

# Higher Dimensional Techniques for the Regularity of Maximal Functions

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## Introduction: Background

For  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  the centered Hardy-Littlewood maximal function is defined by

$$M^c f(x) = \sup_{r>0} f_{B(x,r)} \quad \text{with} \quad f_{B(x,r)} = \frac{1}{\mathcal{L}(B(x,r))} \int_{B(x,r)} |f|.$$

Theorem (Hardy-Littlewood maximal function theorem)

$$\|M^c f\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|f\|_{L^p(\mathbb{R}^n)}$$

*if and only if  $p > 1$ .*

$$\|M^c f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim_n \|f\|_{L^1(\mathbb{R}^n)}$$

## Introduction: Background

Theorem (Juha Kinnunen (1997))

For  $p > 1$  we have

$$\|\nabla M^c f\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|\nabla f\|_{L^p(\mathbb{R}^n)}$$

**Proof:** For  $e \in \mathbb{R}^n$  by the sublinearity of  $M^c$

$$\begin{aligned} \partial_e M^c f(x) &\sim \frac{M^c f(x + he) - M^c f(x)}{h} \\ &\leq \frac{M^c(f(\cdot + he) - f)(x)}{h} \\ &= M^c\left(\frac{f(\cdot + he) - f}{h}\right)(x) \sim M^c(\partial_e f)(x) \end{aligned}$$

By the Hardy-Littlewood maximal function theorem for  $p > 1$

$$\|\nabla M^c f\|_{L^p(\mathbb{R}^n)} \leq \|M^c(|\nabla f|)\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|\nabla f\|_{L^p(\mathbb{R}^n)}$$

## Introduction: Background

Question (Hajlasz and Onninen 2004)

*Is it true that*

$$\|\nabla M^c f\|_{L^1(\mathbb{R}^n)} \lesssim_n \|\nabla f\|_{L^1(\mathbb{R}^n)}?$$

Uncentered Hardy-Littlewood maximal function

$$\tilde{M}f(x) = \sup_{B \ni x} f_B.$$

Endpoint question by Hajlasz and Onninen is interesting for  $\tilde{M}$  and other maximal operators.

## Introduction: In one dimension

Theorem (Tanaka 2002, Aldaz and Pérez Lázaro 2007)

For  $f : \mathbb{R} \rightarrow \mathbb{R}$  we have

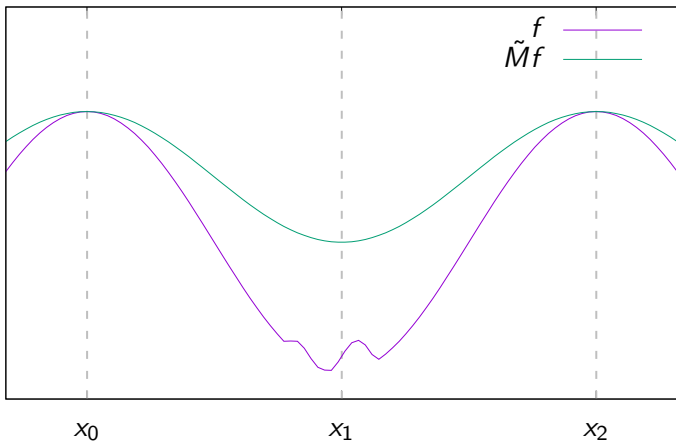
$$\|\nabla \tilde{M}f\|_1 \leq \|\nabla f\|_1$$

**Proof:**

- In one dimension

$$\|\nabla f\|_1 = \sup_{x_1 < x_2 < \dots} \sum_i |f(x_{i+1}) - f(x_i)| = \text{var } f.$$

- For almost all  $x \in \mathbb{R}^d$ :  $\tilde{M}f(x) \geq f(x)$
- and  $\tilde{M}f(x) = f(x)$  at a strict local maximum of  $Mf$ .



$$\begin{aligned}
 \text{var}_{[x_0, x_2]} \tilde{M}f &= |\tilde{M}f(x_1) - \tilde{M}f(x_0)| + |\tilde{M}f(x_2) - \tilde{M}f(x_1)| \\
 &\leq |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| \\
 &\leq \text{var}_{[x_0, x_2]} f
 \end{aligned}$$

