

Regularity of maximal functions in higher dimensions

Julian Weigt

2025 CMS Winter Meeting

07.12.2025

Background

For $f : \mathbb{R}^n \rightarrow \mathbb{R}$ the centered Hardy-Littlewood maximal function is defined by

$$M^c f(x) = \sup_{r>0} f_{B(x,r)} \quad \text{with} \quad f_{B(x,r)} = \frac{1}{\mathcal{L}(B(x,r))} \int_{B(x,r)} |f|.$$

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Theorem (Hardy-Littlewood maximal function theorem)

$$\|M^c f\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|f\|_{L^p(\mathbb{R}^n)}$$

if and only if $p > 1$.

$$\|M^c f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim_n \|f\|_{L^1(\mathbb{R}^n)}$$

Theorem (Juha Kinnunen (1997))

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Proof: For $e \in \mathbb{R}^n$ by the sublinearity of M^c

$$\begin{aligned} \partial_e M^c f(x) &\sim \frac{M^c f(x + he) - M^c f(x)}{h} \\ &\leq \frac{M^c(f(\cdot + he) - f)(x)}{h} \\ &= M^c\left(\frac{f(\cdot + he) - f}{h}\right)(x) \sim M^c(\partial_e f)(x) \end{aligned}$$

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By the Hardy-Littlewood maximal function theorem for $p > 1$

$$\|\nabla M^c f\|_{L^p(\mathbb{R}^n)} \leq \|M^c(|\nabla f|)\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|\nabla f\|_{L^p(\mathbb{R}^n)}$$

Question (Hajłasz and Onninen 2004)

Is it true that

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Uncentered Hardy-Littlewood maximal function

$$Mf(x) = \sup_{B \ni x} f_B.$$

Endpoint question by Hajłasz and Onninen is interesting for M and other maximal operators.

In one dimension

Theorem (Tanaka 2002, Aldaz and Pérez Lázaro 2007)

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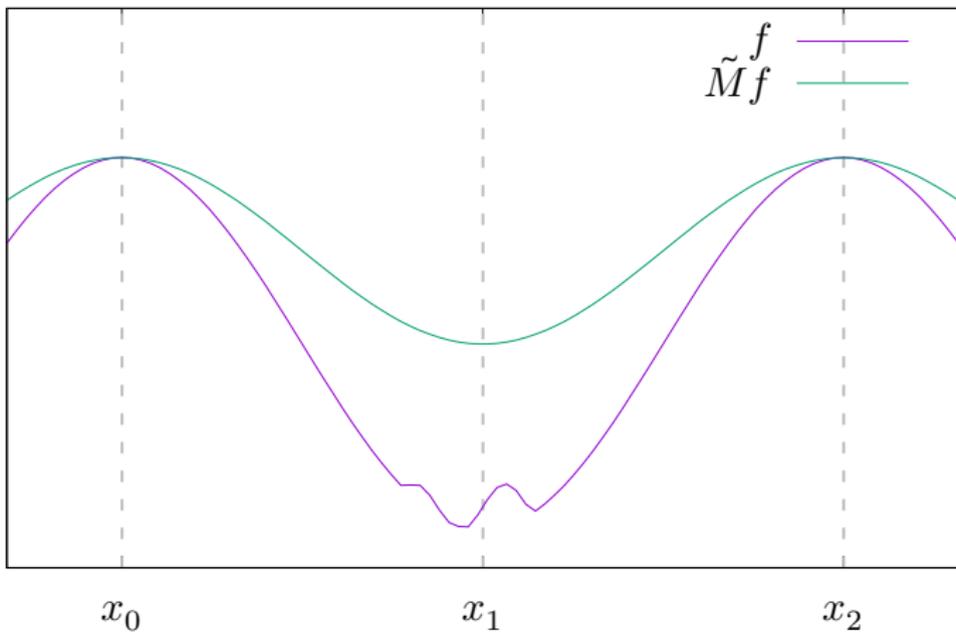
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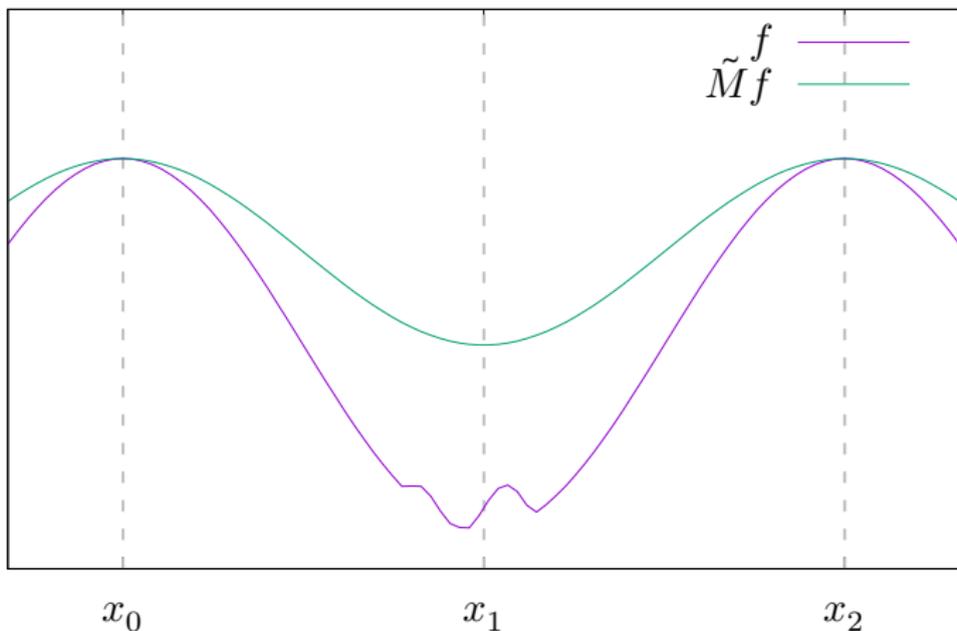
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- In one dimension

