Endpoint regularity of maximal functions

Julian Weigt

Mathematics Essex seminar

08.02.2024

The Hardy-Littlewood maximal function theorem

For $f : \mathbb{R}^n \to \mathbb{R}$ the centered Hardy-Littlewood maximal function is defined by

$$\mathrm{M}^{\mathrm{c}} f(x) = \sup_{r>0} f_{B(x,r)} \quad \text{with} \quad f_{B(x,r)} = \frac{1}{\mathcal{L}(B(x,r))} \int_{B(x,r)} |f|.$$

Theorem (Hardy-Littlewood maximal function theorem)

$$\|\mathrm{M}^{\mathrm{c}} f\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|f\|_{L^p(\mathbb{R}^n)}$$

if and only if $p > 1$.

$$\|\mathrm{M}^{\mathrm{c}} f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim_n \|f\|_{L^1(\mathbb{R}^n)}$$

The Hardy-Littlewood maximal function theorem

Proof strategy:

1. $p = 1$: prove weak bound i.e. for every $\lambda > 0$ show

$$\mathcal{L}(\{x \in \mathbb{R}^n : \mathrm{M}^{\mathrm{c}} f(x) > \lambda\}) \lesssim_n \frac{\|f\|_{L^1(\mathbb{R}^n)}}{\lambda}$$

2. $p = \infty$ ✓.
3. Conclude $1 < p < \infty$ by interpolation ✓.

The Hardy-Littlewood maximal function theorem

$p = 1$:

Theorem (Vitali covering lemma)

Let $\mathcal{B}$ be a (bounded) set of balls. Then it has a subset $\mathcal{S} \subset \mathcal{B}$ of disjoint balls with

$$\bigcup \mathcal{B} \subset \bigcup_{B \in \mathcal{S}} 5B.$$

Then for every $\lambda > 0$ we estimate

$$\begin{aligned} \mathcal{L}(\{x \in \mathbb{R}^n : \mathrm{M^c} f(x) > \lambda\}) &\leq \mathcal{L}\Big(\bigcup \{B : f_B > \lambda\}\Big) \\ &\leq \sum_{B \in \mathcal{S}} 5^n \mathcal{L}(B) \leq 5^n \sum_{B \in \mathcal{S}} \frac{1}{\lambda} \int_B |f| \\ &\leq 5^n \frac{\|f\|_{L^1(\mathbb{R}^n)}}{\lambda} \ \checkmark \end{aligned}$$

The endpoint regularity question

Theorem (Juha Kinnunen (1997))

For $p > 1$ we have

$$\|\nabla \mathrm{M}^{\mathrm{c}} f\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|\nabla f\|_{L^p(\mathbb{R}^n)}$$

Proof: For $e \in \mathbb{R}^n$ by the sublinearity of M^{c}

$$\begin{aligned} \partial_e \mathrm{M}^{\mathrm{c}} f(x) &\sim \frac{\mathrm{M}^{\mathrm{c}} f(x+he) - \mathrm{M}^{\mathrm{c}} f(x)}{h} \\ &\leq \frac{\mathrm{M}^{\mathrm{c}}(f(\cdot + he) - f)(x)}{h} \\ &= \mathrm{M}^{\mathrm{c}}\Big(\frac{f(\cdot + he) - f)}{h}\Big)(x) \sim \mathrm{M}^{\mathrm{c}}(\partial_e f)(x) \end{aligned}$$

By the Hardy-Littlewood maximal function theorem for $p > 1$

$$\|\nabla \mathrm{M}^{\mathrm{c}} f\|_{L^p(\mathbb{R}^n)} \leq \|\mathrm{M}^{\mathrm{c}}(|\nabla f|)\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|\nabla f\|_{L^p(\mathbb{R}^n)}$$

The endpoint regularity question

Question (Hajłasz and Onninen 2004)

Is it true that

$$\|\nabla \mathrm{M}^{\mathrm{c}} f\|_{L^1(\mathbb{R}^n)} \lesssim_n \|\nabla f\|_{L^1(\mathbb{R}^n)}?$$

Uncentered Hardy-Littlewood maximal function

$$\widetilde{\mathrm{M}} f(x) = \sup_{B \ni x} f_B.$$

Endpoint question by Hałjasz and Onninen is interesting for $\widetilde{\mathrm{M}}$ and other maximal operators.

