Endpoint regularity bounds of maximal functions in any dimensions

Julian Weigt

Aalto University

17.06.2022

Outline

Introduction

Background

Background

For $f : \mathbb{R}^d \to \mathbb{R}$ the centered Hardy-Littlewood maximal function is defined by

$$\mathrm{M}^{\mathrm{c}} f(x) = \sup_{r>0} f_{B(x,r)} \quad \text{with} \quad f_{B(x,r)} = \frac{1}{\mathcal{L}(B(x,r))} \int_{B(x,r)} |f|.$$

Background

For $f : \mathbb{R}^d \to \mathbb{R}$ the centered Hardy-Littlewood maximal function is defined by

$$\mathrm{M}^{\mathrm{c}} f(x) = \sup_{r>0} f_{B(x,r)} \quad \text{with} \quad f_{B(x,r)} = \frac{1}{\mathcal{L}(B(x,r))} \int_{B(x,r)} |f|.$$

Theorem (Hardy-Littlewood maximal function theorem)

$$\|\mathrm{M}^{\mathrm{c}} f\|_{L^p(\mathbb{R}^d)} \leq C_{d,p} \|f\|_{L^p(\mathbb{R}^d)}$$

if and only if $p > 1$.

Proof: Interpolation. It suffices to prove $\|\mathrm{M}^{\mathrm{c}} f\|_{1,\infty} \lesssim \|f\|_1$ and $\|\mathrm{M}^{\mathrm{c}} f\|_{\infty} \lesssim \|f\|_{\infty}$.

Proof: Interpolation. It suffices to prove $\|\mathrm{M}^{\mathrm{c}} f\|_{1,\infty} \lesssim \|f\|_1$ and $\|\mathrm{M}^{\mathrm{c}} f\|_{\infty} \lesssim \|f\|_{\infty}$.

$$\{\mathrm{M}^{\mathrm{c}} f > \lambda\} = \{x \in \mathbb{R}^d : \mathrm{M}^{\mathrm{c}} f(x) > \lambda\} \subset \bigcup \{B : f_B > \lambda\}.$$

Proof: Interpolation. It suffices to prove $\|\mathrm{M}^{\mathrm{c}} f\|_{1,\infty} \lesssim \|f\|_1$ and $\|\mathrm{M}^{\mathrm{c}} f\|_\infty \lesssim \|f\|_\infty$.

$$\{\mathrm{M}^{\mathrm{c}} f > \lambda\} = \{x \in \mathbb{R}^d : \mathrm{M}^{\mathrm{c}} f(x) > \lambda\} \subset \bigcup \{B : f_B > \lambda\}.$$

By the Vitali covering theorem there is a disjoint set $\mathcal{B}$ of balls B with $f_B > \lambda$ and

$$\begin{aligned}\mathcal{L}\Big(\bigcup\{B : f_B > \lambda\}\Big) &\leq \mathcal{L}\Big(\bigcup\{5B : B \in \mathcal{B}\}\Big) \\ &\leq 5^d \sum_{B\in\mathcal{B}} \mathcal{L}(B) \leq 5^d \sum_{B\in\mathcal{B}} \frac{1}{\lambda} \int_B |f| \\ &\leq \frac{5^d}{\lambda} \int_{\mathbb{R}^d} |f|\end{aligned}$$

Proof: Interpolation. It suffices to prove $\|\mathrm{M}^{\mathrm{c}} f\|_{1,\infty} \lesssim \|f\|_1$ and $\|\mathrm{M}^{\mathrm{c}} f\|_\infty \lesssim \|f\|_\infty$.

$$\{\mathrm{M}^{\mathrm{c}} f > \lambda\} = \{x \in \mathbb{R}^d : \mathrm{M}^{\mathrm{c}} f(x) > \lambda\} \subset \bigcup\{B : f_B > \lambda\}.$$

By the Vitali covering theorem there is a disjoint set $\mathcal{B}$ of balls B with $f_B > \lambda$ and

$$\begin{aligned}\mathcal{L}\Big(\bigcup\{B : f_B > \lambda\}\Big) &\le \mathcal{L}\Big(\bigcup\{5B : B \in \mathcal{B}\}\Big) \\ &\le 5^d \sum_{B\in\mathcal{B}} \mathcal{L}(B) \le 5^d \sum_{B\in\mathcal{B}} \frac{1}{\lambda}\int_B |f| \\ &\le \frac{5^d}{\lambda}\int_{\mathbb{R}^d} |f|\end{aligned}$$

$\mathrm{M}^{\mathrm{c}} f(x) \le \|f\|_\infty \checkmark$.

Theorem (Juha Kinnunen (1997))

If $p > 1$ then

$$\|\nabla \mathrm{M}^{\mathrm{c}} f\|_{L^p(\mathbb{R}^d)} \leq C_{d,p} \|\nabla f\|_{L^p(\mathbb{R}^d)}$$

Theorem (Juha Kinnunen (1997))

If $p > 1$ then

$$\|\nabla \mathrm{M^c} f\|_{L^p(\mathbb{R}^d)} \leq C_{d,p} \|\nabla f\|_{L^p(\mathbb{R}^d)}$$

Proof: For $e \in \mathbb{R}^d$ by the sublinearity of $\mathrm{M^c}$

$$\begin{aligned} \partial_e \mathrm{M^c} f(x) &\sim \frac{\mathrm{M^c} f(x+he) - \mathrm{M^c} f(x)}{h} \\ &\leq \frac{\mathrm{M^c}(f(\cdot + he) - f)(x)}{h} \end{aligned}$$

Theorem (Juha Kinnunen (1997))

If $p > 1$ then

$$\|\nabla \mathrm{M}^{\mathrm{c}} f\|_{L^p(\mathbb{R}^d)} \leq C_{d,p} \|\nabla f\|_{L^p(\mathbb{R}^d)}$$

Proof: For $e \in \mathbb{R}^d$ by the sublinearity of M^{c}

$$\begin{aligned}
\partial_e \mathrm{M}^{\mathrm{c}} f(x) &\sim \frac{\mathrm{M}^{\mathrm{c}} f(x+he) - \mathrm{M}^{\mathrm{c}} f(x)}{h} \\
&\leq \frac{\mathrm{M}^{\mathrm{c}}(f(\cdot + he) - f)(x)}{h} \\
&= \mathrm{M}^{\mathrm{c}}\Big(\frac{f(\cdot + he) - f)}{h}\Big)(x)
\end{aligned}$$

Theorem (Juha Kinnunen (1997))

If $p > 1$ then

$$\|\nabla \mathrm{M^c} f\|_{L^p(\mathbb{R}^d)} \leq C_{d,p} \|\nabla f\|_{L^p(\mathbb{R}^d)}$$

Proof: For $e \in \mathbb{R}^d$ by the sublinearity of $\mathrm{M^c}$

$$\begin{aligned} \partial_e \mathrm{M^c} f(x) &\sim \frac{\mathrm{M^c} f(x+he) - \mathrm{M^c} f(x)}{h} \\ &\leq \frac{\mathrm{M^c}(f(\cdot + he) - f)(x)}{h} \\ &= \mathrm{M^c}\Big(\frac{f(\cdot + he) - f)}{h}\Big)(x) \sim \mathrm{M^c}(\partial_e f)(x) \end{aligned}$$

Theorem (Juha Kinnunen (1997))

