, but is finite for $q = 1$.

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- Constant blows up for $q \searrow 1$, but is finite for $q = 1$.
- Generalizes Meyers and Ziemer 1977 who consider $\mu(x) \lesssim |x|^{-\alpha}$ which implies $\mathrm{M}_\alpha \mu \lesssim 1$.

Fractional with weights

Theorem (Myyryläinen, Pérez, and Weigt 2023)

Let $0 \le \delta < 1$ *and* $1 \le q \le \frac{d}{d-\delta}$. *Then*

$$\left(\int_Q |f - f_Q|^q \mu\right)^{\frac{1}{q}} \lesssim_d$$

$$(1-\delta) \int_Q \int_Q \frac{|f(x) - f(y)|}{|x-y|^{n+\delta}} \,\mathrm{d}y \, \mathrm{M}_{d-q(d-\delta)}\mu(x)^{\frac{1}{q}} \,\mathrm{d}x$$

- Hurri-Syrjänen, Javier C. Martínez-Perales, et al. 2022: case $q = \frac{d}{d-\delta}$.

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- $d - q(d-\delta)$ is optimal.

Fractional with weights to classical

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- Also have version for $p > 1$.

A_p-weights

Theorem (Myyryläinen, Pérez, and Weigt 2023)

Let $0 < \delta < 1$, $1 \le r \le p < \frac{d}{\delta}$ *and let* q *be defined by*

$$\frac{1}{p} - \frac{1}{q} = \frac{\delta}{nr}.$$

Then for all cubes $Q \subset \mathbb{R}^d$, $f \in L^1(Q)$ *and* $w \in A_r$ *we have*

$$\begin{aligned}
&\inf_{c\in\mathbb{R}} \left(\frac{1}{w(Q)} \int_Q |f - c|^q \, w \, \mathrm{d}x \right)^{\frac{1}{q}} \\
&\quad \lesssim q[w]_{A_r}^{\frac{1}{p}+\frac{\delta}{nr}+1} \frac{(1-\delta)^{\frac{1}{p}}}{\delta^{1-\frac{1}{p}}} \, \mathrm{l}(Q)^{\delta} \\
&\qquad \left(\frac{1}{w(Q)} \int_Q \int_Q \frac{|f(x) - f(y)|^p}{|x-y|^{n+\delta p}} \, \mathrm{d}y \, w(x) \, \mathrm{d}x \right)^{\frac{1}{p}}.
\end{aligned}$$

$p > 1$? no, for δ large

Counterexample (Myyryläinen, Pérez, and Weigt 2023)

For any $1 < p < n$, $p \leq q \leq \frac{np}{n-p}$, $\alpha = n - \frac{q}{p}(n-p)$, and $C > 0$ there is a Radon measure $\mu \ll \mathcal{L}$ and a Lipschitz function f with

$$\left(\int_Q |f - f_Q|^q \, \mathrm{d}\mu \right)^{\frac{1}{q}} > C \left(\int_Q |\nabla f|^p \, (\mathrm{M}_\alpha \mu)^{\frac{p}{q}} \, \mathrm{d}x \right)^{\frac{1}{p}},$$

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- Is also counterexample against weighted fractional p-Poincaré for δ near 1.

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- Is also counterexample against weighted fractional p-Poincaré for δ near 1.
- What about $\delta = 0$?

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Theorem (Hurri-Syrjänen, Javier C. Martínez-Perales, et al. 2022)

$$\left(\int_Q |f - f_Q|^p \right)^{\frac{1}{p}} \lesssim_d$$

$$(1-\delta)^{\frac{1}{p}} \frac{\mathrm{l}(Q)^{\varepsilon}}{\varepsilon} \left(\int_Q \int_Q \frac{|f(x) - f(y)|^p}{|x-y|^{d+\delta p}} \, \mathrm{d}x \, \mathrm{M}_{(\delta-\varepsilon)p}\mu(y) \, \mathrm{d}y \right)^{\frac{1}{p}}$$

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- Not optimal for $p = 1$ by our result.

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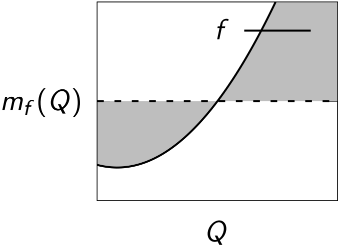
- Not optimal for $p = 1$ by our result.
- Is there a unified result for all p?

