The Variation of the Uncentered Maximal Operator with respect to Cubes

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Abstract

We consider the maximal operator with respect to uncentered cubes on a Euclidean space of arbitrary dimension. We prove that for any function with bounded variation, the variation of its maximal function is bounded by the variation of the function times a dimensional constant. We also prove the corresponding result for maximal operators with respect to more general sets of cubes.

1 Introduction

For a locally integrable function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, with $d \in \mathbb{N}$, we consider the uncentered Hardy-Littlewood maximal function over cubes, defined by

$$\mathrm{M}f(x) = \sup_{x \in Q} \frac{1}{\mathcal{L}(Q)} \int_Q f(y) \, \mathrm{d}y,$$

where the supremum is taken over all open axes-parallel cubes Q which contain $x \in \mathbb{R}^d$. We discuss maximal operators with respect to more general sets of cubes in Section 2. Usually, the maximal operator integrates over $|f|$ instead of f, because the maximal function is used for L^p estimates, but we also discuss sign-changing functions. The regularity of a maximal operator was first studied in [Kin97], where Kinnunen proved that for $p > 1$ and $f \in W^{1,p}(\mathbb{R}^d)$ the bound

$$\|\nabla \mathrm{M}f\|_p \leq C_{d,p} \|\nabla f\|_p \tag{1.1}$$

holds, from which it follows that the Hardy-Littlewood maximal operator is bounded on $W^{1,p}(\mathbb{R}^d)$. Originally, Kinnunen proved (1.1) only for the Hardy-Littlewood maximal operator which averages over all balls centered in x, but his arguments work for a variety of maximal operators, including the operator M defined above. His strategy relies on the L^p-boundedness of the maximal operator, and thus fails for $p = 1$. The question of whether (1.1) nevertheless holds with $p = 1$ for any maximal operator has become a well known problem and has been subject to considerable research. However, it has so far remained mostly unanswered, except in one dimension. Our main result is the following.

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Theorem 1.1. *Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ with $\operatorname{var} f < \infty$. Then $\mathrm{M}f$ is locally integrable to the power of $d/(d-1)$ and*

$$\operatorname{var} \mathrm{M}f \leq C_d \operatorname{var} f \tag{1.2}$$

where the constant C_d depends only on the dimension d.

Theorem 1.1 answers a question in the paper [HO04] of Hajłasz and Onninen from 2004 for the uncentered maximal operator over cubes. Lahti showed in [Lah20] that the variation bound (1.2) implies that for functions $f \in W^{1,1}(\mathbb{R}^d)$ we have $\nabla \mathrm{M}f \in L^1(\mathbb{R}^d)$. This is proved in [Lah20] only for the Hardy-Littlewood maximal operator with respect to balls, but the result continues to hold for the maximal operator with respect to cubes, with essentially the same proof. We can conclude that the bound

$$\|\nabla \mathrm{M}f\|_{L^1(\mathbb{R}^d)} \leq C_d \|\nabla f\|_{L^1(\mathbb{R}^d)}$$

holds. We prove the variation bound corresponding to (1.2) also for maximal operators which average over more general sets of cubes than M, see Theorems 2.4 and 2.5, Remarks 2.6, 2.7 and 2.9, and Proposition 2.8.

