Regularity of maximal functions in higher dimensions

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Background

For $f : \mathbb{R}^n \to \mathbb{R}$ the centered Hardy-Littlewood maximal function is defined by

$$\mathrm{M^c} f(x) = \sup_{r>0} f_{B(x,r)} \qquad \text{with} \qquad f_{B(x,r)} = \frac{1}{\mathcal{L}(B(x,r))} \int_{B(x,r)} |f|.$$

Theorem (Hardy-Littlewood maximal function theorem)

$$\|\mathrm{M^c} f\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|f\|_{L^p(\mathbb{R}^n)}$$

if and only if $p > 1$.

$$\|\mathrm{M^c} f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim_n \|f\|_{L^1(\mathbb{R}^n)}$$

Background

Theorem (Juha Kinnunen (1997))

For $p > 1$ we have

$$\|\nabla \mathrm{M}^{\mathrm{c}} f\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|\nabla f\|_{L^p(\mathbb{R}^n)}$$

Proof: For $e \in \mathbb{R}^n$ by the sublinearity of M^{c}

$$\begin{aligned} \partial_e \mathrm{M}^{\mathrm{c}} f(x) &\sim \frac{\mathrm{M}^{\mathrm{c}} f(x+he) - \mathrm{M}^{\mathrm{c}} f(x)}{h} \\ &\leq \frac{\mathrm{M}^{\mathrm{c}}(f(\cdot + he) - f)(x)}{h} \\ &= \mathrm{M}^{\mathrm{c}}\Big(\frac{f(\cdot + he) - f)}{h}\Big)(x) \sim \mathrm{M}^{\mathrm{c}}(\partial_e f)(x) \end{aligned}$$

By the Hardy-Littlewood maximal function theorem for $p > 1$

$$\|\nabla \mathrm{M}^{\mathrm{c}} f\|_{L^p(\mathbb{R}^n)} \leq \|\mathrm{M}^{\mathrm{c}}(|\nabla f|)\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|\nabla f\|_{L^p(\mathbb{R}^n)}$$

Background

Question (Hajłasz and Onninen 2004)

Is it true that

$$\|\nabla \mathrm{M}^{\mathrm{c}} f\|_{L^1(\mathbb{R}^n)} \lesssim_n \|\nabla f\|_{L^1(\mathbb{R}^n)}?$$

Uncentered Hardy-Littlewood maximal function

$$\mathrm{M} f(x) = \sup_{B \ni x} f_B.$$

Endpoint question by Hałjasz and Onninen is interesting for M and other maximal operators.

In one dimension

Theorem (Tanaka 2002, Aldaz and Pérez Lázaro 2007)

For $f : \mathbb{R} \to \mathbb{R}$ *we have*

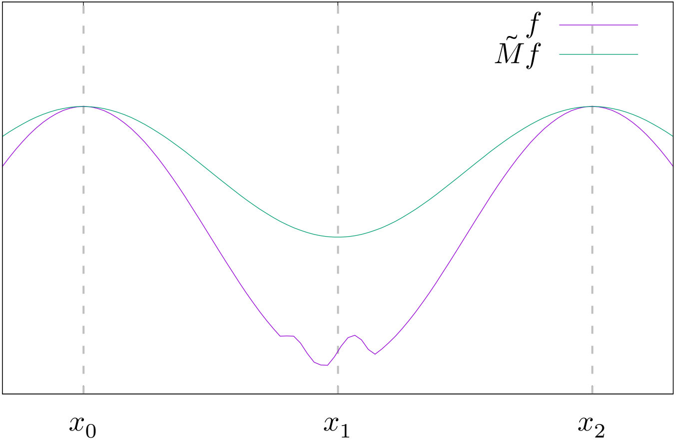
$$\|\nabla \mathrm{M} f\|_1 \leq \|\nabla f\|_1$$

Proof:

- In one dimension

$$\|\nabla f\|_1 = \sup_{x_1 < x_2 < \dots} \sum_i |f(x_{i+1}) - f(x_i)| = \operatorname{var} f.$$