In one dimension

Theorem (Tanaka 2002, Aldaz and Pérez Lázaro 2007)

For $f : \mathbb{R} \to \mathbb{R}$ we have

$$\|\nabla \widetilde{\mathrm{M}} f\|_1 \leq \|\nabla f\|_1$$

1. For almost all $x \in \mathbb{R}^d$: $\widetilde{\mathrm{M}} f(x) \geq f(x)$
2. In one dimension

   $$\|\nabla f\|_1 = \sup_{x_1 < x_2 < \dots} \sum_i |f(x_{i+1}) - f(x_i)| = \operatorname{var} f$$

   and the supremum is attained if $\dots < x_{-1} < x_0 < x_1 < \dots$ are the strict local extrema of f.
3. and $\widetilde{\mathrm{M}} f(x) = f(x)$ at every strict local maximum x of $\widetilde{\mathrm{M}} f$.

In one dimension

1. If f is continuous in x then

$$\widetilde{\mathrm{M}}f(x) \geq \lim_{r\to 0} f_{B(x,r)} = f(x).$$

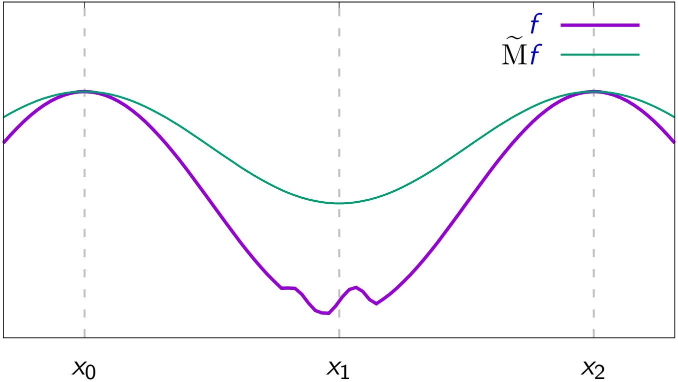
2. For any $x_1 < x_2 < \ldots$ we have

$$\sum_i |f(x_{i+1}) - f(x_i)| = \sum_i \left| \int_{x_i}^{x_{i+1}} f' \right| \leq \sum_i \int_{x_i}^{x_{i+1}} |f'| \leq \|f'\|_{L^1(\mathbb{R})}.$$

Conversely, if $\ldots < x_{-1} < x_0 < x_1 < \ldots$ are the strict local extrema then for $x_i \leq x \leq x_{i+1}$ we have $(-1)^i f'(x) \geq 0$. Then

$$\begin{aligned}\|f'\|_{L^1(\mathbb{R})} &= \sum_i \int_{x_i}^{x_{i+1}} |f'| = \sum_i (-1)^i \int_{x_i}^{x_{i+1}} f' \\ &= \sum_i (-1)^i \big(f(x_{i+1}) - f(x_i)\big) = \sum_i |f(x_{i+1}) - f(x_i)|\end{aligned}$$

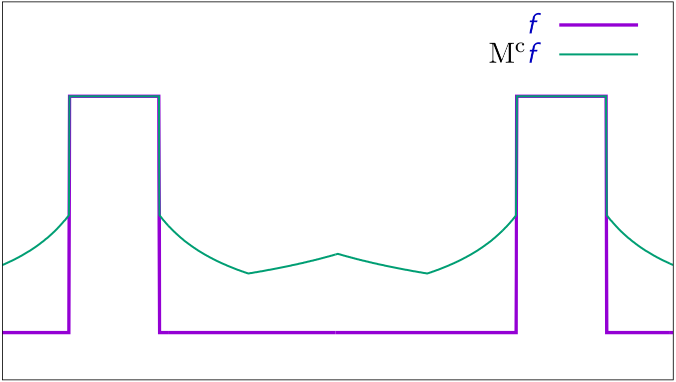
In one dimension



$$\begin{aligned}\mathrm{var}_{[x_0,x_2]}\,\widetilde{\mathrm{M}}f &= |\widetilde{\mathrm{M}}f(x_1) - \widetilde{\mathrm{M}}f(x_0)| + |\widetilde{\mathrm{M}}f(x_2) - \widetilde{\mathrm{M}}f(x_1)| \\ &\leq |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| \\ &\leq \mathrm{var}_{[x_0,x_2]}\, f\end{aligned}$$

The centered maximal function M^c

~~$\mathrm{M}^c f(x) = f(x)$ at every strict local maximum x of $\mathrm{M}^c f$~~.