For a function $f : \mathbb{R} \to \mathbb{R}$, the variation bound for the uncentered maximal function has already been proven in [Tan02] by Tanaka and in [APL07] by Aldaz and Pérez Lázaro. Note that in one dimension, balls and cubes are the same. For the centered Hardy-Littlewood maximal function Kurka proved the bound in [Kur15]. The latter proof turned out to be much more complicated. In [APL09] Aldaz and Pérez Lázaro have proven the gradient bound for the uncentered maximal operator for block decreasing functions in $W^{1,1}(\mathbb{R}^d)$ and any dimension d. In [Lui18] Luiro has done the same for radial functions. More endpoint results are available for related maximal operators, for example convolution maximal operators [CS13, CGR19], fractional maximal operators [KS03, CM17, CM17, BM19, BRS19, Wei21, HKKT15], and discrete maximal operators [CH12], as well as maximal operators on different spaces, such as in the metric setting [KT07] and on Hardy-Sobolev spaces [PPSS18]. For more background information on the regularity of maximal operators there is a survey [Car19] by Carneiro. Local regularity properties of the maximal function, which are weaker than the gradient bound of the maximal operator have also been studied [HM10, ACPL12]. The question whether the maximal operator is a continuous operator on the gradient level is even more difficult to answer than its boundedness because the maximal operator is not linear. Some progress has already been made on this question in [Lui07, CMP17, CGRM20, BGRMW21].

This is the fourth paper in a series [Wei20b, Wei20a, Wei21] on higher dimensional variation bounds of maximal operators, using geometric measure theory and covering arguments. In [Wei20b] we prove (1.2) for the uncentered Hardy-Littlewood maximal function of characteristic functions, in [Wei20a] we prove it for the dyadic maximal operator for general functions, and in [Wei21] we prove the corresponding result for the fractional maximal operator. Here we apply tools developed in [Wei20b, Wei20a]. Note that it is not possible to extend the variation bound from characteristic to simple and then general functions, using only the sublinearity of the maximal function. The pitfall in that strategy is that while the maximal function is sublinear, this is not true on the gradient level: There are characteristic functions f_1, f_2 such that $\operatorname{var} \mathrm{M}(f_1 + f_2) > \operatorname{var} \mathrm{M}f_1 + \operatorname{var} \mathrm{M}f_2$, see [Wei20a, Example 5.2].

The starting point here and in [Wei20b, Wei20a] is the coarea formula, which expresses the variation of the maximal function in terms of the boundary of the distribution set. We observe that the distribution set of the uncentered maximal function is the union of all cubes on which the function has the corresponding average. We divide the cubes of the distribution set of the maximal function, into two groups: We say that those which intersect the distribution set of the function a lot have *high density*, and the others have *low density*. The union of the high density cubes looks similar to the distribution set of the function, and for characteristic functions we have already bounded its boundary in [Wei20b] due to a result in the spirit of the isoperimetric inequality. The motivation for that bound came from [KKST08, Theorem 3.1] by Kinnunen, Korte, Shanmugalingam and Tuominen. In [Wei20a] and in this paper the high density cubes are bounded using the same argument. This bound is even strong enough to control the low density balls for characteristic functions in the global setting in [Wei20b]. But in the local setting in [Wei20b] dealing with the low density balls is more involved. It requires a careful decomposition of the function in parallel with the low density balls of the maximal function by dyadic scales. In that paper it also relies on the fact that the function is a characteristic function. In [Wei20a] we devise a strategy for dealing with the low density cubes for general functions in the dyadic cube setting. The advantage of the dyadic setting is that the decomposition of the low density cubes and the function are a lot more straightforward because dyadic cubes only intersect in trivial ways. Furthermore, the argument contains a sum over side lengths of cubes which converges as a geometric sum for dyadic cubes. In [Wei21] we bound the fractional operator, using that it disregards small balls, which allows for a reduction from balls to dyadic cubes so that we can apply the result from [Wei20a]. Non-fractional maximal operators are much more sensitive, in that we have to deal with complicated intersections of balls or cubes of any size. In this paper we represent the low density cubes of the maximal function by a subfamily of cubes with dyadic properties, which allows to apply the key dyadic result from [Wei20a]. In order to make the rest of the dyadic strategy of [Wei20a] work here, the function is decomposed in a similar way as in the local case of [Wei20b].