$$\|\nabla f\|_1 = \sup_{x_1 < x_2 < \dots} \sum_i |f(x_{i+1}) - f(x_i)| = \text{var } f.$$

- For almost all $x \in \mathbb{R}^d$: $Mf(x) \geq f(x)$
- and $Mf(x) = f(x)$ at a strict local maximum of Mf .





$$\begin{aligned}
 \text{var}_{[x_0, x_2]} Mf &= |Mf(x_1) - Mf(x_0)| + |Mf(x_2) - Mf(x_1)| \\
 &\leq |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| \\
 &\leq \text{var}_{[x_0, x_2]} f
 \end{aligned}$$

Past progress

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$$n = 1$$

block decreasing f

centered M , $n = 1$

radial f

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- best constants in one dimension

Theorem (W. 2022)

For $E \subset \mathbb{R}^n$ we have

$$\|\nabla M(1_E)\|_1 \leq C_n \|\nabla 1_E\|_1.$$

Higher dimensions

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Theorem (W. 2023)

For $f : \mathbb{R}^n \rightarrow \mathbb{R}$ we have

$$\|\nabla M^d f\|_1 \leq C_n \|\nabla f\|_1$$

for the dyadic maximal function

$$M^d f(x) = \sup_{\text{dyadic cube } Q, Q \ni x} f_Q.$$

Higher dimensions

Combining tools from both leads to the same bound for cube maximal operator given by

$$M^{\square} f(x) = \sup_{\text{cube } Q, Q \ni x} f_Q.$$

Theorem (W. 2024)

For the cube maximal operator M^{\square} and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ we have

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Proof works for more general sets with a tiling property, but not for balls and certainly not for centered M° .

Coarea formula

$$\|\nabla f\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}} \mathcal{H}^{n-1}(\partial\{x \in \mathbb{R}^n : f(x) > \lambda\}) \, d\lambda$$

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Superlevel sets

$$\{x \in \mathbb{R}^n : Mf(x) > \lambda\} = \bigcup \{B : f_B > \lambda\}$$

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Relative isoperimetric inequality

$$\min\{\mathcal{L}(Q \cap E), \mathcal{L}(Q \setminus E)\}^{n-1} \lesssim_n \mathcal{H}^{n-1}(Q \cap \partial E)^n.$$

Approach

Since $Mf \geq f$ a.e. we have

$$\begin{aligned} \mathcal{H}^{n-1}(\partial\{Mf > \lambda\}) &\leq \mathcal{H}^{n-1}(\partial\{Mf > \lambda\} \setminus \overline{\{f > \lambda\}}) \\ &\quad + \mathcal{H}^{n-1}(\partial\{f > \lambda\}) \end{aligned}$$

and we can bound the second summand in terms of $\|\nabla f\|_1$.

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and we can bound the second summand in terms of $\|\nabla f\|_1$. For the first summand define

$$\mathcal{B}_{\lambda, \geq} := \{B : f_B > \lambda, \mathcal{L}(B \cap \{f > \lambda\}) \geq 1/2\},$$

$\mathcal{B}_{\lambda, <}$ similarly. Then

$$\mathcal{H}^{n-1}(\partial\{Mf > \lambda\} \setminus \overline{\{f > \lambda\}}) \leq \mathcal{H}^{n-1}\left(\partial\bigcup \mathcal{B}_{\lambda, \geq} \setminus \overline{\{f > \lambda\}}\right) \\ + \mathcal{H}^{n-1}\left(\partial\bigcup \mathcal{B}_{\lambda, <} \setminus \overline{\{f > \lambda\}}\right)$$

and we deal with each summand separately.

Important tool:

Proposition

Let \mathcal{B}_λ be a set of balls B with $\mathcal{L}(B \cap E) = \lambda \mathcal{L}(B)$. Then

$$\mathcal{H}^{n-1}\left(\partial \bigcup \mathcal{B}_\lambda \setminus \overline{E}\right) \lesssim |\log \lambda| \lambda^{-\frac{n-1}{n}} \mathcal{H}^{n-1}(\partial E).$$

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Can also be used to prove

Theorem (Vitali for boundary, W. 2025)

Any bounded set \mathcal{B} of balls has a disjoint subset $\tilde{\mathcal{B}}$ with

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This can in turn be used to remove $\log \lambda$ term.