Proof strategy for uncentered $\widetilde{\mathrm{M}}$ fails for centered M^c.

The centered maximal function M^{c}

$\operatorname{var} \mathrm{M}^{\mathrm{c}} f \leq 35000 \operatorname{var} f$ [Kurka 2015]

$\operatorname{var} \mathrm{M}^{\mathrm{c}}(1_E) \leq \operatorname{var} 1_E$ for any $E \subset \mathbb{R}$ [Bilz, W 2022]

$\operatorname{var} \mathrm{M}^{\mathrm{c}} f \leq \operatorname{var} f$? open

Continuity

Stronger property than boundedness:

Operator continuity of M

f close to g $\quad\Rightarrow\quad$ $\mathrm{M}f$ close to $\mathrm{M}g$?

By sublinearity $\mathrm{M}f(x) - \mathrm{M}g(x) \leq \mathrm{M}(f-g)(x) + \mathrm{M}g(x) - \mathrm{M}g(x)$.

$$\|\mathrm{M}f - \mathrm{M}g\|_{L^p(\mathbb{R}^n)} \leq \|\mathrm{M}(f-g)\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|f-g\|_{L^p(\mathbb{R}^n)}.$$

However, $|\nabla \mathrm{M}f(x) - \nabla \mathrm{M}g(x)| \not\leq |\nabla \mathrm{M}(f-g)(x)|$. Nevertheless, [Luiro, 2004] proved for $p > 1$ that

$$\|f_n - f\|_{W^{1,p}(\mathbb{R}^n)} \to 0 \quad \Longrightarrow \quad \|\nabla \mathrm{M}f_n - \nabla \mathrm{M}f\|_{L^p(\mathbb{R}^n)} \to 0.$$

For $p = 1$ continuity is now known in the same cases as the gradient bound.

The fractional maximal function

For $0 < \alpha < n$ the centered fractional Hardy-Littlewood maximal function is

$$\mathrm{M}_\alpha^{\mathrm{c}} f(x) = \sup_{r>0} r^\alpha f_{B(x,r)}.$$

Corresponding Hardy-Littlewood theorem

$$\|\mathrm{M}_\alpha f\|_{L^{p_\alpha}(\mathbb{R}^n)} \lesssim_{n,\alpha,p} \|f\|_{L^p(\mathbb{R}^n)}$$

with $p_\alpha = \frac{pn}{n-\alpha p} > p$ if and only if $p > 1$. Corresponding regularity bound

$$\|\nabla \mathrm{M}_\alpha f\|_{L^{p_\alpha}(\mathbb{R}^n)} \lesssim_{n,\alpha,p} \|\nabla f\|_{L^p(\mathbb{R}^n)},$$

proven for $p > 1$.

The fractional maximal function

Theorem (Kinnunen and Saksman 2003)

For $\alpha \geq 1$

$$|\nabla \mathrm{M}^{\mathrm{c}}_{\alpha} f(x)| \lesssim_n |\mathrm{M}^{\mathrm{c}}_{\alpha-1} f(x)|.$$

Corollary (Carneiro and Madrid 2016)

For $\alpha \geq 1$ we have $1_\alpha = \frac{n}{n-\alpha} = \left(\frac{n}{n-1}\right)_{\alpha-1}$ and $\frac{n}{n-1} > 1$ and therefore

$$\|\nabla \mathrm{M}^{\mathrm{c}}_{\alpha} f\|_{L^{1_\alpha}(\mathbb{R}^n)} \lesssim_n \|\mathrm{M}^{\mathrm{c}}_{\alpha-1} f\|_{L^{1_\alpha}(\mathbb{R}^n)} \lesssim_n \|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)}$$

$$\lesssim_n \|\nabla f\|_{L^1(\mathbb{R}^n)}.$$

The fractional maximal function

- $0 < \alpha < 1$ radial f [Beltran, Madrid, Luiro 2017+2019]
- most fractional results are known both for the centered and uncentered maximal function.
- Continuity of fractional maximal operators is known in the same cases as boundedness.

