Lemma (Vitali covering lemma)

Any bounded set $\mathcal{B}$ of balls has a subset $\tilde{\mathcal{B}}$ of pairwise disjoint balls with

$$\mathcal{L}\Big(\bigcup \mathcal{B}\Big) \lesssim_n \mathcal{L}\Big(\bigcup \tilde{\mathcal{B}}\Big).$$

For $f : \mathbb{R}^n \to \mathbb{R}$ the uncentered Hardy-Littlewood maximal function is defined by

$$\mathrm{M}f(x) = \sup_{B \ni x} f_B \qquad \text{with} \qquad f_B = \frac{1}{\mathcal{L}(B)} \int_B |f|.$$

Theorem (Hardy-Littlewood maximal function theorem)

$$\|\mathrm{M}f\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|f\|_{L^p(\mathbb{R}^n)} \qquad \|\mathrm{M}f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim_n \|f\|_{L^1(\mathbb{R}^n)}$$

if and only if $p > 1$.

Theorem (Covering lemma for boundary, W. 2025)

Any bounded set $\mathcal{B}$ of balls has a subset $\tilde{\mathcal{B}}$ of pairwise disjoint balls with

$$\mathcal{H}^{n-1}\Big(\partial\bigcup\mathcal{B}\Big) \lesssim_n \mathcal{H}^{n-1}\Big(\partial\bigcup\tilde{\mathcal{B}}\Big).$$

Theorem (Kinnunen 1997)

For $p > 1$ we have

$$\|\nabla \mathrm{M} f\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|\nabla f\|_{L^p(\mathbb{R}^n)}$$

Question (Hajłasz and Onninen 2004)

Is it true that

$$\|\nabla \mathrm{M} f\|_{L^1(\mathbb{R}^n)} \lesssim_n \|\nabla f\|_{L^1(\mathbb{R}^n)}?$$

Question of Hajłasz-Onninen is mostly solved in one dimension [Aldaz, Carneiro, Gonzalez-Riquelme, Kinnunen, Kurka, Luiro, Madrid, Pérez Lázaro, Saksman, Tanaka,...], few results in higher dimensions [W.].

Coarea formula

$$\|\nabla f\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}} \mathcal{H}^{n-1}(\partial\{x \in \mathbb{R}^n : f(x) > \lambda\})\,\mathrm{d}\lambda$$

Superlevel sets

$$\{x \in \mathbb{R}^n : \mathrm{M}f(x) > \lambda\} = \bigcup \{B : f_B > \lambda\}$$

for *uncentered* maximal operators.