- For almost all $x \in \mathbb{R}^d$: $\mathrm{M} f(x) \geq f(x)$
- and $\mathrm{M} f(x) = f(x)$ at a strict local maximum of $\mathrm{M} f$.



$$\begin{aligned}\operatorname{var}_{[x_0,x_2]} \mathrm{M}f &= |\mathrm{M}f(x_1) - \mathrm{M}f(x_0)| + |\mathrm{M}f(x_2) - \mathrm{M}f(x_1)| \\ &\leq |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| \\ &\leq \operatorname{var}_{[x_0,x_2]} f\end{aligned}$$

Past progress

$n = 1$	[Tanaka 2002, Aldaz +Pérez Lázaro 2007]
block decreasing f	[Aldaz+Pérez Lázaro 2009]
centered M, $n = 1$	[Kurka 2015]
radial f	[Luiro 2018]
fractional maximal function $\alpha \geq 1$	[Kinnunen + Saksman 2003, Carneiro + Madrid 2016]

- bounds on other maximal operators, such as local,... ,
- local regularity and smoothing, i.e. does $f \mapsto \nabla \mathrm{M} f$ map $\mathrm{BV}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ or only into Radon measures?
- operator continuity of $f \mapsto \nabla \mathrm{M} f$ on $W^{1,1}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$, stronger than boundedness.
- best constants in one dimension

Higher dimensions

Theorem (W. 2022)

For $E \subset \mathbb{R}^n$ *we have*

$$\|\nabla \mathrm{M}(1_E)\|_1 \leq C_n \|\nabla 1_E\|_1.$$

Theorem (W. 2023)

For $f : \mathbb{R}^n \to \mathbb{R}$ *we have*

$$\|\nabla \mathrm{M}^{\mathrm{d}} f\|_1 \leq C_n \|\nabla f\|_1$$

for the dyadic maximal function

$$\mathrm{M}^{\mathrm{d}} f(x) = \sup_{\textit{dyadic cube } Q,\ Q \ni x} f_Q.$$

Higher dimensions

Combining tools from both leads to the same bound for cube maximal operator given by

$$\mathrm{M}^{\square} f(x) = \sup_{\text{cube } Q,\ Q \ni x} f_Q .$$

Theorem (W. 2024)

For the cube maximal operator $\mathrm{M}^{\square}$ *and* $f : \mathbb{R}^n \to \mathbb{R}$ *we have*

$$\|\nabla \mathrm{M}^{\square} f\|_1 \leq C_n \|\nabla f\|_1 .$$

Proof works for more general sets with a tiling property, but not for balls and certainly not for centered M^{c}.

Proof ingredients

Coarea formula

$$\|\nabla f\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}} \mathcal{H}^{n-1}(\partial\{x \in \mathbb{R}^n : f(x) > \lambda\})\,\mathrm{d}\lambda$$

Superlevel sets

$$\{x \in \mathbb{R}^n : \mathrm{M}f(x) > \lambda\} = \bigcup\{B : f_B > \lambda\}$$

for *uncentered* maximal operators.

Relative isoperimetric inequality

$$\min\{\mathcal{L}(Q \cap E), \mathcal{L}(Q \setminus E)\}^{n-1} \lesssim_n \mathcal{H}^{n-1}(Q \cap \partial E)^n.$$

Approach

Since $\mathrm{M}f \geq f$ a.e. we have

$$\mathcal{H}^{n-1}(\partial\{\mathrm{M}f > \lambda\}) \leq \mathcal{H}^{n-1}(\partial\{\mathrm{M}f > \lambda\} \setminus \overline{\{f > \lambda\}}) + \mathcal{H}^{n-1}(\partial\{f > \lambda\})$$

and we can bound the second summand in terms of $\|\nabla f\|_1$. For the first summand define

$$\mathcal{B}_{\lambda,\geq} := \{B : f_B > \lambda,\ \mathcal{L}(B \cap \{f > \lambda\}) \geq 1/2\},$$

$\mathcal{B}_{\lambda,<}$ similarly. Then

$$\mathcal{H}^{n-1}(\partial\{\mathrm{M}f > \lambda\} \setminus \overline{\{f > \lambda\}}) \leq \mathcal{H}^{n-1}\Big(\partial \bigcup \mathcal{B}_{\lambda,\geq} \setminus \overline{\{f > \lambda\}}\Big) + \mathcal{H}^{n-1}\Big(\partial \bigcup \mathcal{B}_{\lambda,<} \setminus \overline{\{f > \lambda\}}\Big)$$

and we deal with each summand separately.