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2 Preliminaries and core results

In this paper we understand a cube to be open, nonempty and with arbitrary orientation. For the maximal function however it plays no role if cubes are open or closed, see Proposition 2.8, and also all other statements and proofs in this paper continue to hold almost verbatim for closed or half open cubes. We denote the side length of a cube Q by $\mathrm{l}(Q) > 0$. We say that Q is of scale 2^n if $\mathrm{l}(Q) \in [2^n, 2^{n+1})$. Recall the definition of the collection of dyadic cubes

$$\bigcup_{n\in\mathbb{Z}} \{(x_1, x_1 + 2^n) \times \ldots \times (x_d, x_d + 2^n) : i = 1, \ldots, n,\ x_i \in 2^n\mathbb{Z}\}.$$

For a cube Q_0 let φ be a linear transformation with $\varphi(Q_0) = (0,1)^d$. We say that Q is dyadic with respect to Q_0 if $\varphi(Q)$ is a dyadic cube contained in $(0,1)^d$. We denote by $\mathfrak{D}(Q)$ the set of dyadic cubes with respect to Q_0. For a cube Q with center c_Q and $K > 0$ we denote the dilated cube by

$$KQ = \{c_Q + Kr : c_Q + r \in Q\}.$$

Definition 2.1. A set $\mathcal{Q}$ of cubes is *dyadically complete* if for every $Q_0, P \in \mathcal{Q}$ with $P \subset Q_0$, all cubes $Q \in \mathfrak{D}(Q_0)$ with $P \subset Q$ also belong to $\mathcal{Q}$.

We work in the setting of functions of bounded variation, as in Evans-Gariepy [EG15], Section 5. Let $\Omega \subset \mathbb{R}^d$ be an open set. A function $f \in L^1_{\mathrm{loc}}(\Omega)$ is said to have locally bounded variation if for every open and compactly contained set $V \subset \Omega$ we have

$$\sup\Big\{\int_V f \operatorname{div}\varphi : \varphi \in C^1_{\mathrm{c}}(V;\mathbb{R}^d),\ |\varphi| \le 1\Big\} < \infty.$$

Such a function comes with a Radon measure μ and a μ-measurable function $\sigma : \Omega \to \mathbb{R}^d$ which satisfies $|\sigma(x)| = 1$ for μ-a.e. $x \in \mathbb{R}^d$ and such that for all $\varphi \in C^1_{\mathrm{c}}(\Omega;\mathbb{R}^d)$ we have

$$\int_V f \operatorname{div}\varphi = \int_V \varphi\sigma \,\mathrm{d}\mu.$$

We define the variation of f in Ω by $\operatorname{var}_\Omega f = \mu(\Omega)$. If $f \notin L^1_{\mathrm{loc}}(\Omega)$ then we set $\operatorname{var}_\Omega f = \infty$. For a measurable set $E \subset \mathbb{R}^d$ denote by $\mathring{E}$, $\overline{E}$ and ∂E the topological interior, closure and boundary of E, respectively. The measure theoretic closure and the measure theoretic boundary of E are defined as

$$\overline{E}^* = \Big\{x : \limsup_{r\to 0} \frac{\mathcal{L}(B(x,r)\cap E)}{r^d} > 0\Big\} \quad \text{and} \quad \partial_* E = \overline{E}^* \cap \overline{\mathbb{R}^d \setminus E}^*.$$

The measure theoretic versions are robust against changes with measure zero. Note that $\overline{E}^* \subset \overline{E}$ and thus $\partial_* E \subset \partial E$. For a cube, its measure theoretic boundary and its closure agree with the respective topological quantities.

For a set $\mathcal{Q}$ of cubes let

$$\bigcup \mathcal{Q} = \bigcup_{Q\in\mathcal{Q}} Q.$$

The integral average of a function $f \in L^1(Q)$ over a cube Q is denoted by

$$f_Q = \frac{1}{\mathcal{L}(Q)} \int_Q f(x)\,\mathrm{d}x.$$

We write

$$\{f > \lambda\} = \{x \in \mathbb{R}^d : f(x) > \lambda\}$$

for the superlevelset of a function $f : \mathbb{R}^d \to \mathbb{R}$. We define $\{f \ge \lambda\}$ similarly. By $a \lesssim b$ we mean that there exists a constant C_d that depends only on the dimension d such that $a \le C_d b$.