If $p > 1$ then

$$\|\nabla \mathrm{M}^{\mathrm{c}} f\|_{L^p(\mathbb{R}^d)} \leq C_{d,p} \|\nabla f\|_{L^p(\mathbb{R}^d)}$$

Proof: For $e \in \mathbb{R}^d$ by the sublinearity of M^{c}

$$\begin{aligned}
\partial_e \mathrm{M}^{\mathrm{c}} f(x) &\sim \frac{\mathrm{M}^{\mathrm{c}} f(x+he) - \mathrm{M}^{\mathrm{c}} f(x)}{h} \\
&\leq \frac{\mathrm{M}^{\mathrm{c}}(f(\cdot + he) - f)(x)}{h} \\
&= \mathrm{M}^{\mathrm{c}}\Big(\frac{f(\cdot + he) - f}{h}\Big)(x) \sim \mathrm{M}^{\mathrm{c}}(\partial_e f)(x)
\end{aligned}$$

By the Hardy-Littlewood maximal function theorem for $p > 1$

$$\|\nabla \mathrm{M}^{\mathrm{c}} f\|_{L^p(\mathbb{R}^d)} \lesssim \|\mathrm{M}^{\mathrm{c}}(|\nabla f|)\|_{L^p(\mathbb{R}^d)} \lesssim \|\nabla f\|_{L^p(\mathbb{R}^d)}$$

Question (Hajłasz and Onninen 2004)

Is it true that

$$\|\nabla \mathrm{M}^{\mathrm{c}} f\|_{L^1(\mathbb{R}^d)} \leq C_d \|\nabla f\|_{L^1(\mathbb{R}^d)}?$$

Question (Hajłasz and Onninen 2004)

Is it true that

$$\|\nabla \mathrm{M}^{\mathrm{c}} f\|_{L^1(\mathbb{R}^d)} \leq C_d \|\nabla f\|_{L^1(\mathbb{R}^d)}?$$

For $f : \mathbb{R}^d \to \mathbb{R}$ the uncentered Hardy-Littlewood maximal function is defined by

$$\widetilde{\mathrm{M}} f(x) = \sup_{B \ni x} f_B.$$

The result by Kinnunen also holds for $\widetilde{\mathrm{M}}$ and various other maximal operators, and the question by Hałjasz and Onninen is being investigated.

Onedimensional case

Onedimensional case

In 2002 Tanaka proved

$$\operatorname{var} \widetilde{\mathrm{M}} f \leq \operatorname{var} f$$

for $f : \mathbb{R} \to \mathbb{R}$, but with a factor 2 on the right hand side. In 2007 Aldaz and Pérez Lázaro reduced that factor to the optimal value 1.

Onedimensional case

In 2002 Tanaka proved

$$\operatorname{var} \widetilde{\mathrm{M}} f \leq \operatorname{var} f$$

for $f : \mathbb{R} \to \mathbb{R}$, but with a factor 2 on the right hand side. In 2007 Aldaz and Pérez Lázaro reduced that factor to the optimal value 1. They use that in one dimension we have

$$\operatorname{var} f = \sup_{n \in \mathbb{N},\ x_1 < \ldots < x_n} \sum_{i=1}^{n-1} |f(x_{n+1}) - f(x_n)|.$$

Onedimensional case

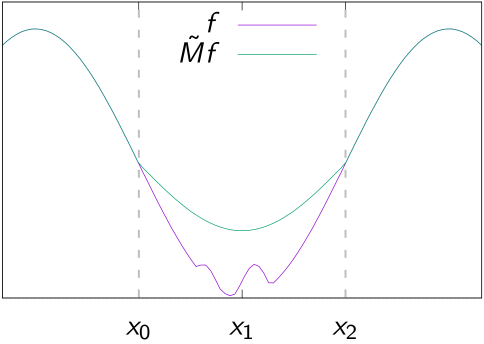
In 2002 Tanaka proved

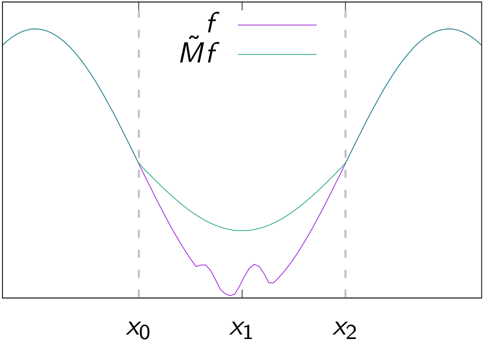
$$\operatorname{var} \widetilde{\mathrm{M}} f \leq \operatorname{var} f$$

for $f : \mathbb{R} \to \mathbb{R}$, but with a factor 2 on the right hand side. In 2007 Aldaz and Pérez Lázaro reduced that factor to the optimal value 1. They use that in one dimension we have

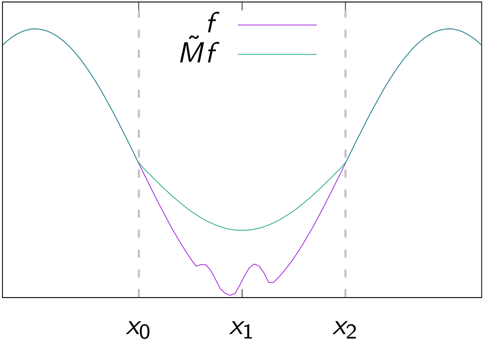
$$\operatorname{var} f = \sup_{n \in \mathbb{N},\ x_1 < \ldots < x_n} \sum_{i=1}^{n-1} |f(x_{n+1}) - f(x_n)|.$$

Main ingredient: $\widetilde{\mathrm{M}} f$ is convex on connected components of $\{x \in \mathbb{R} : \widetilde{\mathrm{M}} f(x) > f(x)\}$.

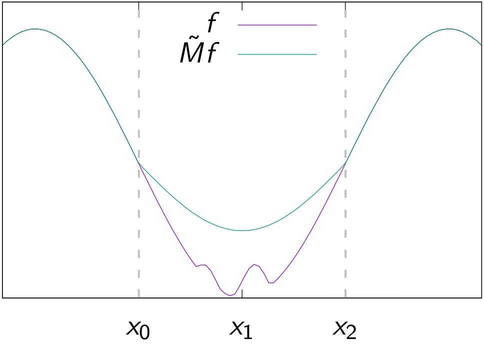

f
$\tilde{M}f$
x_0
x_1
x_2



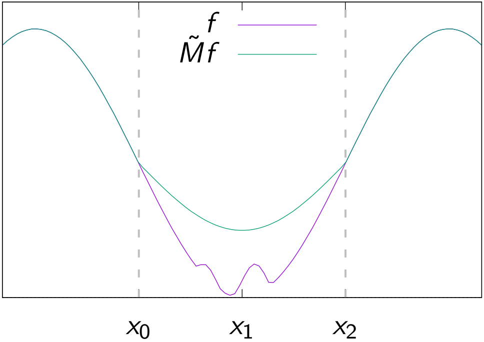
$$\operatorname{var} \widetilde{\mathrm{M}} f = \operatorname{var}_{[0,x_0]} \widetilde{\mathrm{M}} f + \operatorname{var}_{[x_2,1]} \widetilde{\mathrm{M}} f$$
$$+ |\widetilde{\mathrm{M}} f(x_0) - \widetilde{\mathrm{M}} f(x_1)| + |\widetilde{\mathrm{M}} f(x_2) - \widetilde{\mathrm{M}} f(x_1)|$$



