Weighted and fractional Poincaré Inequalities

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based on ongoing work with

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The Poincaré inequality bounds the oscillation of a function by its gradient

$$\int_Q |f - f_Q| \lesssim_d \mathrm{l}(Q) \int_Q |\nabla f|, \qquad (1)$$

where

$$f_Q = \frac{1}{\mathcal{L}(Q)} \int_Q f$$

and $\mathrm{l}(Q)$ is the sidelength of the cube Q.

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$\underline{eq.\ (2) \implies eq.\ (1)}$ by taking $E = 1_{f > \lambda}$ and using the coarea formula.

The Poincaré inequality also comes in its strengthened form

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For $1 \leq p \leq d$ there exist L^p-versions of the Poincaré inequality

$$\Big(\int_Q |f - f_Q|^{p^*}\Big)^{\frac{1}{p^*}} \lesssim_d \Big(\int_Q |\nabla f|^p\Big)^{\frac{1}{p}}, \tag{4}$$

where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$.

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where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$.
Note, that with $Q = [-r, r]^d$ and $r \to \infty$ we obtain Sobolev embedding from eq. (4),

$$\|f\|_{p^*} \lesssim_d \|\nabla f\|_p.$$

Fractional

Theorem (Bourgain, Brezis, and Mironescu 2002; Maz'ya and Shaposhnikova 2002)

Let $0 \leq \delta < 1$. *Then*

$$\int_Q |f - f_Q| \lesssim_d (1-\delta)\, l(Q)^\delta \int_Q \fint_Q \frac{|f(x) - f(y)|}{|x-y|^{d+\delta}} \, \mathrm{d}x \, \mathrm{d}y$$

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- Without the factor $(1-\delta)$ this follows directly from the triangle inequality and $|x-y| \lesssim l(Q)$.
- In fact they prove an L^p version, with exponent p^*_δ given by $\frac{1}{p^*_\delta} = \frac{1}{p} - \frac{\delta}{d}$.

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Theorem (Ponce 2004)

For a bounded Lipschitz domain Ω *and certain radial kernels* $\rho \geq 0$ *with* $\int_{\mathbb{R}^d} \rho = 1$ *we have*

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Milman 2005 also proves the result from Bourgain, Brezis, and Mironescu 2002, using some general interpolation.

With weights

For $0 \leq \alpha \leq d$ we define the fractional maximal function of a weight w by

$$\mathrm{M}_\alpha w(x) = \sup_{r>0} r^\alpha \frac{1}{\mathcal{L}(B(x,r))} \int_{B(x,r)} w.$$

Theorem (Franchi, Pérez, and Wheeden 2000)

Let w be a weight. Then for all $1 \leq q \leq \frac{d}{d-1}$ we have

$$\left(\int_Q |f - f_Q|^q w \right)^{\frac{1}{q}} \lesssim_{d,q} \int_Q |\nabla f| (\mathrm{M}_{d-q(d-1)} w)^{\frac{1}{q}}.$$

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- Recall the term $\mathrm{l}(Q)^\alpha$ in front of the original Poincaré. That term is absorbed in the fractional maximal function, and in an optimal way due to $\mathrm{M}_\alpha w(x) \le \mathrm{diam}(Q)^\alpha \mathrm{M}_0 w(x)$.

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- WIP: version with $|\nabla f|^p$, $p > 1$.

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Theorem (WIP)

$$\Big(\int_Q |f - f_Q|^q w\Big)^{\frac{1}{q}} \lesssim_d$$
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- Can the right hand side again be bounded by an integral over the gradient? Yes, for A_1-weights, and with loss of additional $(1-\delta)$ due to Hurri-Syrjänen et al. 2022.

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- also version with $|f(x) - f(y)|^p$, $p > 1$ in progress

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Our proofs are in the spirit of the following proof of the Poincaré inequality by the relative isoperimetric inequality.

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$$m_f(Q) = \sup\{\lambda \in \mathbb{R} : \mathcal{L}(\{x \in Q : f(x) > \lambda\}) \geq \mathcal{L}(Q)\}$$

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$$\int_Q |f - m_f(Q)| = \int_{m_f(Q)}^{\infty} \mathcal{L}(\{x \in Q : f(x) > \lambda\})\,\mathrm{d}\lambda + \int_{-\infty}^{m_f(Q)} \mathcal{L}(\{x \in Q : f(x) < \lambda\})$$

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By potentially replacing f by $-f$ we can assume that the first summand is larger, so it suffices to bound that one.

$$\int_{m_f(Q)}^{\infty} \mathcal{L}(\{x \in Q : f(x) > \lambda\}) \,\mathrm{d}\lambda$$
$$\leq \mathsf{I}(Q) \int_{m_f(Q)}^{\infty} \mathcal{L}(\{x \in Q : f(x) > \lambda\})^{\frac{d-1}{d}} \,\mathrm{d}\lambda$$

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and by the relative isoperimetric inequality and the definition of $m_f(Q)$

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To prove weighted versions of the fractional Poincaré inequality the relative isoperimetric inequality is of not much use since we don't estimate by the gradient.

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Lemma

Let $a > 0$ and $A \subset Q$ with $a^d \leq \mathcal{L}(A) \leq \mathcal{L}(Q)/2$. Then

$$\int_Q \int_{Q \cap B(x,a) \setminus B(x,a/2)} 1_{A \times (Q \setminus A)}(x,y) \, \mathrm{d}y \, \mathrm{d}x \gtrsim a^{d+1} \mathcal{L}(A)^{\frac{d-1}{d}}.$$

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- By the relative isoperimetric inequality we have $\mathcal{L}(A)^{\frac{d-1}{d}} \lesssim \mathcal{H}^{d-1}(Q \cap \partial A)$. So the Lemma detects the rough size of the boundary of A by integrating over pairs $x, y \in Q$ with $a/2 < |x - y| < a$.

Thank you