Overview of past results

$n = 1$ uncentered $\widetilde{\mathrm{M}}$	[Tanaka 2002, Aldaz + Pérez Lázaro 2007]
$n = 1$ centered M^{c}	[Kurka 2015]
fractional M_α for $\alpha \geq 1$	[Beltran, Carneiro, Madrid, Kinnunen, Ramos, Saari,...]
block decreasing f	[Aldaz+Pérez Lázaro 2009]
radial f	[Luiro 2018]

New results in higher dimensions

- uncentered maximal function of characteristic function $\widetilde{\mathrm{M}} 1_E$
- dyadic maximal operator
- fractional maximal operator M_α for all $\alpha > 0$ (also continuity)
  Fractional: complete!
- cube maximal operator

New results: Coarea formula

$$
\begin{aligned}
\|\nabla f\|_{L^1(\mathbb{R}^1)} &= \sum_i |f(x_{i+1}) - f(x_i)| \\
&= \sum_i (f(x_{2i+1}) - f(x_{2i})) + (f(x_{2i-1}) - f(x_{2i})) \\
&= \sum_i \int_{f(x_{2i})}^{f(x_{2i+1})} 1 \,\mathrm{d}\lambda + \dots \\
&= \sum_i \int_{\mathbb{R}} 1_{[f(x_{2i}), f(x_{2i+1})]}(\lambda) \,\mathrm{d}\lambda + \dots \\
&= \sum_i \int_{\mathbb{R}} \#\big([x_i, x_{i+1}] \cap \partial\{x \in \mathbb{R}^n : f(x) > \lambda\}\big) \,\mathrm{d}\lambda \\
&= \int_{\mathbb{R}} \#\partial\{x \in \mathbb{R} : f(x) > \lambda\} \,\mathrm{d}\lambda \\
&= \int_{\mathbb{R}} \mathcal{H}^0(\partial\{x \in \mathbb{R} : f(x) > \lambda\}) \,\mathrm{d}\lambda
\end{aligned}
$$

New results: Reformulation and decomposition

Coarea formula

$$\|\nabla f\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}} \mathcal{H}^{n-1}(\partial\{x \in \mathbb{R}^n : f(x) > \lambda\})\,\mathrm{d}\lambda$$

Compare with layer cake formula/Cavalieri's principle

$$\|f\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}} \mathcal{L}(\{x \in \mathbb{R}^n : f(x) > \lambda\})\,\mathrm{d}\lambda$$

Superlevel sets

$$\{\mathrm{M}f > \lambda\} = \{x \in \mathbb{R}^n : \mathrm{M}f(x) > \lambda\} = \bigcup\{B : f_B > \lambda\}$$

for <u>uncentered</u> maximal operators.

New results: Tools

Decomposition of the boundary

Denote

$$\mathcal{B}_\lambda^{<} = \{B : f_B > \lambda,\ \mathcal{L}(B \cap \{f > \lambda\}) < 2^{-n-1}\mathcal{L}(B)\}$$

and $\mathcal{B}_\lambda^{\geq}$ accordingly.

1. **relative isoperimetric inequality**:

$$\min\{\mathcal{L}(Q \cap E), \mathcal{L}(Q \setminus E)\}^{n-1} \lesssim_n \mathcal{H}^{n-1}(Q \cap \partial E)^n.$$

2. **Vitali covering** and similar: general balls $\to$ separated balls
3. **Besicovitch covering for boundary**
4. **superlevelset estimate**: $f < 0$ on most of $B \Rightarrow$ most mass of f lies far above f_B

New results: Reformulation and decomposition

We have

$$\{\mathrm{M}f > \lambda\} = \bigcup \mathcal{B}_\lambda^{<} \cup \bigcup \mathcal{B}_\lambda^{\geq}.$$

Since $\{f > \lambda\} \subset \{\mathrm{M}f > \lambda\}$ we have

$$\partial\{\mathrm{M}f > \lambda\} \subset (\partial\{\mathrm{M}f > \lambda\} \setminus \overline{\{f > \lambda\}}) \cup \partial\{f > \lambda\}.$$