$$
\begin{aligned}
\operatorname{var} \widetilde{\mathrm{M}} f &= \operatorname{var}_{[0,x_0]} \widetilde{\mathrm{M}} f + \operatorname{var}_{[x_2,1]} \widetilde{\mathrm{M}} f \\
&\quad + |\widetilde{\mathrm{M}} f(x_0) - \widetilde{\mathrm{M}} f(x_1)| + |\widetilde{\mathrm{M}} f(x_2) - \widetilde{\mathrm{M}} f(x_1)| \\
&\leq \operatorname{var}_{[0,x_0]} f + \operatorname{var}_{[x_2,1]} f \\
&\quad + |f(x_0) - f(x_1)| + |f(x_2) - f(x_1)|
\end{aligned}
$$



$$
\begin{aligned}
\operatorname{var} \widetilde{\mathrm{M}} f &= \operatorname{var}_{[0,x_0]} \widetilde{\mathrm{M}} f + \operatorname{var}_{[x_2,1]} \widetilde{\mathrm{M}} f \\
&\quad + |\widetilde{\mathrm{M}} f(x_0) - \widetilde{\mathrm{M}} f(x_1)| + |\widetilde{\mathrm{M}} f(x_2) - \widetilde{\mathrm{M}} f(x_1)| \\
&\le \operatorname{var}_{[0,x_0]} f + \operatorname{var}_{[x_2,1]} f \\
&\quad + |f(x_0) - f(x_1)| + |f(x_2) - f(x_1)| \\
&\le \operatorname{var}_{[0,x_0]} f + \operatorname{var}_{[x_2,1]} f + \operatorname{var}_{[x_0,x_2]} f
\end{aligned}
$$



$$\begin{aligned}
\operatorname{var} \widetilde{\mathrm{M}} f &= \operatorname{var}_{[0,x_0]} \widetilde{\mathrm{M}} f + \operatorname{var}_{[x_2,1]} \widetilde{\mathrm{M}} f \\
&\quad + |\widetilde{\mathrm{M}} f(x_0) - \widetilde{\mathrm{M}} f(x_1)| + |\widetilde{\mathrm{M}} f(x_2) - \widetilde{\mathrm{M}} f(x_1)| \\
&\leq \operatorname{var}_{[0,x_0]} f + \operatorname{var}_{[x_2,1]} f \\
&\quad + |f(x_0) - f(x_1)| + |f(x_2) - f(x_1)| \\
&\leq \operatorname{var}_{[0,x_0]} f + \operatorname{var}_{[x_2,1]} f + \operatorname{var}_{[x_0,x_2]} f = \operatorname{var} f
\end{aligned}$$

Onedimensional case

For the centered maximal function $\mathrm{M}^{\mathrm{c}} f$ the convexity property does not hold. Nevertheless,

centered

Kurka proved $\operatorname{var} \mathrm{M}^{\mathrm{c}} f \leq C \operatorname{var} f$ for $f : \mathbb{R} \to \mathbb{R}$ in a very involved paper in 2015.

Onedimensional case

For the centered maximal function $\mathrm{M^c} f$ the convexity property does not hold. Nevertheless,

centered

Kurka proved $\operatorname{var} \mathrm{M^c} f \leq C \operatorname{var} f$ for $f : \mathbb{R} \to \mathbb{R}$ in a very involved paper in 2015.

He did case distinctions with respect to the shape of triples $x_0 < x_1 < x_2$ with $\mathrm{M^c} f(x_0) < \mathrm{M^c} f(x_1) > \mathrm{M^c} f(x_2)$ and a decomposition in scales.

Onedimensional case

For radial functions $f : \mathbb{R}^d \to \mathbb{R}$ with $f(x) = f(|x|)$ we have

$$\|\nabla f\|_{L^1(\mathbb{R}^d)} = \int_0^\infty |\nabla f(r)| r^{d-1} \, \mathrm{d}r$$

and also $\widetilde{\mathrm{M}} f$ is radial.

Onedimensional case

For radial functions $f : \mathbb{R}^d \to \mathbb{R}$ with $f(x) = f(|x|)$ we have

$$\|\nabla f\|_{L^1(\mathbb{R}^d)} = \int_0^\infty |\nabla f(r)| r^{d-1} \, \mathrm{d}r$$

and also $\widetilde{\mathrm{M}} f$ is radial.

radial

In 2018 Luiro used this one-dimensional representation to prove $\|\nabla \widetilde{\mathrm{M}} f\|_{L^1(\mathbb{R}^d)} \leq C_d \|\nabla f\|_{L^1(\mathbb{R}^d)}$ for radial functions $f : \mathbb{R}^d \to \mathbb{R}$.

Onedimensional case

For radial functions $f : \mathbb{R}^d \to \mathbb{R}$ with $f(x) = f(|x|)$ we have

$$\|\nabla f\|_{L^1(\mathbb{R}^d)} = \int_0^\infty |\nabla f(r)| r^{d-1} \,\mathrm{d}r$$

and also $\widetilde{\mathrm{M}} f$ is radial.

radial

In 2018 Luiro used this one-dimensional representation to prove $\|\nabla \widetilde{\mathrm{M}} f\|_{L^1(\mathbb{R}^d)} \le C_d \|\nabla f\|_{L^1(\mathbb{R}^d)}$ for radial functions $f : \mathbb{R}^d \to \mathbb{R}$.

block-decreasing

In 2009 Aldaz and Pérez Lázaro proved $\|\nabla \widetilde{\mathrm{M}} f\|_{L^1(\mathbb{R}^d)} \le C_d \|\nabla f\|_{L^1(\mathbb{R}^d)}$ for block-decreasing $f : \mathbb{R}^d \to \mathbb{R}$,

which are to some extent similar to radially decreasing functions.

The fractional maximal function

The fractional maximal function

For $0 < \alpha < d$ the centered fractional Hardy-Littlewood maximal function is

$$\mathrm{M}_\alpha^{\mathrm{c}} f(x) = \sup_{r>0} r^\alpha f_{B(x,r)},$$

and similarly the uncentered version $\widetilde{\mathrm{M}}_\alpha f$.

The fractional maximal function

For $0 < \alpha < d$ the centered fractional Hardy-Littlewood maximal function is

$$\mathrm{M}_\alpha^{\mathrm{c}} f(x) = \sup_{r>0} r^\alpha f_{B(x,r)},$$

and similarly the uncentered version $\widetilde{\mathrm{M}}_\alpha f$. The corresponding Hardy-Littlewood theorem is

$$\|\mathrm{M}_\alpha f\|_{L^{\frac{pd}{d-\alpha p}}(\mathbb{R}^d)} \leq C_{d,\alpha,p} \|f\|_{L^p(\mathbb{R}^d)}$$

if and only if $p > 1$,

The fractional maximal function

For $0 < \alpha < d$ the centered fractional Hardy-Littlewood maximal function is

$$\mathrm{M}^{\mathrm{c}}_{\alpha} f(x) = \sup_{r>0} r^{\alpha} f_{B(x,r)},$$

and similarly the uncentered version $\widetilde{\mathrm{M}}_{\alpha} f$. The corresponding Hardy-Littlewood theorem is

$$\|\mathrm{M}_{\alpha} f\|_{L^{\frac{pd}{d-\alpha p}}(\mathbb{R}^d)} \leq C_{d,\alpha,p} \|f\|_{L^p(\mathbb{R}^d)}$$

if and only if $p > 1$, and the corresponding regularity bound is

$$\|\nabla \mathrm{M}_{\alpha} f\|_{L^{\frac{pd}{d-\alpha p}}(\mathbb{R}^d)} \leq C_{d,\alpha,p} \|\nabla f\|_{L^p(\mathbb{R}^d)},$$

which for $p > 1$ follows by the same proof as for $\alpha = 0$ in Kinnunen (1997).