The following coarea formula gives a useful interpretation of the variation.

Lemma 2.2 ([EG15, Theorem 3.40])**.** Let $\Omega \subset \mathbb{R}^d$ be an open set and assume that $f \in L^1_{\mathrm{loc}}(\Omega)$. Then

$$\operatorname{var}_\Omega f = \int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial_* \{f \ge \lambda\} \cap \Omega)\,\mathrm{d}\lambda.$$

In [EG15, Theorem 3.40] the formula is stated with the set $\{f \ge \lambda\}$ in place of $\{f > \lambda\}$, but it can be proven for $\{f \ge \lambda\}$ using the same proof. Our core result is the following.

Theorem 2.3. *Let $\mathcal{Q}$ be a finite set of cubes which is dyadically complete and let $f \in L^1(\mathbb{R}^d)$ be a function with $\operatorname{var}_{\bigcup \mathcal{Q}} f < \infty$. For $\lambda \in \mathbb{R}$ denote $\mathcal{Q}^\lambda = \{Q \in \mathcal{Q} : f_Q \ge \lambda\}$. Then*

$$\int_{-\infty}^{\infty} \mathcal{H}^{d-1}\Big(\partial \bigcup \mathcal{Q}^\lambda \setminus \overline{\{f \ge \lambda\}}^*\Big)\,\mathrm{d}\lambda \le C_d \int_{-\infty}^{\infty} \mathcal{H}^{d-1}\Big(\partial_* \{f \ge \lambda\} \cap \bigcup \mathcal{Q}\Big)\,\mathrm{d}\lambda, \tag{2.1}$$

where C_d depends only on the dimension.

The set $\mathcal{Q}^\lambda$ is of interest because

$$\{\mathrm{M}f > \lambda\} = \bigcup \{Q : f_Q > \lambda\}.$$

In light of the coarea formula Lemma 2.2 and because $\{f > \lambda\} \setminus \{\mathrm{M}f > \lambda\}$ has measure zero, (2.1) is essentially a version of

$$\operatorname{var} \mathrm{M}f \leq C_d \operatorname{var} f$$

for a finite set of cubes. We prove Theorem 2.3 in Section 3. That section contains the key arguments of this paper. Section 4 is a more technical section where we deal mostly with integrability and convergence issues. In Section 4.1 we deduce the following theorem from Theorem 2.3 by approximation.

Theorem 2.4. *Given an open set $\Omega \subset \mathbb{R}^d$, let $\mathcal{Q}$ be a dyadically complete set of cubes Q with $Q \subset \Omega$ and let $f \in L^1_{\mathrm{loc}}(\Omega)$ be a function with $\operatorname{var}_\Omega f < \infty$. Then for every $Q \in \mathcal{Q}$ we have $\int_Q |f| < \infty$, and the maximal function defined by*

$$\mathrm{M}_{\mathcal{Q}} f(x) = \max\left\{ f(x), \sup_{Q \in \mathcal{Q},\ x \in Q} \frac{1}{\mathcal{L}(Q)} \int_Q f(y)\,\mathrm{d}y \right\} \tag{2.2}$$

belongs to $L^{d/(d-1)}_{\mathrm{loc}}(\Omega)$ and satisfies

$$\operatorname{var}_\Omega \mathrm{M}_{\mathcal{Q}} f \leq C_d \operatorname{var}_\Omega f,$$

where the constant C_d depends only on the dimension d.

In order to deduce Theorem 1.1, let $\Omega = \mathbb{R}^d$ and let $\mathcal{Q}$ be the set of all axes-parallel cubes. By the Lebesgue differentiation theorem for almost every $x \in \mathbb{R}^d$ we have $\mathrm{M}f(x) \geq f(x)$ and consequently $\mathrm{M}_{\mathcal{Q}} f(x) = \mathrm{M}f(x)$. Thus we can conclude Theorem 1.1 from Theorem 2.4.

