Regularity of maximal functions in higher dimensions

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For $f : \mathbb{R}^n \to \mathbb{R}$ the centered Hardy-Littlewood maximal function is defined by

$$\mathrm{M^c} f(x) = \sup_{r>0} f_{B(x,r)} \quad \text{with} \quad f_{B(x,r)} = \frac{1}{\mathcal{L}(B(x,r))} \int_{B(x,r)} |f|.$$

Theorem (Hardy-Littlewood maximal function theorem)

$$\|\mathrm{M^c} f\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|f\|_{L^p(\mathbb{R}^n)}$$

if and only if $p > 1$.

$$\|\mathrm{M^c} f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim_n \|f\|_{L^1(\mathbb{R}^n)}$$

Introduction: Background

Theorem (Juha Kinnunen (1997))

For $p > 1$ we have

$$\|\nabla \mathrm{M}^{\mathrm{c}} f\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|\nabla f\|_{L^p(\mathbb{R}^n)}$$

Proof: For $e \in \mathbb{R}^n$ by the sublinearity of M^{c}

$$\begin{aligned}
\partial_e \mathrm{M}^{\mathrm{c}} f(x) &\sim \frac{\mathrm{M}^{\mathrm{c}} f(x+he) - \mathrm{M}^{\mathrm{c}} f(x)}{h} \\
&\leq \frac{\mathrm{M}^{\mathrm{c}}(f(\cdot + he) - f)(x)}{h} \\
&= \mathrm{M}^{\mathrm{c}}\Big(\frac{f(\cdot + he) - f)}{h}\Big)(x) \sim \mathrm{M}^{\mathrm{c}}(\partial_e f)(x)
\end{aligned}$$

By the Hardy-Littlewood maximal function theorem for $p > 1$

$$\|\nabla \mathrm{M}^{\mathrm{c}} f\|_{L^p(\mathbb{R}^n)} \leq \|\mathrm{M}^{\mathrm{c}}(|\nabla f|)\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|\nabla f\|_{L^p(\mathbb{R}^n)}$$

Question (Hajłasz and Onninen 2004)

Is it true that

$$\|\nabla \mathrm{M}^{\mathrm{c}} f\|_{L^1(\mathbb{R}^n)} \lesssim_n \|\nabla f\|_{L^1(\mathbb{R}^n)}?$$

Uncentered Hardy-Littlewood maximal function

$$\mathrm{M} f(x) = \sup_{B \ni x} f_B.$$

Endpoint question by Hałjasz and Onninen is interesting for M and other maximal operators.

Introduction: In one dimension

Theorem (Tanaka 2002, Aldaz and Pérez Lázaro 2007)

For $f : \mathbb{R} \to \mathbb{R}$ *we have*

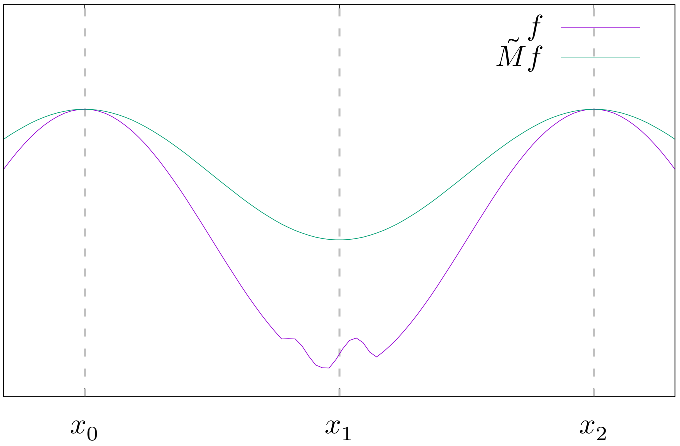
$$\|\nabla \mathrm{M} f\|_1 \leq \|\nabla f\|_1$$

Proof:

