Higher Dimensional Techniques for the Regularity of Maximal Functions

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For $f : \mathbb{R}^n \to \mathbb{R}$ the centered Hardy-Littlewood maximal function is defined by

$$\mathrm{M}^{\mathrm{c}} f(x) = \sup_{r>0} f_{B(x,r)} \quad \text{with} \quad f_{B(x,r)} = \frac{1}{\mathcal{L}(B(x,r))} \int_{B(x,r)} |f|.$$

Introduction: Background

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The Hardy-Littlewood maximal function theorem:

$$\|\mathrm{M^c} f\|_{L^p(\mathbb{R}^n)} \leq C_{n,p} \|f\|_{L^p(\mathbb{R}^n)} \qquad \text{if and only if } p > 1$$

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Juha Kinnunen (1997):

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Question (Hajłasz and Onninen 2004)

Is it true that

$$\|\nabla \mathrm{M}^{\mathrm{c}} f\|_{L^1(\mathbb{R}^n)} \leq C_n \|\nabla f\|_{L^1(\mathbb{R}^n)}?$$

For $e \in \mathbb{R}^n$ by the sublinearity of M^{c} Kinnunen proved

$$|\nabla \mathrm{M}^{\mathrm{c}} f(x)| \leq \mathrm{M}^{\mathrm{c}} |\nabla f|(x).$$

Thus by the Hardy-Littlewood maximal function theorem for $p > 1$

$$\|\nabla \mathrm{M}^{\mathrm{c}} f\|_{L^p(\mathbb{R}^n)} \lesssim \|\mathrm{M}^{\mathrm{c}}(|\nabla f|)\|_{L^p(\mathbb{R}^n)} \lesssim \|\nabla f\|_{L^p(\mathbb{R}^n)}$$

Introduction: Motivation

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In 2002 Tanaka proved

$$\|\nabla \widetilde{\mathrm{M}} f\|_1 \leq 2\|\nabla f\|_1$$

for the uncentered maximal function of a function $f : \mathbb{R} \to \mathbb{R}$. The proof depends strongly on one-dimensional geometry.

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$$\|\mathrm{M}_{\alpha} f\|_{L^{\frac{pn}{n-\alpha p}}(\mathbb{R}^n)} \leq C_{n,\alpha,p} \|f\|_{L^p(\mathbb{R}^n)}$$

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if and only if $p > 1$. The corresponding regularity bound is

$$\|\nabla \mathrm{M}_\alpha f\|_{L^{\frac{pn}{n-\alpha p}}(\mathbb{R}^n)} \leq C_{n,\alpha,p} \|\nabla f\|_{L^p(\mathbb{R}^n)}.$$

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For example: Continuity of the operator given by $f \mapsto \nabla \mathrm{M} f$ on $W^{1,1}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$. This is a stronger property than boundedness.

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- dyadic maximal operator
- fractional maximal operator
- cube maximal operator

Proof: Reformulation and decomposition

Coarea formula

$$\|\nabla f\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial\{x \in \mathbb{R}^n : f(x) > \lambda\})\, \mathrm{d}\lambda$$

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for *uncentered* maximal operators.

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Decomposition of the boundary

Denote

$$\mathcal{B}_\lambda^< = \{B : f_B > \lambda,\ \mathcal{L}(B \cap \{f > \lambda\}) < 2^{-n-1}\mathcal{L}(B)\}$$

and $\mathcal{B}_\lambda^\geq$ accordingly.

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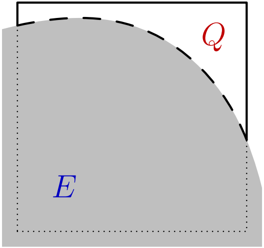
We will estimate the perimeter of $\bigcup \mathcal{B}_\lambda^<$ and $\bigcup \mathcal{B}_\lambda^\geq$ separately.

Proof: High density case $\mathcal{B}_\lambda^{\geq}$

Proposition

$$\mathcal{L}(Q \cap E) \geq 2^{-n-1}\mathcal{L}(Q) \quad \Longrightarrow$$
$$\mathcal{H}^{d-1}(\partial Q \setminus \overline{E}) \lesssim \mathcal{H}^{d-1}(Q \cap \partial E)$$



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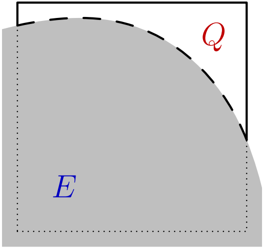

relative isoperimetric inequality

If $\mathcal{L}(Q \cap E) \leq \mathcal{L}(Q)/2$ then

$$\mathcal{L}(Q \cap E)^{n-1} \lesssim \mathcal{H}^{d-1}(Q \cap \partial E)^n$$

Proof: Low density case $\mathcal{B}_\lambda^<$

dyadic maximal operator

$$\mathrm{M}^{\mathrm{d}} f(x) = \sup_{Q \ni x,\ Q \text{ dyadic}} f_Q.$$

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$$\int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial \bigcup \mathcal{Q}_\lambda^<)\,\mathrm{d}\lambda$$