## Introduction: The fractional maximal function

For  $0 < \alpha < n$  the centered fractional Hardy-Littlewood maximal function is

$$M_{\alpha}^c f(x) = \sup_{r>0} r^{\alpha} f_{B(x,r)}.$$

Corresponding Hardy-Littlewood theorem

$$\|M_{\alpha} f\|_{L^{p_{\alpha}}(\mathbb{R}^n)} \lesssim_{n,\alpha,p} \|f\|_{L^p(\mathbb{R}^n)}$$

with  $p_{\alpha} = \frac{pn}{n-\alpha p} > p$  if and only if  $p > 1$ . Corresponding regularity bound

$$\|\nabla M_{\alpha} f\|_{L^{p_{\alpha}}(\mathbb{R}^n)} \lesssim_{n,\alpha,p} \|\nabla f\|_{L^p(\mathbb{R}^n)},$$

proven for  $p > 1$ .

## Introduction: The fractional maximal function

### Theorem (Kinnunen and Saksman 2003)

For  $\alpha \geq 1$

$$|\nabla M_\alpha^c f(x)| \lesssim_n |M_{\alpha-1}^c f(x)|.$$

### Corollary (Carneiro and Madrid 2016)

For  $\alpha \geq 1$  we have  $1_\alpha = \frac{n}{n-\alpha} = \left(\frac{n}{n-1}\right)_{\alpha-1}$  and  $\frac{n}{n-1} > 1$  and therefore

$$\begin{aligned} \|\nabla M_\alpha^c f\|_{L^{1_\alpha}(\mathbb{R}^n)} &\lesssim_n \|M_{\alpha-1}^c f\|_{L^{1_\alpha}(\mathbb{R}^n)} \lesssim_n \|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \\ &\lesssim_n \|\nabla f\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

- What about  $0 < \alpha < 1$ ?
- Same result for  $\tilde{M}_\alpha$ .



## Introduction: Past progress

$n = 1$

[Tanaka 2002, Aldaz  
+Pérez Lázaro 2007]

block decreasing  $f$

[Aldaz+Pérez Lázaro 2009]

centered  $M$ ,  $n = 1$

[Kurka 2015]

radial  $f$

[Luiro 2018]

### **fractional:**

$n = 1$

[Beltran + Madrid 2016]

$1 \leq \alpha$

[Kinnunen + Saksman 2003,  
Carneiro + Madrid 2016]

radial  $f$

[Luiro + Madrid 2017]

lacunary

[Beltran + Ramos + Saari 2018]

$n = 1$ , radial for centered  $f$

[Beltran + Madrid 2019]

more related bounds, bounds on other maximal operators, such as local, . . . , for example: Continuity of  $f \mapsto \nabla Mf$  on  $W^{1,1}(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n)$ , stronger than boundedness.

## Introduction: New results

We prove the endpoint regularity bound for the maximal function for

- ① uncentered maximal function of characteristic  $f$
- ② dyadic maximal operator
- ③ fractional maximal operator (uncentered & centered)
- ④ cube maximal operator

## Introduction: Reformulation and decomposition

### Coarea formula

$$\|\nabla f\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}} \mathcal{H}^{n-1}(\partial\{x \in \mathbb{R}^n : f(x) > \lambda\}) \, d\lambda$$

### Superlevel sets

$$\{Mf > \lambda\} = \{x \in \mathbb{R}^n : Mf(x) > \lambda\} = \bigcup \{B : f_B > \lambda\}$$

for *uncentered* maximal operators.

### Decomposition of the boundary

Denote

$$\mathcal{B}_\lambda^< = \{B : f_B > \lambda, \mathcal{L}(B \cap \{f > \lambda\}) < 2^{-n-1} \mathcal{L}(B)\}$$

and  $\mathcal{B}_\lambda^>$  accordingly.