Classical Poincaré by isoperimetric inequality

Consider median $m_f(Q)$ instead of the average



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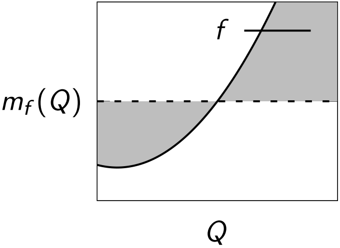

$$\blacksquare = \int_Q |f - m_f(Q)| = \int_{m_f(Q)}^{\infty} \mathcal{L}(\{x \in Q : f(x) > \lambda\})\,\mathrm{d}\lambda + \int_{-\infty}^{m_f(Q)} \mathcal{L}(\{x \in Q : f(x) < \lambda\})\,\mathrm{d}\lambda$$

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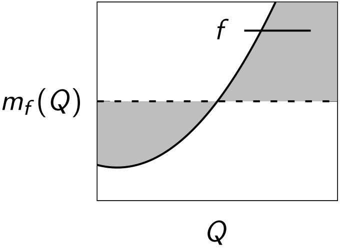

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Classical Poincaré by isoperimetric inequality

$$\int_{m_f(Q)}^{\infty} \mathcal{L}(\{x \in Q : f(x) > \lambda\})\,\mathrm{d}\lambda \leq \mathrm{I}(Q) \int_{m_f(Q)}^{\infty} \mathcal{L}(\{x \in Q : f(x) > \lambda\})^{\frac{d-1}{d}}\,\mathrm{d}\lambda$$

Classical Poincaré by isoperimetric inequality

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\begin{aligned}
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$\square$

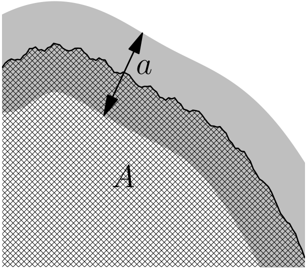
general measure: replace $\mathcal{L} \to \mu$, weigh $\mathcal{H}^{d-1}$ with $\mathrm{M}_\alpha \mu$.

Fractional Poincaré

Lemma (Fractional relative isoperimetric inequality)

Let $a > 0$ and $A \subset Q$ with $a^d \leq \mathcal{L}(A) \leq \mathcal{L}(Q)/2$. Then

$$a\mathcal{L}(Q \cap A)^{\frac{d-1}{d}} \lesssim \int_Q \fint_{Q \cap B(x,a)} |1_A(x) - 1_A(y)| \, \mathrm{d}y \, \mathrm{d}x$$



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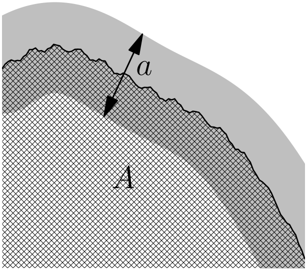

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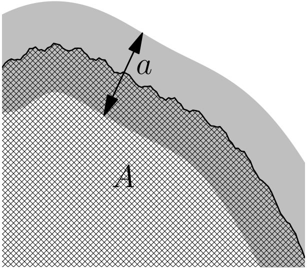

Proof

Theorem (Myyryläinen, Pérez, and Weigt 2023)

$$\left(\int_Q |f - f_Q|^q \mu\right)^{\frac{1}{q}} \lesssim_d$$

$$(1-\delta) \int_Q \int_Q \frac{|f(x) - f(y)|}{|x-y|^{n+\delta}} \, \mathrm{d}y \, \mathrm{M}_{d-q(d-\delta)}\mu(x)^{\frac{1}{q}} \, \mathrm{d}x$$

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2. insert geometric sum over $\sum_{k\geq 0} 2^{-k(1-\delta)} \sim \frac{1}{1-\delta}$
3. apply fractional relative isoperimetric inequality with $a \sim 2^{-k}\,\mathrm{l}(Q)$

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$$(1-\delta) \int_Q \int_Q \frac{|f(x) - f(y)|}{|x-y|^{n+\delta}} \,\mathrm{d}y \,\mathrm{M}_{d-q(d-\delta)}\mu(x)^{\frac{1}{q}} \,\mathrm{d}x$$

1. write LHS in terms of levelsets
2. insert geometric sum over $\sum_{k \geq 0} 2^{-k(1-\delta)} \sim \frac{1}{1-\delta}$
3. apply fractional relative isoperimetric inequality with $a \sim 2^{-k} \,\mathrm{l}(Q)$
4. evalute integral over levelsets to recover difference quotient with $|x-y| \sim a \sim 2^{-k} \,\mathrm{l}(Q)$, using Fubini and

$$\int_{\mathbb{R}} |1_{\{f>\lambda\}}(x) - 1_{\{f>\lambda\}}(y)| \,\mathrm{d}\lambda = |f(x) - f(y)|.$$

□

Thank you