Another maximal operator which is essentially of the form (2.2) is the local maximal operator. For an open set Ω and a function $f \in L^1_{\mathrm{loc}}(\Omega)$ we define the local maximal function by

$$\mathrm{M}_\Omega f(x) = \sup_{x \in Q,\ \overline{Q} \subset \Omega} \fint_Q f(y)\,\mathrm{d}y,$$

where the supremum is taken over all axes-parallel cubes Q which are compactly contained in Ω and contain x. Similarly as above, we may also consider the local maximal operator which considers cubes with arbitrary orientation. It is usually more difficult to prove regularity results for local maximal operators than for global maximal operators, because some arguments use that the maximal operator also takes into account certain blow-ups of balls or cubes. In fact for the fractional maximal operator, gradient bounds which hold for the global operator fail for the local operator, see [HKKT15, Example 4.2]. Not so here, we also obtain Theorem 1.1 for general domains.

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^d$ open and $f \in L^1_{\mathrm{loc}}(\Omega)$ with $\operatorname{var}_\Omega f < \infty$. Then $\mathrm{M}_\Omega f \in L^{d/(d-1)}_{\mathrm{loc}}(\Omega)$ and*

$$\operatorname{var}_\Omega \mathrm{M}_\Omega f \leq C_d \operatorname{var}_\Omega f.$$

For the proof of Theorem 2.5 let $\mathcal{Q}$ be the set of cubes Q with $\overline{Q} \subset \Omega$. Then $\mathcal{Q}$ is dyadically complete and Theorem 2.5 follows from Theorem 2.4 by the same argument as Theorem 1.1.

Remark 2.6. The proofs of Theorems 1.1 and 2.5 work the same way for the maximal operators that average over all cubes with arbitrary orientation which contain the point x, it is not necessary to consider only axes-parallel cubes.

Remark 2.7. Theorem 2.4 also applies to the global and the local dyadic operator, so it supersedes the main results in [Wei20a].

Proposition 2.8. Denote by $\overline{\mathrm{M}}_{\mathcal{Q}} f$ the maximal function given by the same definition as $\mathrm{M}_{\mathcal{Q}} f$, except the supremum is taken over all cubes $Q \in \mathcal{Q}$ with $x \in \overline{Q}$. Then for almost every $x \in \Omega$ we have $\overline{\mathrm{M}}_{\mathcal{Q}} f(x) = \mathrm{M}_{\mathcal{Q}} f(x)$. In particular, Theorem 2.4 also holds for $\overline{\mathrm{M}}_{\mathcal{Q}} f$.

We prove Proposition 2.8 in Section 4.2.

Remark 2.9. Every cube Q with $x \in Q$ and $Q \subset \Omega$ can be approximated from the inside by cubes P with $x \in P$ and $\overline{P} \subset \Omega$. That means in the definition of $\mathrm{M}_\Omega f$ we may replace the condition $\overline{Q} \subset \Omega$ by $Q \subset \Omega$ without changing the maximal function. Together with Proposition 2.8 we can conclude that if we define cubes to be closed instead of open, or replace $\overline{Q}$ by Q, then the definitions of the maximal functions barely change, and hence also Theorem 2.4 and its consequences continue to hold. Furthermore, also the proofs in this paper continue to work almost verbatim if we define cubes to be closed, open, or half-open.

In [Wei20a] we assumed that the dyadic maximal operator averages only over cubes which are compactly contained in the domain Ω in order to ensure local integrability of the maximal function. This condition is not necessary, provided that $\operatorname{var}_\Omega f < \infty$.

Remark 2.10. Everything in the proof of Theorem 1.1 also holds not only for cubes but also almost verbatim for rectangles with a bounded ratio of sidelengths, tetrahedrons and other convex sets that can be written as disjoint unions of finitely many smaller versions of themselves. Except from Proposition 3.12, everything also works for balls instead of cubes.