# Introduction: Tools

- ① **relative isoperimetric inequality:**

$$\min\{\mathcal{L}(Q \cap E), \mathcal{L}(Q \setminus E)\}^{n-1} \lesssim_n \mathcal{H}^{n-1}(Q \cap \partial E)^n.$$

- ② **Vitaly covering** and similar: general balls  $\rightarrow$  separated balls
- ③ **Besicovitch covering for boundary**
- ④ **superlevelset estimate:**  $f < 0$  on most of  $B \Rightarrow$  most mass of  $f$  lies far above  $f_B$

used in proof	isoperimetric, Vitali	boundary Besicovitch	superlevel
dyadic char. f.	x		
char. f.	x	x	
dyadic	x		x
fractional	x		x
cube	x	x	x

## Proof: Reformulation and decomposition

We have

$$\{Mf > \lambda\} = \bigcup \mathcal{B}_\lambda^< \cup \bigcup \mathcal{B}_\lambda^>.$$

Since  $\{f > \lambda\} \subset \{Mf > \lambda\}$  we have

$$\partial\{Mf > \lambda\} \subset (\partial\{Mf > \lambda\} \setminus \overline{\{f > \lambda\}}) \cup \partial\{f > \lambda\}.$$

We conclude

### Decomposition

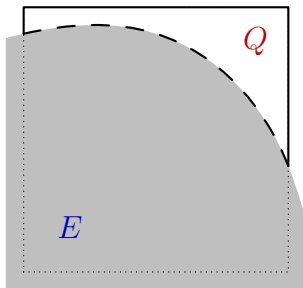
$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla Mf| &\leq \int_0^\infty \mathcal{H}^{n-1}(\partial \bigcup \mathcal{B}_\lambda^> \setminus \overline{\{f > \lambda\}}) \, d\lambda \\ &\quad + \int_0^\infty \mathcal{H}^{n-1}(\partial \bigcup \mathcal{B}_\lambda^<) \, d\lambda \\ &\quad + \int_{\mathbb{R}^d} |\nabla f| \end{aligned}$$

# Proof: High density case $\mathcal{B}_\lambda^{\geq}$

## Proposition

For  $Q, E$  with  $\mathcal{L}(Q \cap E) \geq 2^{-n-1} \mathcal{L}(Q)$  we have

$$\mathcal{H}^{n-1}(\partial Q \setminus \bar{E}) \lesssim_n \mathcal{H}^{n-1}(Q \cap \partial E)$$



## dyadic maximal operator

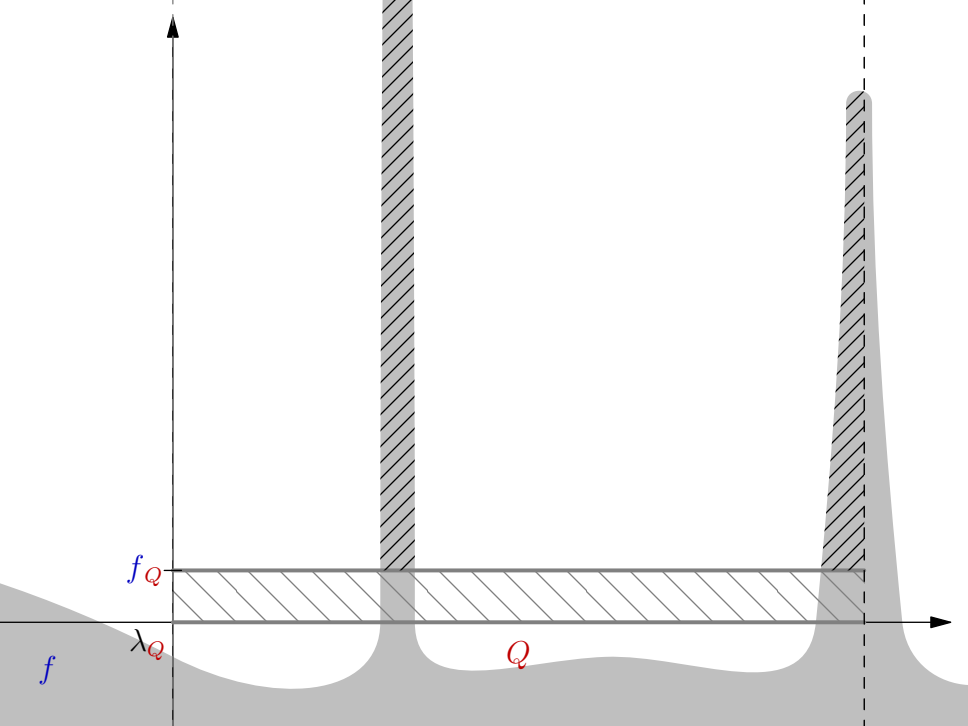
$$M^d f(x) = \sup_{\text{dyadic } Q, Q \ni x} f_Q.$$