Important tool:

Proposition

Let $\mathcal{B}_\lambda$ be a set of balls B with $\mathcal{L}(B \cap E) = \lambda \mathcal{L}(B)$. Then

$$\mathcal{H}^{n-1}\Big(\partial \bigcup \mathcal{B}_\lambda \setminus \overline{E}\Big) \lesssim |\log \lambda| \lambda^{-\frac{n-1}{n}} \mathcal{H}^{n-1}(\partial E).$$

Can also be used to prove

Theorem (Vitali for boundary, W. 2025)

Any bounded set $\mathcal{B}$ of balls has a disjoint subset $\tilde{\mathcal{B}}$ with

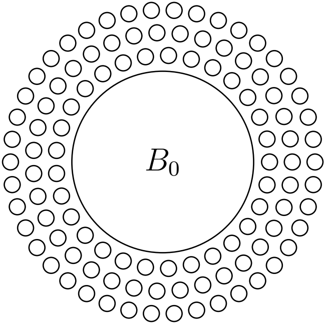
$$\mathcal{H}^{n-1}\Big(\partial \bigcup \mathcal{B}\Big) \lesssim_n \mathcal{H}^{n-1}\Big(\partial \bigcup \tilde{\mathcal{B}}\Big).$$

This can in turn be used to remove $\log \lambda$ term.

Lemma (Vitali covering lemma)

Any bounded set $\mathcal{B}$ of balls has a disjoint subset $\tilde{\mathcal{B}}$ with

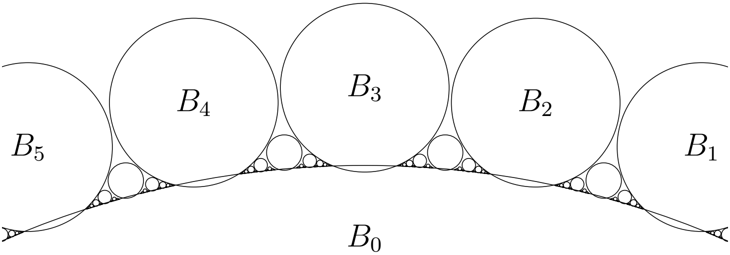
$$\mathcal{L}\Big(\bigcup \mathcal{B}\Big) \lesssim_n \mathcal{L}\Big(\bigcup \tilde{\mathcal{B}}\Big).$$



That means Vitali selection strategy doesn't work for boundary. As it turns out, the Besicovitch covering theorem strategy combined with $\log \lambda$-proposition does, however.

Hybrids

Can we get a disjoint subset $\tilde{\mathcal{B}} \subset \mathcal{B}$ that witnesses both the Vitali covering lemma and the Vitali covering lemma for the boundary? No.



Hybrids

Theorem (W. 2025)

For each $\mathcal{B}$ *and any* $\varepsilon > 0$ *exists a subset* $\tilde{\mathcal{B}} \subset \mathcal{B}$ *such that for any distinct* $B_1, B_2 \in \tilde{\mathcal{B}}$ *we have*

$$\mathcal{L}(B_1 \cap B_2) \leq \varepsilon \min\{\mathcal{L}(B_1), \mathcal{L}(B_2)\}$$

and with

$$\mathcal{L}\Big(\bigcup \mathcal{B}\Big) \lesssim \mathcal{L}\Big(\bigcup \tilde{\mathcal{B}}\Big), \quad \mathcal{H}^{n-1}\Big(\partial \bigcup \mathcal{B}\Big) \lesssim_n \varepsilon^{-\frac{n-1}{n+1}} \mathcal{H}^{n-1}\Big(\partial \bigcup \tilde{\mathcal{B}}\Big).$$

The rate $\varepsilon^{-\frac{n-1}{n+1}}$ is sharp.

Thank you