- In one dimension

$$\|\nabla f\|_1 = \sup_{x_1 < x_2 < \dots} \sum_i |f(x_{i+1}) - f(x_i)| = \operatorname{var} f.$$

- For almost all $x \in \mathbb{R}^d$: $\mathrm{M} f(x) \geq f(x)$
- and $\mathrm{M} f(x) = f(x)$ at a strict local maximum of $\mathrm{M} f$.



$$
\begin{aligned}
\operatorname{var}_{[x_0,x_2]} \mathrm{M}f &= |\mathrm{M}f(x_1) - \mathrm{M}f(x_0)| + |\mathrm{M}f(x_2) - \mathrm{M}f(x_1)| \\
&\le |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| \\
&\le \operatorname{var}_{[x_0,x_2]} f
\end{aligned}
$$

Introduction: Progress

$n = 1$	[Tanaka 2002, Aldaz +Pérez Lázaro 2007]
block decreasing f	[Aldaz+Pérez Lázaro 2009]
centered M, $n = 1$	[Kurka 2015]
radial f	[Luiro 2018]
fractional maximal function $\alpha \geq 1$	[Kinnunen + Saksman 2003, Carneiro + Madrid 2016]
characteristic f	[W 2020]
dyadic maximal function	[W 2020]
fractional maximal function	[W 2020]
cube maximal function $\alpha > 0$	[W 2021]

- bounds on other maximal operators, such as local,... ,
- local regularity and smoothing, i.e. does $f \mapsto \nabla \mathrm{M} f$ map $\mathrm{BV}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ or only into Radon measures?
- operator continuity of $f \mapsto \nabla \mathrm{M} f$ on $W^{1,1}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$, stronger than boundedness.
- best constants in one dimension

Characteristic functions

Coarea formula

$$\|\nabla f\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}} \mathcal{H}^{n-1}(\partial\{x \in \mathbb{R}^n : f(x) > \lambda\})\,\mathrm{d}\lambda$$
$$= \mathcal{H}^{n-1}(\partial E) \qquad \text{if } f = 1_E$$

Superlevel sets

$$\{\mathrm{M}f > \lambda\} = \{x \in \mathbb{R}^n : \mathrm{M}f(x) > \lambda\} = \bigcup\{B : f_B > \lambda\}$$

for *uncentered* maximal operators. $(= \bigcup\{B : f_B = \lambda\}$ for balls)

Case distinction

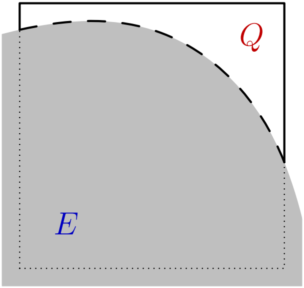
$$\lambda \geq 1/2 : \qquad \mathcal{L}(B \cap E) \geq \mathcal{L}(B)/2,$$
$$\lambda \leq 1/2 : \qquad \mathcal{L}(B \cap E) \leq \mathcal{L}(B)/2$$

Proof: High density case $\lambda \geq 1/2$

Proposition

For Q, E with $\mathcal{L}(Q \cap E) \geq \mathcal{L}(Q)/2$ we have

$$\mathcal{H}^{n-1}(\partial Q \setminus \overline{E}) \lesssim_n \mathcal{H}^{n-1}(Q \cap \partial E)$$



Proof: Low density case $\lambda \leq 1/2$

relative isoperimetric inequality

$$\min\{\mathcal{L}(Q \cap E), \mathcal{L}(Q \setminus E)\}^{n-1} \lesssim_n \mathcal{H}^{n-1}(Q \cap \partial E)^n.$$

For Q, E with $\lambda := \mathcal{L}(Q \cap E)/\mathcal{L}(Q) \leq 1/2$ we have

$$\begin{aligned}
\mathcal{H}^{n-1}(\partial Q) &\sim_n \mathcal{L}(Q)^{\frac{n-1}{n}} \\
&= \lambda^{-\frac{n-1}{n}} \min\{\mathcal{L}(Q \cap E), \mathcal{L}(Q \setminus E)\}^{\frac{n-1}{n}} \\
&\lesssim_n \lambda^{-\frac{n-1}{n}} \mathcal{H}^{n-1}(Q \cap \partial E)
\end{aligned}$$