$$\begin{aligned} \mathcal{H}^{n-1}(\partial \bigcup Q_\lambda^{\geq} \setminus \overline{\{f > \lambda\}}) &\leq \sum_{Q \in Q_\lambda^{\geq}} \mathcal{H}^{n-1}(\partial Q \setminus \overline{\{f > \lambda\}}) \\ &\lesssim_n \sum_{Q \in Q_\lambda^{\geq}} \mathcal{H}^{n-1}(Q \cap \partial\{f > \lambda\}) \\ &\leq \mathcal{H}^{n-1}(\partial\{f > \lambda\}) \end{aligned}$$

## Proposition

For a set  $B$  of balls  $B$  with  $\mathcal{L}(B \cap E) \geq 2^{-n-1} \mathcal{L}(B)$  we have

$$\mathcal{H}^{n-1}(\partial \bigcup B \setminus \overline{E}) \lesssim_n \mathcal{H}^{n-1}(\bigcup B \cap \partial E).$$





**Proof:** Low density case  $\mathcal{B}_\lambda^<$ , dyadic

$$\int_{\mathbb{R}} \mathcal{H}^{n-1}(\partial \cup \mathcal{Q}_\lambda^<) d\lambda \leq \sum_{Q \text{ dyadic}} (f_Q - \lambda_Q) \mathcal{H}^{n-1}(\partial Q)$$

with

$$\mathcal{L}(Q \cap \{f > \lambda_Q\}) = 2^{-n-1} \mathcal{L}(Q)$$

Proposition

$$(f_Q - \lambda_Q) \mathcal{L}(Q) \lesssim_n \int_{f_Q}^{\infty} \mathcal{L}(Q \cap \{f > \lambda\}) d\lambda$$

**Proof:** Low density case  $\mathcal{B}_\lambda^<$ , dyadic

$$\int_{\mathbb{R}} \mathcal{H}^{n-1}(\partial \bigcup \mathcal{Q}_\lambda^<) d\lambda \leq \sum_{Q \text{ dyadic}} (f_Q - \lambda_Q) \mathcal{H}^{n-1}(\partial Q)$$

with

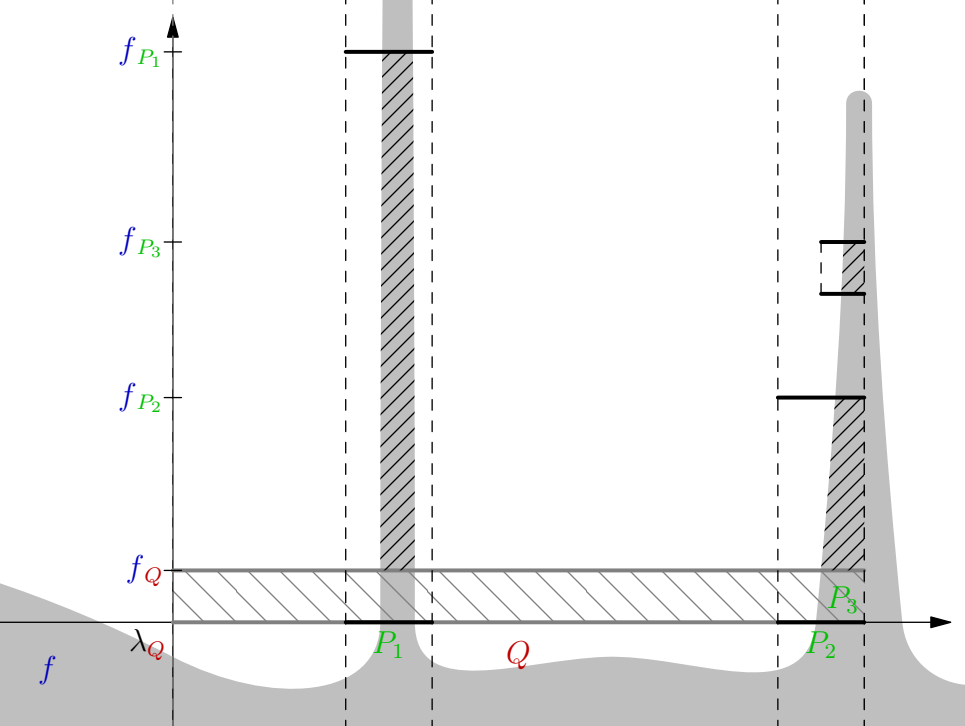
$$\mathcal{L}(Q \cap \{f > \lambda_Q\}) = 2^{-n-1} \mathcal{L}(Q)$$