Fractional: $d = 1$

Fractional endpoint

Do we have

$$\|\nabla \mathrm{M}_\alpha f\|_{L^{\frac{d}{d-\alpha}}(\mathbb{R}^d)} \leq C_{d,\alpha}\|\nabla f\|_{L^1(\mathbb{R}^d)}?$$

Fractional: $d = 1$

Fractional endpoint

Do we have

$$\|\nabla \mathrm{M}_\alpha f\|_{L^{\frac{d}{d-\alpha}}(\mathbb{R}^d)} \leq C_{d,\alpha} \|\nabla f\|_{L^1(\mathbb{R}^d)}?$$

Also already known for

Fractional: $d = 1$

Fractional endpoint

Do we have

$$\|\nabla \mathrm{M}_\alpha f\|_{L^{\frac{d}{d-\alpha}}(\mathbb{R}^d)} \leq C_{d,\alpha} \|\nabla f\|_{L^1(\mathbb{R}^d)}?$$

Also already known for

- $d = 1$

Fractional: $d = 1$

Fractional endpoint

Do we have

$$\|\nabla \mathrm{M}_\alpha f\|_{L^{\frac{d}{d-\alpha}}(\mathbb{R}^d)} \leq C_{d,\alpha} \|\nabla f\|_{L^1(\mathbb{R}^d)}?$$

Also already known for

- $d = 1$
- radial f

Fractional: Improvements over nonfractional

For $\alpha \geq 1$ Kinnunen and Saksman proved in 2003

$$|\nabla \mathrm{M}_\alpha f(x)| \lesssim |\mathrm{M}_{\alpha-1} f(x)|.$$

Fractional: Improvements over nonfractional

For $\alpha \geq 1$ Kinnunen and Saksman proved in 2003

$$|\nabla \mathrm{M}_\alpha f(x)| \lesssim |\mathrm{M}_{\alpha-1} f(x)|.$$

This implies

$$\|\nabla \mathrm{M}_\alpha f\|_{L^{\frac{d}{d-\alpha}}(\mathbb{R}^d)} \lesssim \|\mathrm{M}_{\alpha-1} f\|_{L^{\frac{d}{d-\alpha}}(\mathbb{R}^d)} \lesssim \|f\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \lesssim \|\nabla f\|_{L^1(\mathbb{R}^d)}.$$

Fractional: Improvements over nonfractional

For $\alpha \geq 1$ Kinnunen and Saksman proved in 2003

$$|\nabla \mathrm{M}_\alpha f(x)| \lesssim |\mathrm{M}_{\alpha-1} f(x)|.$$

This implies

$$\begin{aligned}\|\nabla \mathrm{M}_\alpha f\|_{L^{\frac{d}{d-\alpha}}(\mathbb{R}^d)} \lesssim \|\mathrm{M}_{\alpha-1} f\|_{L^{\frac{d}{d-\alpha}}(\mathbb{R}^d)} \lesssim \|f\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \\ \lesssim \|\nabla f\|_{L^1(\mathbb{R}^d)}.\end{aligned}$$

Endpoint bound is known for

- $\alpha \geq 1$

Fractional: Improvements over nonfractional

For $\alpha \geq 1$ Kinnunen and Saksman proved in 2003

$$|\nabla \mathrm{M}_\alpha f(x)| \lesssim |\mathrm{M}_{\alpha-1} f(x)|.$$

This implies

$$\|\nabla \mathrm{M}_\alpha f\|_{L^{\frac{d}{d-\alpha}}(\mathbb{R}^d)} \lesssim \|\mathrm{M}_{\alpha-1} f\|_{L^{\frac{d}{d-\alpha}}(\mathbb{R}^d)} \lesssim \|f\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)}$$
$$\lesssim \|\nabla f\|_{L^1(\mathbb{R}^d)}.$$

Endpoint bound is known for

- $\alpha \geq 1$
- maximal operator that only averages over balls with radii $2^n,\ n \in \mathbb{Z}$

Fractional: Improvements over nonfractional

For $\alpha \geq 1$ Kinnunen and Saksman proved in 2003

$$|\nabla \mathrm{M}_\alpha f(x)| \lesssim |\mathrm{M}_{\alpha-1} f(x)|.$$

This implies

$$\begin{aligned}\|\nabla \mathrm{M}_\alpha f\|_{L^{\frac{d}{d-\alpha}}(\mathbb{R}^d)} &\lesssim \|\mathrm{M}_{\alpha-1} f\|_{L^{\frac{d}{d-\alpha}}(\mathbb{R}^d)} \lesssim \|f\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \\ &\lesssim \|\nabla f\|_{L^1(\mathbb{R}^d)}.\end{aligned}$$

Endpoint bound is known for

- $\alpha \geq 1$
- maximal operator that only averages over balls with radii $2^n, \ n \in \mathbb{Z}$
- maximal operator that only averages against a smooth kernel

Fractional: Improvements over nonfractional

For $\alpha \geq 1$ Kinnunen and Saksman proved in 2003

$$|\nabla \mathrm{M}_\alpha f(x)| \lesssim |\mathrm{M}_{\alpha-1} f(x)|.$$

This implies

$$\|\nabla \mathrm{M}_\alpha f\|_{L^{\frac{d}{d-\alpha}}(\mathbb{R}^d)} \lesssim \|\mathrm{M}_{\alpha-1} f\|_{L^{\frac{d}{d-\alpha}}(\mathbb{R}^d)} \lesssim \|f\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \lesssim \|\nabla f\|_{L^1(\mathbb{R}^d)}.$$

Endpoint bound is known for

- $\alpha \geq 1$
- maximal operator that only averages over balls with radii $2^n,\ n \in \mathbb{Z}$
- maximal operator that only averages against a smooth kernel
- All previous results also known for $\mathrm{M}_\alpha^{\mathrm{c}}$.

Other maximal operators and related questions

- convolution operators
- local maximal operators
- discrete maximal operators
- bilinear maximal operators

Other maximal operators and related questions

- convolution operators
- local maximal operators
- discrete maximal operators
- bilinear maximal operators
- any combinations of the above

Other maximal operators and related questions

- convolution operators
- local maximal operators
- discrete maximal operators
- bilinear maximal operators
- any combinations of the above
- bounds on other spaces than Sobolev spaces

Other maximal operators and related questions

- convolution operators
- local maximal operators
- discrete maximal operators
- bilinear maximal operators
- any combinations of the above
- bounds on other spaces than Sobolev spaces

related: Continuity of the operator given by $f \mapsto \nabla \mathrm{M} f$ on $W^{1,1}(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$. This is a stronger property than boundedness.