Conclusion

For $\lambda := \mathcal{L}(Q \cap E)/\mathcal{L}(Q)$ we have

$$\mathcal{H}^{n-1}(\partial Q \setminus \overline{E}) \lesssim \lambda^{-\frac{n-1}{n}} \mathcal{H}^{n-1}(Q \cap \partial E).$$

dyadic maximal operator M^{d}. For $0 < \lambda < 1$ let $\mathcal{Q}_\lambda$ be the set of maximal dyadic cubes Q with $\mathcal{L}(Q \cap E) \geq \lambda \mathcal{L}(Q)$. Then

$$\begin{aligned} \operatorname{var}(\mathrm{M}^{\mathrm{d}} 1_E) &= \int_0^1 \mathcal{H}^{n-1}\Big(\partial \bigcup \mathcal{Q}_\lambda\Big) \, \mathrm{d}\lambda \\ &\leq \int_0^1 \mathcal{H}^{n-1}\Big(\partial \bigcup \mathcal{Q}_\lambda \setminus \overline{E}\Big) \, \mathrm{d}\lambda + \int_0^1 \mathcal{H}^{n-1}(\partial E) \, \mathrm{d}\lambda \end{aligned}$$

$$\begin{aligned} \sum_{Q \in \mathcal{Q}_\lambda} \mathcal{H}^{n-1}(\partial Q \setminus \overline{E}) &\lesssim \lambda^{-\frac{n-1}{n}} \sum_{Q \in \mathcal{Q}_\lambda} \mathcal{H}^{n-1}(Q \cap \partial E) \\ &\leq \lambda^{-\frac{n-1}{n}} \mathcal{H}^{n-1}(E) \end{aligned}$$

Thus,

$$\operatorname{var}(\mathrm{M}^{\mathrm{d}}1_E) \lesssim_n \int_0^1 \lambda^{-\frac{n-1}{n}} \mathcal{H}^{n-1}(E)\,\mathrm{d}\lambda \lesssim_n \operatorname{var}(1_E).$$

Balls? Not disjoint, but

Proposition

Let $\mathcal{B}_\lambda$ be a set of balls with $\mathcal{L}(B \cap E) = \lambda \mathcal{L}(B)$. Then

$$\mathcal{H}^{n-1}\Big(\partial \bigcup \mathcal{B}_\lambda \setminus \overline{E}\Big) \lesssim |\log \lambda| \lambda^{-\frac{n-1}{n}} \mathcal{H}^{n-1}(\partial E).$$

Same proof for $\mathrm{M}1_E$ can be done, despite $\log \lambda$ term. Can also be used to prove

Theorem (Vitali for boundary)

Any bounded set $\mathcal{B}$ of balls has a disjoint subset $\tilde{\mathcal{B}}$ with

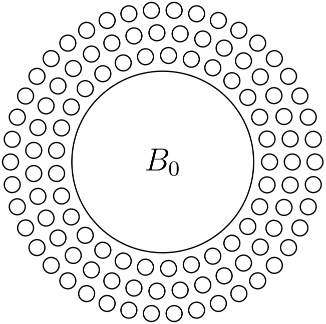
$$\mathcal{H}^{n-1}\Big(\partial \bigcup \mathcal{B}\Big) \lesssim_n \mathcal{H}^{n-1}\Big(\partial \bigcup \tilde{\mathcal{B}}\Big).$$

This can in turn be used to remove $\log \lambda$ term.