Proposition

$$(f_Q - \lambda_Q) \mathcal{L}(Q) \lesssim_n \int_{f_Q}^{\infty} \sum_{P \subsetneq Q: \bar{\lambda}_P < \lambda < f_P} \mathcal{L}(P \cap \{f > \lambda\}) d\lambda$$

where  $P$  is maximal above  $\bar{\lambda}_P$  and

$$\mathcal{L}(P \cap \{f > \bar{\lambda}_P\}) = 2^{-1} \mathcal{L}(P)$$



## Proof: Low density case $\mathcal{B}_\lambda^<$ , dyadic

Combining, we obtain

$$\int_{\mathbb{R}} \mathcal{H}^{n-1}(\partial \bigcup \mathcal{Q}_\lambda^<) d\lambda \\ \lesssim_n \int_{\mathbb{R}} \sum_{Q \text{ dyadic}} \sum_{P \subsetneq Q: \bar{\lambda}_P < \lambda < f_P} \frac{\mathcal{L}(P \cap \{f > \lambda\})}{l(Q)} d\lambda$$

- 1 change the order of summation
- 2 convergence of the geometric sum
- 3 apply the relative isoperimetric inequality to  $P$ .
- 4 coarea formula to recover  $\|\nabla f\|_1$

## Proof: Low density case $\mathcal{B}_\lambda^<$ , fractional

$1 \leq \alpha$  [Kinnunen + Saksman, Carneiro + Madrid]

$$\|\nabla M_\alpha f\|_{\frac{n}{n-\alpha}} \lesssim_n \|M_{\alpha-1} f\|_{\frac{n}{n-\alpha}} \lesssim_n \|f\|_{\frac{n}{n-1}} \lesssim_n \|\nabla f\|_1.$$

$0 < \alpha$

$$\|\nabla M_\alpha f\|_{\frac{n}{n-\alpha}} \lesssim_n \|M_{\alpha,-1} f\|_{\frac{n}{n-\alpha}} \lesssim_{n,\alpha} \|\nabla f\|_1,$$

$M_{\alpha,-1}$  replacement for  $M_{\alpha-1}$  if  $0 < \alpha < 1$ .

Can bound  $M_{\alpha,-1} f$  both centered and uncentered

- using low density arguments from the dyadic proof
- extra flexibility coming from  $\alpha > 0$ , allowing for rough Vitali covering arguments

## Proof: Low density case $\mathcal{B}_\lambda^<$ , general cubes

cube maximal function

$$Mf(x) = \sup_{\text{cube } Q, Q \ni x} f_Q.$$

We reduce to almost dyadic cubes, using

Proposition (Vitali/Besicovitch for perimeter)

For any (finite) set of cubes  $\mathcal{Q}$  there is a subset  $\mathcal{S} \subset \mathcal{Q}$  of disjoint cubes such that

$$\mathcal{H}^{n-1}(\partial \cup \mathcal{Q}) \lesssim_n \sum_{S \in \mathcal{S}} \mathcal{H}^{n-1}(\partial S).$$

# Uncentered HL $\tilde{M}f$ (balls)?

- All arguments work
- except low density bound  $(f_B - \lambda_B)\mathcal{L}(B) \lesssim_n?$

Thank you