New results

New results

We prove the endpoint regularity bound for the maximal function for

- characteristic f

New results

We prove the endpoint regularity bound for the maximal function for

- characteristic f
- dyadic maximal operator

New results

We prove the endpoint regularity bound for the maximal function for

- characteristic f
- dyadic maximal operator
- fractional maximal operator

New results

We prove the endpoint regularity bound for the maximal function for

- characteristic f
- dyadic maximal operator
- fractional maximal operator
- cube maximal operator

Proof strategy

Reduction and decomposition

reformulations

definition

$$\operatorname{var} f = \sup\left\{ \int f \operatorname{div} \varphi : \varphi \in C^1_c(\mathbb{R}^d;\mathbb{R}^d),\ |\varphi| \le 1 \right\}$$

reformulations

definition

$$\operatorname{var} f = \sup\left\{ \int f \operatorname{div} \varphi : \varphi \in C^1_{\mathrm{c}}(\mathbb{R}^d;\mathbb{R}^d),\ |\varphi| \le 1 \right\}$$
$$= \|\nabla f\|_{L^1(\mathbb{R}^d)} \qquad \text{if } f \in W^{1,1}(\mathbb{R}^d).$$

reformulations

definition

$$\operatorname{var} f = \sup\left\{ \int f \operatorname{div} \varphi : \varphi \in C_c^1(\mathbb{R}^d;\mathbb{R}^d),\ |\varphi| \le 1 \right\}$$
$$= \|\nabla f\|_{L^1(\mathbb{R}^d)} \qquad \text{if } f \in W^{1,1}(\mathbb{R}^d).$$

coarea formula

$$\operatorname{var} f = \int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial\{x \in \mathbb{R}^d : f(x) > \lambda\})\,\mathrm{d}\lambda$$

reformulations

definition

$$\operatorname{var} f = \sup\left\{ \int f \operatorname{div} \varphi : \varphi \in C_c^1(\mathbb{R}^d;\mathbb{R}^d),\ |\varphi| \le 1 \right\}$$
$$= \|\nabla f\|_{L^1(\mathbb{R}^d)} \qquad \text{if } f \in W^{1,1}(\mathbb{R}^d).$$

coarea formula

$$\operatorname{var} f = \int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial\{x \in \mathbb{R}^d : f(x) > \lambda\})\,\mathrm{d}\lambda$$

superlevel sets

$$\{x \in \mathbb{R}^d : \mathrm{M}f(x) > \lambda\} = \bigcup\{B : f_B > \lambda\}$$

for *uncentered* maximal operators.

reformulations

definition

$$\operatorname{var} f = \sup\left\{\int f \operatorname{div}\varphi : \varphi \in C^1_c(\mathbb{R}^d;\mathbb{R}^d),\ |\varphi| \le 1\right\}$$
$$= \|\nabla f\|_{L^1(\mathbb{R}^d)} \qquad \text{if } f \in W^{1,1}(\mathbb{R}^d).$$

coarea formula

$$\operatorname{var} f = \int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial\{x \in \mathbb{R}^d : f(x) > \lambda\})\,\mathrm{d}\lambda$$

superlevel sets

$$\{\mathrm{M}f > \lambda\} = \{x \in \mathbb{R}^d : \mathrm{M}f(x) > \lambda\} = \bigcup\{B : f_B > \lambda\}$$

for *uncentered* maximal operators.

Denote

$$\mathcal{B}_\lambda^{<} = \{B : f_B > \lambda,\ \mathcal{L}(B \cap \{f > \lambda\}) < \mathcal{L}(B)/2\}$$

and $\mathcal{B}_\lambda^{\geq}$ accordingly. We split the boundary

$$\partial \bigcup \{B : f_B > \lambda\} \subset \partial \bigcup \mathcal{B}_\lambda^{<} \cup \partial \bigcup \mathcal{B}_\lambda^{\geq}. \tag{1}$$

Denote

$$\mathcal{B}_\lambda^{<} = \{B : f_B > \lambda,\ \mathcal{L}(B \cap \{f > \lambda\}) < \mathcal{L}(B)/2\}$$

and $\mathcal{B}_\lambda^{\geq}$ accordingly. We split the boundary

$$\partial \bigcup \{B : f_B > \lambda\} \subset \partial \bigcup \mathcal{B}_\lambda^{<} \cup \partial \bigcup \mathcal{B}_\lambda^{\geq}. \tag{1}$$

Since $\mathrm{M}f \geq f$ a.e. we have $\{f > \lambda\} \subset \{\mathrm{M}f > \lambda\}$ up to measure zero, and thus

$$\partial \bigcup \{B : f_B > \lambda\} \subset \left(\partial \bigcup \{B : f_B > \lambda\}\right) \setminus \overline{\{f > \lambda\}} \cup \partial \{f > \lambda\}. \tag{2}$$

Denote

$$\mathcal{B}_\lambda^{<} = \{B : f_B > \lambda,\ \mathcal{L}(B \cap \{f > \lambda\}) < \mathcal{L}(B)/2\}$$

and $\mathcal{B}_\lambda^{\geq}$ accordingly. We split the boundary

$$\partial \bigcup \{B : f_B > \lambda\} \subset \partial \bigcup \mathcal{B}_\lambda^{<} \cup \partial \bigcup \mathcal{B}_\lambda^{\geq}. \tag{1}$$

Since $\mathrm{M}f \geq f$ a.e. we have $\{f > \lambda\} \subset \{\mathrm{M}f > \lambda\}$ up to measure zero, and thus

$$\partial \bigcup \{B : f_B > \lambda\} \subset \Big(\partial \bigcup \{B : f_B > \lambda\}\Big) \setminus \overline{\{f > \lambda\}} \cup \partial \{f > \lambda\}. \tag{2}$$

Plug (1) into (2) and that into the coarea formula

$$\operatorname{var} \mathrm{M}f = \int_0^\infty \mathcal{H}^{d-1}\Big(\partial \bigcup \{B : f_B > \lambda\}\Big)\, \mathrm{d}\lambda.$$

Decomposition of the boundary

decomposition

$$\operatorname{var} \mathrm{M}f \leq \int_0^\infty \mathcal{H}^{d-1}\Big(\partial \bigcup \mathcal{B}_\lambda^<\Big)\,\mathrm{d}\lambda + \int_0^\infty \mathcal{H}^{d-1}\Big(\Big(\partial \bigcup \mathcal{B}_\lambda^\geq\Big) \setminus \overline{\{f > \lambda\}}\Big)\,\mathrm{d}\lambda + \operatorname{var} f$$

Decomposition of the boundary

decomposition

$$\operatorname{var} \mathrm{M}f \leq \int_0^\infty \mathcal{H}^{d-1}\Big(\partial \bigcup \mathcal{B}_\lambda^<\Big)\,\mathrm{d}\lambda + \int_0^\infty \mathcal{H}^{d-1}\Big(\Big(\partial \bigcup \mathcal{B}_\lambda^\geq\Big) \setminus \overline{\{f > \lambda\}}\Big)\,\mathrm{d}\lambda + \operatorname{var} f \quad \checkmark$$

High density case

High density case

Relative isoperimetric inequality

Let B be a cube or a ball and $\mathcal{L}(B \cap E) \leq \mathcal{L}(B)/2$. Then

$$\mathcal{L}(B \cap E)^{\frac{d-1}{d}} \lesssim \mathcal{H}^{d-1}(B \cap \partial E)$$