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$$\mathrm{M^d} f(x) = \sup_{Q \ni x,\ Q \text{ dyadic}} f_Q.$$

$$\int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial \bigcup \mathcal{Q}_\lambda^<)\,\mathrm{d}\lambda \leq \sum_{Q \text{ dyadic}} (f_Q - \lambda_Q)\mathcal{H}^{d-1}(\partial Q)$$

with

$$\mathcal{L}(Q \cap \{f > \lambda_Q\}) = 2^{-n-1}\mathcal{L}(Q)$$

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Proposition

$$(f_Q - \lambda_Q)\mathcal{L}(Q) \lesssim \int_{\mathbb{R}} \sum_{P \subsetneq Q : \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\})\, \mathrm{d}\lambda$$

where P is maximal above $\bar{\lambda}_P$ and

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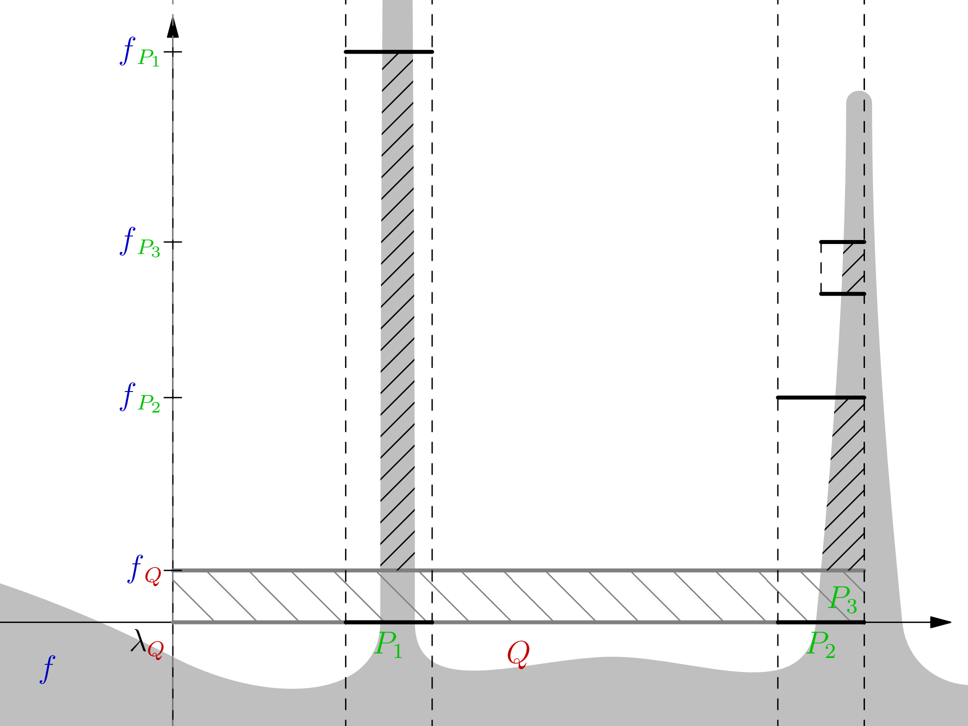
Then we sum over all Q and change the order of summation, use the convergence of a geometric sum and apply the relative isoperimetric inequality to P. We recover $\|\nabla f\|_1$ on the right hand side.



f_{P_1}
f_{P_3}
f_{P_2}
f_Q
λ_Q
f
P_1
Q
P_2
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Proof: Low density case $\mathcal{B}_\lambda^<$, general cubes

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Proposition (Vitali for perimeter)

For any (finite) set of cubes $\mathcal{Q}$ there is a subset $\mathcal{S} \subset \mathcal{Q}$ of disjoint cubes such that

$$\mathcal{H}^{d-1}\left(\partial \bigcup \mathcal{Q}\right) \lesssim \sum_{S \in \mathcal{S}} \mathcal{H}^{d-1}(\partial S).$$

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The Vitali covering for the perimeter also works for balls, however we do not have the earlier bound on $(f_Q - \lambda_Q)\mathcal{L}(Q)$ for balls.

Proof: Low density case $\mathcal{B}_\lambda^<$, fractional

$1 \leq \alpha$ [Kinnunen + Saksman, Carneiro + Madrid]

$$\|\nabla \mathrm{M}_\alpha f\|_{\frac{n}{n-\alpha}} \lesssim \|\mathrm{M}_{\alpha-1} f\|_{\frac{n}{n-\alpha}} \lesssim \|f\|_{\frac{n}{n-1}} \lesssim \|\nabla f\|_1.$$

Proof: Low density case $\mathcal{B}_\lambda^<$, fractional

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$0 < \alpha$

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Thank you