Lemma (Vitali covering lemma)

Any bounded set $\mathcal{B}$ of balls has a disjoint subset $\tilde{\mathcal{B}}$ with

$$\mathcal{L}\Big(\bigcup \mathcal{B}\Big) \lesssim_n \mathcal{L}\Big(\bigcup \tilde{\mathcal{B}}\Big).$$



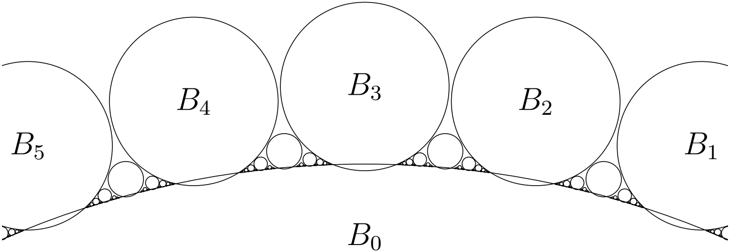
That means Vitali selection strategy doesn't work for boundary. As it turns out, the Besicovitch covering theorem strategy combined with $\log \lambda$-proposition does, however.

Proof of Vitali for boundary

Apply Besicovitch covering theorem, which gives $N \lesssim n$ families $\mathcal{B}_1, \ldots, \mathcal{B}_N$ of disjoint balls that together contain all centers of balls in $\mathcal{B}$. In fact, for each $B(x,r) \in \mathcal{B}$ exists a $B(y,s) \in \mathcal{B}_1 \cup \ldots \cup \mathcal{B}_n$ with $s \gtrsim r$ and $x \in B(y,s)$. Thus, $\mathcal{L}(B(x,r) \cap B(y,s)) \gtrsim_n \mathcal{L}(B(x,r))$. Using $\log\lambda$-proposition,

$$
\begin{aligned}
\mathcal{H}^{n-1}\Big(\partial \bigcup \mathcal{B}\Big) &\lesssim_n \mathcal{H}^{n-1}\Big(\partial \bigcup \mathcal{B}_1 \cup \ldots \cup \mathcal{B}_N\Big) \\
&\leq \sum_{k=1}^{N} \mathcal{H}^{n-1}\Big(\partial \bigcup \mathcal{B}_k\Big) \\
&\leq N \max_{k=1,\ldots,N} \mathcal{H}^{n-1}\Big(\partial \bigcup \mathcal{B}_k\Big)
\end{aligned}
$$

Can we get a disjoint subset $\tilde{\mathcal{B}} \subset \mathcal{B}$ that witnesses both the Vitali covering lemma and the Vitali covering lemma for the boundary?
No.



Hybrids

Theorem

For each $\mathcal{B}$ and any $\varepsilon > 0$ exists a subset $\tilde{\mathcal{B}} \subset \mathcal{B}$ such that for any distinct $B_1, B_2 \in \tilde{\mathcal{B}}$ we have

$$\mathcal{L}(B_1 \cap B_2) \leq \varepsilon \min\{\mathcal{L}(B_1), \mathcal{L}(B_2)\}$$

and with

$$\mathcal{L}\Big(\bigcup \mathcal{B}\Big) \lesssim \mathcal{L}\Big(\bigcup \tilde{\mathcal{B}}\Big), \quad \mathcal{H}^{n-1}\Big(\partial \bigcup \mathcal{B}\Big) \lesssim_n \varepsilon^{-\frac{n-1}{n+1}} \mathcal{H}^{n-1}\Big(\partial \bigcup \tilde{\mathcal{B}}\Big).$$

The rate $\varepsilon^{-\frac{n-1}{n+1}}$ is sharp. But:

Theorem

In one dimension exists a subset $\tilde{\mathcal{B}} \subset \mathcal{B}$ of intervals with disjoint closures such that

$$\mathcal{L}\Big(\bigcup \mathcal{B}\Big) \leq 5\mathcal{L}\Big(\bigcup \tilde{\mathcal{B}}\Big), \quad \mathcal{H}^{n-1}\Big(\partial \bigcup \mathcal{B}\Big) \leq \mathcal{H}^{n-1}\Big(\partial \bigcup \tilde{\mathcal{B}}\Big).$$

Thank you