High density case

Relative isoperimetric inequality

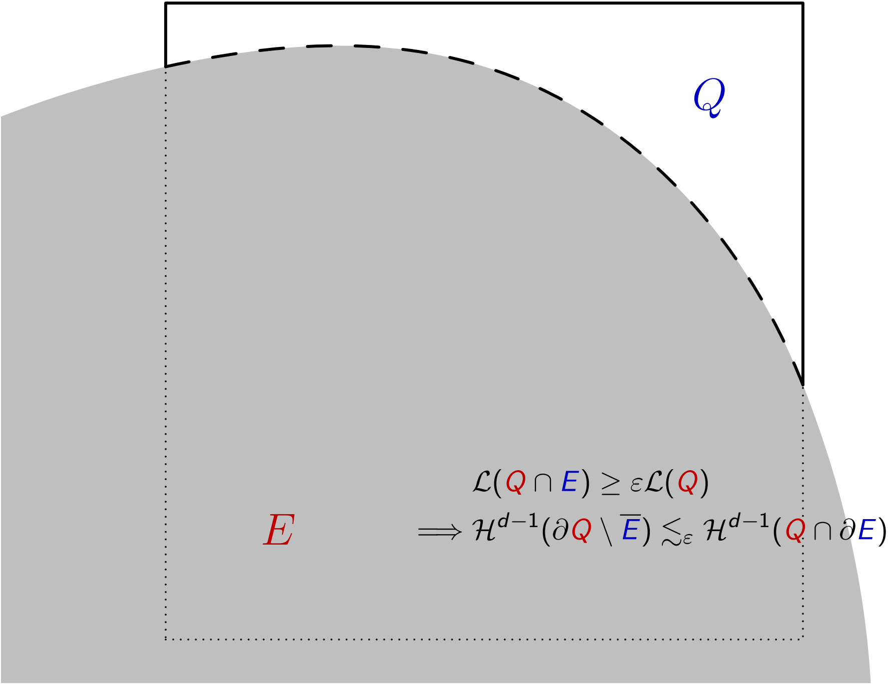
Let B be a cube or a ball and $\mathcal{L}(B \cap E) \le \mathcal{L}(B)/2$. Then

$$\mathcal{L}(B \cap E)^{\frac{d-1}{d}} \lesssim \mathcal{H}^{d-1}(B \cap \partial E)$$

Proposition (High density)

For $\mathcal{L}(B \cap E) \ge \mathcal{L}(B)/2$ we have

$$\mathcal{H}^{d-1}(\partial B \setminus \overline{E}) \lesssim \mathcal{H}^{d-1}(B \cap \partial E).$$


Q
$\mathcal{L}(Q \cap E) \geq \varepsilon \mathcal{L}(Q)$
E
$\implies \mathcal{H}^{d-1}(\partial Q \setminus \overline{E}) \lesssim_\varepsilon \mathcal{H}^{d-1}(Q \cap \partial E)$

High density case

Proposition (High density, general version)

Let $\mathcal{B}$ be a set of balls B with $\mathcal{L}(B \cap E) \geq \varepsilon \mathcal{L}(B)$. Then

$$\mathcal{H}^{d-1}\Big(\partial \bigcup \mathcal{B} \setminus \overline{E}\Big) \lesssim_\varepsilon \mathcal{H}^{d-1}\Big(\bigcup \mathcal{B} \cap \partial E\Big).$$

High density case

Proposition (High density, general version)

Let $\mathcal{B}$ be a set of balls B with $\mathcal{L}(B \cap E) \geq \varepsilon \mathcal{L}(B)$. Then

$$\mathcal{H}^{d-1}\Big(\partial \bigcup \mathcal{B} \setminus \overline{E}\Big) \lesssim_\varepsilon \mathcal{H}^{d-1}\Big(\bigcup \mathcal{B} \cap \partial E\Big).$$

$$\begin{aligned}\int_0^\infty \mathcal{H}^{d-1}\Big(\Big(\partial \bigcup \mathcal{B}_\lambda^{\geq}\Big) \setminus \overline{\{f > \lambda\}}\Big)\,\mathrm{d}\lambda \\ \lesssim \int_0^\infty \mathcal{H}^{d-1}\Big(\bigcup \mathcal{B}_\lambda^{\geq} \cap \partial\{f > \lambda\}\Big)\,\mathrm{d}\lambda \\ \leq \operatorname{var} f.\end{aligned}$$

Low density case

Low density case

Have to bound

$$\int_0^\infty \mathcal{H}^{d-1}\left(\partial \bigcup \mathcal{B}_\lambda^<\right) \mathrm{d}\lambda \lesssim \operatorname{var} f,$$

where

$$\mathcal{B}_\lambda^< = \{B : f_B > \lambda,\ \mathcal{L}(B \cap \{f > \lambda\}) < \mathcal{L}(B)/2\}.$$

Low density case

Have to bound

$$\int_0^\infty \mathcal{H}^{d-1}\Big(\partial \bigcup \mathcal{B}_\lambda^<\Big)\,\mathrm{d}\lambda \lesssim \mathsf{var}\, f,$$

where

$$\mathcal{B}_\lambda^< = \{B : f_B > \lambda,\ \mathcal{L}(B \cap \{f > \lambda\}) < \mathcal{L}(B)/2\}.$$

I can't :(

Low density case

Have to bound

$$\int_0^\infty \mathcal{H}^{d-1}\left(\partial \bigcup \mathcal{B}_\lambda^<\right) \mathrm{d}\lambda \lesssim \operatorname{var} f,$$

where

$$\mathcal{B}_\lambda^< = \{B : f_B > \lambda,\ \mathcal{L}(B \cap \{f > \lambda\}) < \mathcal{L}(B)/2\}.$$

I can't :(

dyadic maximal operator

$$\mathrm{M}^{\mathrm{d}} f(x) = \sup_{Q \ni x,\ Q \text{ dyadic}} f_Q .$$

Low density case

Have to bound

$$\int_0^\infty \mathcal{H}^{d-1}\left(\partial \bigcup \mathcal{B}_\lambda^<\right) \mathrm{d}\lambda \lesssim \text{var}\, f,$$

where

$$\mathcal{B}_\lambda^< = \{B : f_B > \lambda,\ \mathcal{L}(B \cap \{f > \lambda\}) < \mathcal{L}(B)/2\}.$$

I can't :(

dyadic maximal operator

$$\mathrm{M}^\mathrm{d} f(x) = \sup_{Q \ni x,\ Q \text{ dyadic}} f_Q.$$

$$\{x : \mathrm{M}^\mathrm{d} f(x) > \lambda\} = \bigcup \{\text{maximal dyadic } Q : f_Q > \lambda\}$$

Low density case

Have to bound

$$\int_0^\infty \mathcal{H}^{d-1}\left(\partial \bigcup \mathcal{B}_\lambda^<\right) \mathrm{d}\lambda \lesssim \operatorname{var} f,$$

where

$$\mathcal{B}_\lambda^< = \{B : f_B > \lambda,\ \mathcal{L}(B \cap \{f > \lambda\}) < \mathcal{L}(B)/2\}.$$

I can't :(

dyadic maximal operator

$$\mathrm{M}^\mathrm{d} f(x) = \sup_{Q \ni x,\ Q \text{ dyadic}} f_Q.$$

$$\{x : \mathrm{M}^\mathrm{d} f(x) > \lambda\} = \bigcup \{\text{maximal dyadic } Q : f_Q > \lambda\} = \bigcup \mathcal{Q}_\lambda^< \cup \mathcal{Q}_\lambda^<$$

Definition

Q is maximal for $\lambda < f_Q$ if for all $P \supsetneq Q$ we have $f_P \leq \lambda$.