Lemma (Vitali covering lemma)

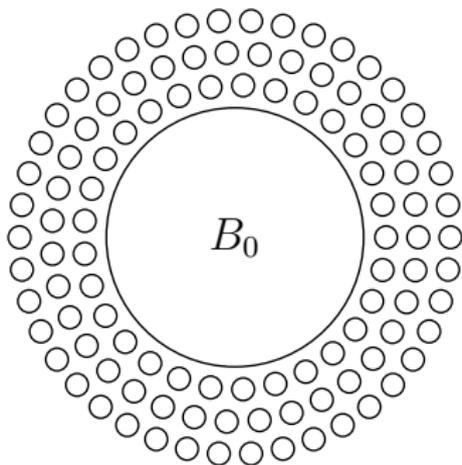
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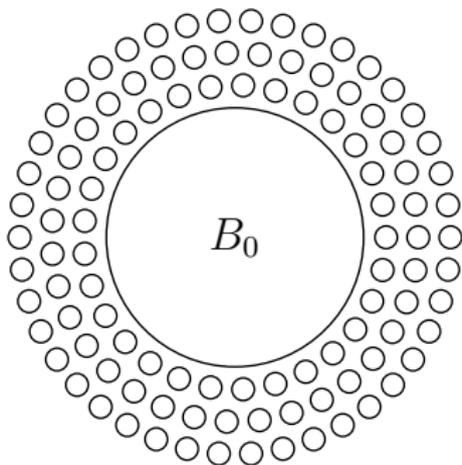
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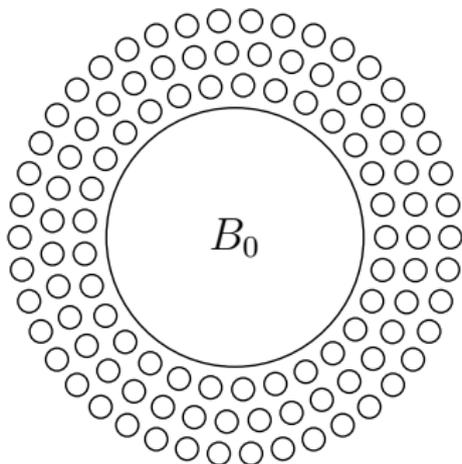


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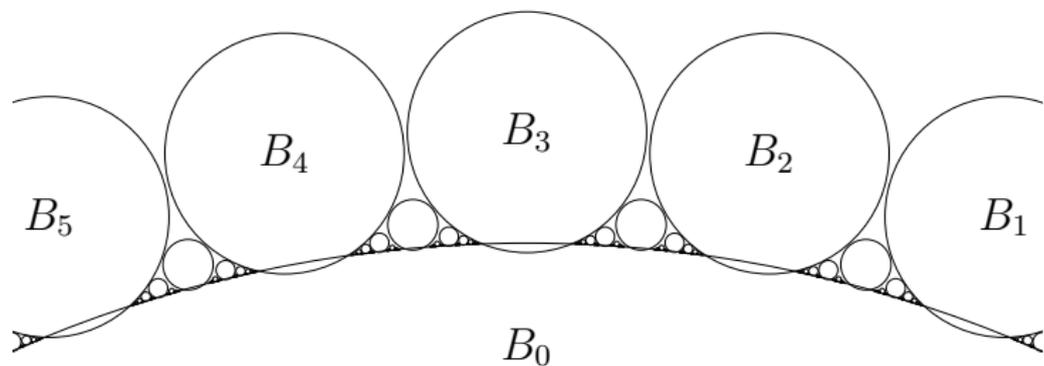
That means Vitali selection strategy doesn't work for boundary. As it turns out, the Besicovitch covering theorem strategy combined with $\log \lambda$ -proposition does, however.

Hybrids

Can we get a disjoint subset $\tilde{\mathcal{B}} \subset \mathcal{B}$ that witnesses both the Vitali covering lemma and the Vitali covering lemma for the boundary?

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Can we get a disjoint subset $\tilde{\mathcal{B}} \subset \mathcal{B}$ that witnesses both the Vitali covering lemma and the Vitali covering lemma for the boundary?
No.



Theorem (W. 2025)

For each \mathcal{B} and any $\varepsilon > 0$ exists a subset $\tilde{\mathcal{B}} \subset \mathcal{B}$ such that for any distinct $B_1, B_2 \in \tilde{\mathcal{B}}$ we have

$$\mathcal{L}(B_1 \cap B_2) \leq \varepsilon \min\{\mathcal{L}(B_1), \mathcal{L}(B_2)\}$$

and with

$$\mathcal{L}\left(\bigcup \mathcal{B}\right) \lesssim \mathcal{L}\left(\bigcup \tilde{\mathcal{B}}\right), \quad \mathcal{H}^{n-1}\left(\partial \bigcup \mathcal{B}\right) \lesssim_n \varepsilon^{-\frac{n-1}{n+1}} \mathcal{H}^{n-1}\left(\partial \bigcup \tilde{\mathcal{B}}\right).$$

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The rate $\varepsilon^{-\frac{n-1}{n+1}}$ is sharp.

Thank you