We conclude

Decomposition

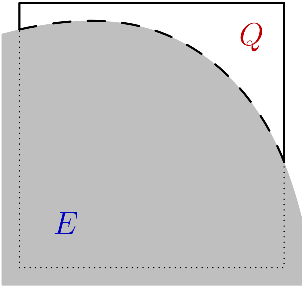
$$\int_{\mathbb{R}^d} |\nabla \mathrm{M}f| \leq \int_0^\infty \mathcal{H}^{n-1}\Big(\partial \bigcup \mathcal{B}_\lambda^{\geq} \setminus \overline{\{f > \lambda\}}\Big)\,\mathrm{d}\lambda + \int_0^\infty \mathcal{H}^{n-1}\Big(\partial \bigcup \mathcal{B}_\lambda^{<}\Big)\,\mathrm{d}\lambda + \int_{\mathbb{R}^d} |\nabla f|$$

New results: High density case $\mathcal{B}_\lambda^{\geq}$

Proposition

For Q, E with $\mathcal{L}(Q \cap E) \geq 2^{-n-1}\mathcal{L}(Q)$ we have

$$\mathcal{H}^{n-1}(\partial Q \setminus \overline{E}) \lesssim_n \mathcal{H}^{n-1}(Q \cap \partial E)$$



New results: High density case $\mathcal{B}_\lambda^{\geq}$

dyadic maximal operator

$$\mathrm{M}^{\mathrm{d}} f(x) = \sup_{\text{dyadic } Q,\ Q \ni x} f_Q.$$

$$\begin{aligned}
\mathcal{H}^{n-1}\Big(\partial \bigcup \mathcal{Q}_\lambda^{\geq} \setminus \overline{\{f > \lambda\}}\Big) &\leq \sum_{Q \in \mathcal{Q}_\lambda^{\geq}} \mathcal{H}^{n-1}\big(\partial Q \setminus \overline{\{f > \lambda\}}\big) \\
&\lesssim_n \sum_{Q \in \mathcal{Q}_\lambda^{\geq}} \mathcal{H}^{n-1}\big(Q \cap \partial \{f > \lambda\}\big) \\
&\leq \mathcal{H}^{n-1}\big(\partial \{f > \lambda\}\big)
\end{aligned}$$

New results: Low density case $\mathcal{B}_\lambda^<$, dyadic

Proposition

For a low density cube Q we have

$$\int_{\lambda_Q}^{f_Q} \mathcal{H}^{n-1}(\partial Q)\,\mathrm{d}\lambda = (f_Q - \lambda_Q)\mathcal{H}^{n-1}(\partial Q)$$
$$\lesssim \int_{f_Q}^{\infty} \dots \mathcal{H}^{n-1}(Q \cap \partial\{f > \lambda\} \cap \dots)\,\mathrm{d}\lambda$$

1. Sum over all low density cubes Q,
2. change order of summation and integration,
3. use convergence of a geometric sum,
4. and recover $\|\nabla f\|_1$ by coarea formula.

New results: Low density case $\mathcal{B}_\lambda^<$, fractional

$1 \leq \alpha$ [Kinnunen + Saksman, Carneiro + Madrid]

$$\|\nabla \mathrm{M}_\alpha f\|_{\frac{n}{n-\alpha}} \lesssim_n \|\mathrm{M}_{\alpha-1} f\|_{\frac{n}{n-\alpha}} \lesssim_n \|f\|_{\frac{n}{n-1}} \lesssim_n \|\nabla f\|_1.$$

$0 < \alpha$

$$\|\nabla \mathrm{M}_\alpha f\|_{\frac{n}{n-\alpha}} \lesssim_n \|\mathrm{M}_{\alpha,-1} f\|_{\frac{n}{n-\alpha}} \lesssim_{n,\alpha} \|\nabla f\|_1,$$

$\mathrm{M}_{\alpha,-1}$ replacement for $\mathrm{M}_{\alpha-1}$ if $0 < \alpha < 1$.
Can bound $\mathrm{M}_{\alpha,-1} f$ both centered and uncentered

- using low density arguments from the dyadic proof
- extra flexibility coming from $\alpha > 0$, allowing for rough Vitali covering arguments

Uncentered HL $\widetilde{\mathrm{M}}f$ (balls)?

- All arguments work except
- low density bound $(f_B - \lambda_B)\mathcal{H}^{n-1}(\partial B) \lesssim_n$?

Thank you