Definition

Q is maximal for $\lambda < f_Q$ if for all $P \supsetneq Q$ we have $f_P \leq \lambda$.

$$\int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial \bigcup \mathcal{Q}_\lambda^<) \, \mathrm{d}\lambda$$

Definition

Q is maximal for $\lambda < f_Q$ if for all $P \supsetneq Q$ we have $f_P \leq \lambda$.

$$\int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial \bigcup \mathcal{Q}_\lambda^<) \,\mathrm{d}\lambda \leq \int_{\mathbb{R}} \sum_{Q \in \mathcal{Q}_\lambda^<} \mathcal{H}^{d-1}(\partial Q) \,\mathrm{d}\lambda$$

Definition

Q is maximal for $\lambda < f_Q$ if for all $P \supsetneq Q$ we have $f_P \leq \lambda$.

$$\int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial \bigcup \mathcal{Q}_\lambda^<) \,\mathrm{d}\lambda \leq \int_{\mathbb{R}} \sum_{Q \in \mathcal{Q}_\lambda^<} \mathcal{H}^{d-1}(\partial Q) \,\mathrm{d}\lambda$$

$$= \int_{\mathbb{R}} \sum_{Q : \tilde{\lambda}_Q < \lambda < f_Q} \mathcal{H}^{d-1}(\partial Q) \,\mathrm{d}\lambda$$

Definition

Q is maximal for $\lambda < f_Q$ if for all $P \supsetneq Q$ we have $f_P \leq \lambda$. Given Q, let λ_Q be the smallest such λ.

$$\int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial \bigcup \mathcal{Q}_\lambda^<) \, \mathrm{d}\lambda \leq \int_{\mathbb{R}} \sum_{Q \in \mathcal{Q}_\lambda^<} \mathcal{H}^{d-1}(\partial Q) \, \mathrm{d}\lambda$$

$$= \int_{\mathbb{R}} \sum_{Q : \tilde{\lambda}_Q < \lambda < f_Q} \mathcal{H}^{d-1}(\partial Q) \, \mathrm{d}\lambda$$

Definition

Q is maximal for $\lambda < f_Q$ if for all $P \supsetneq Q$ we have $f_P \leq \lambda$. Given Q, let λ_Q be the smallest such λ.

$$\int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial \bigcup \mathcal{Q}_\lambda^<)\,\mathrm{d}\lambda \leq \int_{\mathbb{R}} \sum_{Q \in \mathcal{Q}_\lambda^<} \mathcal{H}^{d-1}(\partial Q)\,\mathrm{d}\lambda$$

$$= \int_{\mathbb{R}} \sum_{Q : \tilde{\lambda}_Q < \lambda < f_Q} \mathcal{H}^{d-1}(\partial Q)\,\mathrm{d}\lambda$$

where

$$\tilde{\lambda}_Q = \quad \sup\{\lambda : \mathcal{L}(Q \cap \{f > \tilde{\lambda}_Q\}) \geq 2^{-1} \cdot \mathcal{L}(Q) \quad \}$$

Definition

Q is maximal for $\lambda < f_Q$ if for all $P \supsetneq Q$ we have $f_P \leq \lambda$. Given Q, let λ_Q be the smallest such λ.

$$\int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial \bigcup \mathcal{Q}_\lambda^<)\,\mathrm{d}\lambda \leq \int_{\mathbb{R}} \sum_{Q \in \mathcal{Q}_\lambda^<} \mathcal{H}^{d-1}(\partial Q)\,\mathrm{d}\lambda$$
$$= \int_{\mathbb{R}} \sum_{Q : \tilde{\lambda}_Q < \lambda < f_Q} \mathcal{H}^{d-1}(\partial Q)\,\mathrm{d}\lambda$$

where

$$\tilde{\lambda}_Q = \sup\Big\{\lambda_Q, \sup\{\lambda : \mathcal{L}(Q \cap \{f > \tilde{\lambda}_Q\}) \geq 2^{-1} \cdot \mathcal{L}(Q) \quad \}\Big\}$$

Definition

Q is maximal for $\lambda < f_Q$ if for all $P \supsetneq Q$ we have $f_P \leq \lambda$. Given Q, let λ_Q be the smallest such λ.

$$\begin{aligned}
\int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial \bigcup \mathcal{Q}_\lambda^<)\,\mathrm{d}\lambda &\leq \int_{\mathbb{R}} \sum_{Q \in \mathcal{Q}_\lambda^<} \mathcal{H}^{d-1}(\partial Q)\,\mathrm{d}\lambda \\
&= \int_{\mathbb{R}} \sum_{Q : \tilde{\lambda}_Q < \lambda < f_Q} \mathcal{H}^{d-1}(\partial Q)\,\mathrm{d}\lambda \\
&= \sum_Q (f_Q - \tilde{\lambda}_Q)\mathcal{H}^{d-1}(\partial Q)
\end{aligned}$$

where

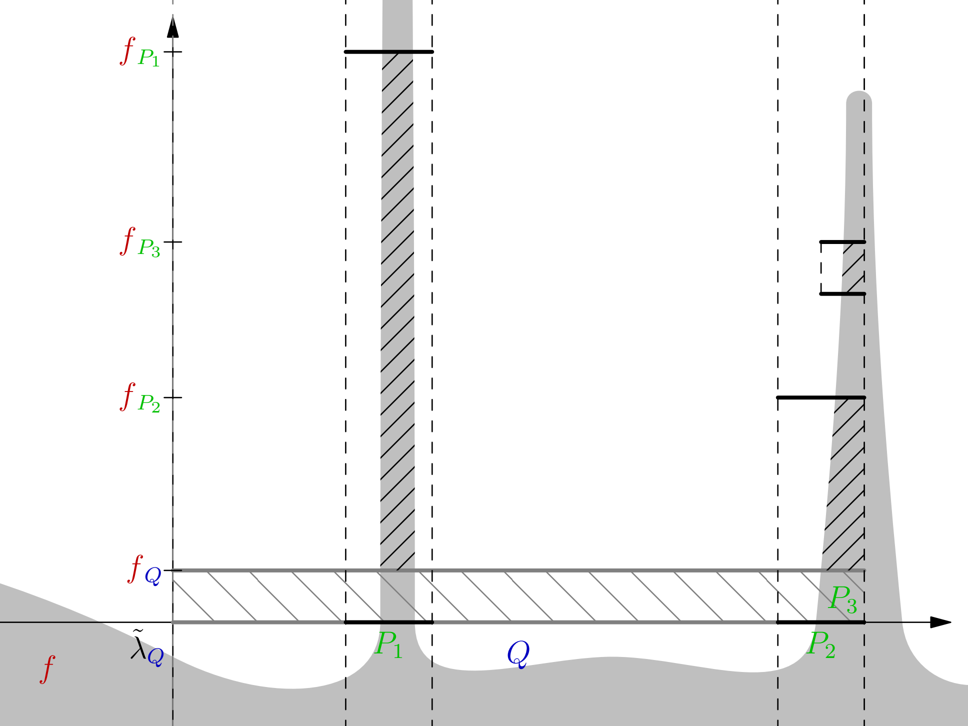
$$\tilde{\lambda}_Q = \sup\Big\{\lambda_Q, \sup\{\lambda : \mathcal{L}(Q \cap \{f > \tilde{\lambda}_Q\}) \geq 2^{-d-2} \cdot \mathcal{L}(Q)\}\Big\}$$

Proposition

$$(f_Q - \tilde{\lambda}_Q)\mathcal{L}(Q) \lesssim \int_{\mathbb{R}} \sum_{P \subsetneq Q : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\}) \, \mathrm{d}\lambda$$

where P is maximal above $\bar{\lambda}_P$ and

$$"\mathcal{L}(P \cap \{f > \bar{\lambda}_P\}) = 2^{-1}\mathcal{L}(P)"$$
$$"\mathcal{L}(Q \cap \{f > \tilde{\lambda}_Q\}) = 2^{-d-2}\mathcal{L}(Q)"$$


f_{P_1}
f_{P_3}
f_{P_2}
f_Q
$\tilde{\lambda}_Q$
f
P_1
Q
P_2
P_3

$$\sum_{Q}(f_Q - \tilde{\lambda}_Q)\mathcal{H}^{d-1}(\partial Q) \lesssim \int_{\mathbb{R}} \sum_{Q} \mathsf{l}(Q)^{-1} \sum_{P \subsetneq Q : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\}) \,\mathrm{d}\lambda$$

$$\sum_{Q}(f_Q - \tilde{\lambda}_Q)\mathcal{H}^{d-1}(\partial Q) \lesssim \int_{\mathbb{R}} \sum_{Q} \mathsf{l}(Q)^{-1} \sum_{P \subsetneq Q : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\}) \, \mathrm{d}\lambda$$

$$= \int_{\mathbb{R}} \sum_{P : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\}) \sum_{Q \supsetneq P} \mathsf{l}(Q)^{-1} \, \mathrm{d}\lambda$$

$$
\begin{aligned}
\sum_{Q}(f_Q - \tilde{\lambda}_Q)\mathcal{H}^{d-1}(\partial Q) &\lesssim \int_{\mathbb{R}} \sum_{Q} \mathrm{l}(Q)^{-1} \sum_{P \subsetneq Q : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\})\, \mathrm{d}\lambda \\
&= \int_{\mathbb{R}} \sum_{P : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\}) \sum_{Q \supsetneq P} \mathrm{l}(Q)^{-1}\, \mathrm{d}\lambda \\
&= \int_{\mathbb{R}} \sum_{P : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\})\, \mathrm{l}(P)^{-1}\, \mathrm{d}\lambda
\end{aligned}
$$

$$
\begin{aligned}
\sum_{Q}(f_Q - \tilde{\lambda}_Q)\mathcal{H}^{d-1}(\partial Q) &\lesssim \int_{\mathbb{R}} \sum_{Q} \mathsf{l}(Q)^{-1} \sum_{P \subsetneq Q : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\}) \,\mathrm{d}\lambda \\
&= \int_{\mathbb{R}} \sum_{P : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\}) \sum_{Q \supsetneq P} \mathsf{l}(Q)^{-1} \,\mathrm{d}\lambda \\
&= \int_{\mathbb{R}} \sum_{P : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\}) \,\mathsf{l}(P)^{-1} \,\mathrm{d}\lambda \\
&\leq \int_{\mathbb{R}} \sum_{P : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\})^{\frac{d-1}{d}} \,\mathrm{d}\lambda
\end{aligned}
$$

$$\sum_{Q}(f_Q-\tilde{\lambda}_Q)\mathcal{H}^{d-1}(\partial Q) \lesssim \int_{\mathbb{R}} \sum_{Q} \mathsf{l}(Q)^{-1} \sum_{P\subsetneq Q:\bar{\lambda}_P<\lambda<f_P} \mathcal{L}(P\cap\{f>\lambda\})\,\mathrm{d}\lambda$$

$$= \int_{\mathbb{R}} \sum_{P:\bar{\lambda}_P<\lambda<f_P} \mathcal{L}(P\cap\{f>\lambda\}) \sum_{Q\supsetneq P} \mathsf{l}(Q)^{-1}\,\mathrm{d}\lambda$$

$$= \int_{\mathbb{R}} \sum_{P:\bar{\lambda}_P<\lambda<f_P} \mathcal{L}(P\cap\{f>\lambda\})\,\mathsf{l}(P)^{-1}\,\mathrm{d}\lambda$$

$$\le \int_{\mathbb{R}} \sum_{P:\bar{\lambda}_P<\lambda<f_P} \mathcal{L}(P\cap\{f>\lambda\})^{\frac{d-1}{d}}\,\mathrm{d}\lambda$$

$$\lesssim \int_{\mathbb{R}} \sum_{P:\bar{\lambda}_P<\lambda<f_P} \mathcal{H}^{d-1}(P\cap\partial\{f>\lambda\})\,\mathrm{d}\lambda$$

$$\begin{aligned}
\sum_{Q} (f_Q - \tilde{\lambda}_Q)\mathcal{H}^{d-1}(\partial Q) &\lesssim \int_{\mathbb{R}} \sum_{Q} \mathsf{l}(Q)^{-1} \sum_{P \subsetneq Q : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\}) \, \mathrm{d}\lambda \\
&= \int_{\mathbb{R}} \sum_{P : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\}) \sum_{Q \supsetneq P} \mathsf{l}(Q)^{-1} \, \mathrm{d}\lambda \\
&= \int_{\mathbb{R}} \sum_{P : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\}) \, \mathsf{l}(P)^{-1} \, \mathrm{d}\lambda \\
&\leq \int_{\mathbb{R}} \sum_{P : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\})^{\frac{d-1}{d}} \, \mathrm{d}\lambda \\
&\lesssim \int_{\mathbb{R}} \sum_{P : \bar{\lambda}_P < \lambda < f_P} \mathcal{H}^{d-1}(P \cap \partial\{f > \lambda\}) \, \mathrm{d}\lambda \\
&\leq \int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial\{f > \lambda\}) \, \mathrm{d}\lambda = \operatorname{var} f
\end{aligned}$$

$$
\begin{aligned}
\sum_{Q} (f_Q - \tilde{\lambda}_Q) \mathcal{H}^{d-1}(\partial Q) &\lesssim \int_{\mathbb{R}} \sum_{Q} \mathsf{l}(Q)^{-1} \sum_{P \subsetneq Q : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\}) \, \mathrm{d}\lambda \\
&= \int_{\mathbb{R}} \sum_{P : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\}) \sum_{Q \supsetneq P} \mathsf{l}(Q)^{-1} \, \mathrm{d}\lambda \\
&= \int_{\mathbb{R}} \sum_{P : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\}) \, \mathsf{l}(P)^{-1} \, \mathrm{d}\lambda \\
&\leq \int_{\mathbb{R}} \sum_{P : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\})^{\frac{d-1}{d}} \, \mathrm{d}\lambda \\
&\lesssim \int_{\mathbb{R}} \sum_{P : \bar{\lambda}_P < \lambda < f_P} \mathcal{H}^{d-1}(P \cap \partial\{f > \lambda\}) \, \mathrm{d}\lambda \\
&\leq \int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial\{f > \lambda\}) \, \mathrm{d}\lambda = \operatorname{var} f \qquad \square
\end{aligned}
$$

Thank you