Remark 2.11. In Theorem 2.4 we take the maximum with f for the following reason. Let $f = 1_{(0,1)^d}$ and for $N \in \mathbb{N}$ let $\mathcal{Q}_N$ be the set of dyadic cubes with side length at least 2, and the cubes $(n_1 2^{-N}, (n_1+1)2^{-N}) \times \ldots \times (n_d 2^{-N}, (n_d+1)2^{-N}) \subset (0,1)^d$ where $n_1, \ldots, n_d$ are integers such that $x_1 + \ldots + x_d$ is even. Then the maximal operator that averages over all cubes in $\mathcal{Q}_N$ has variation of the order $2^{Nd} \cdot 2^{-N(d-1)} = 2^N$ which goes to infinity for $N \to \infty$. It is not clear however if or when the dyadic completeness of $\mathcal{Q}$ is necessary.

The space $L^1_{\mathrm{loc}}(\Omega)$ is not the correct domain for M_Ω because $f \in L^1_{\mathrm{loc}}(\Omega)$ does not imply that $\mathrm{M}_\Omega f$ is finite almost everywhere, as has already been observed in [HO04, footnote (2), p. 170]. If we strengthen the assumption to $f \in L^1(\Omega)$, then $\mathrm{M}_\Omega f$ is finite almost everywhere by the weak bound for the maximal operator. Proposition 4.1 shows that an alternative way ensure the almost everywhere finiteness of $\mathrm{M}_\Omega f$ is to demand $\operatorname{var}_\Omega f < \infty$ in addition to $f \in L^1_{\mathrm{loc}}(\Omega)$.

Remark 2.12. Theorem 2.4 and its consequences also extend to the maximal function of the absolute value due to $\operatorname{var} \mathrm{M}(|f|) \le C_d \operatorname{var} |f| \le C_d \operatorname{var} f$.

Lahti proved in [Lah20] for the local Hardy-Littlewood maximal operator $\widetilde{\mathrm{M}}_\Omega$ with respect to balls that if we have the local variation bound $\operatorname{var}_\Omega \widetilde{\mathrm{M}} f \le C_d \operatorname{var}_\Omega f$, then for all Sobolev functions $f \in W^{1,1}(\Omega)$ their maximal function is a local Sobolev function with $\|\nabla \widetilde{\mathrm{M}}_\Omega f\|_{L^1(\Omega)} \lesssim \|\nabla f\|_{L^1(\Omega)}$. In the global setting $\Omega = \mathbb{R}^d$ this continues to hold for the Hardy-Littlewoood maximal operator M with respect to cubes by a similar proof. The following example, which is also due to Lahti, however shows that for cubes it fails in the local setting.

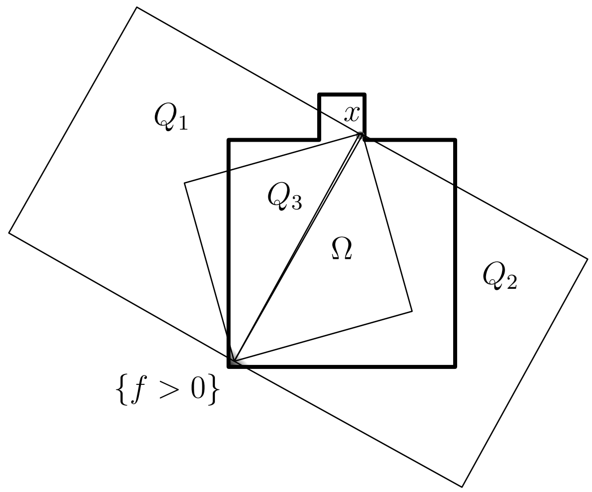

Figure 1: For $x \in \Omega$ with $x_2 \geq 0$, the typical cubes Q_1, Q_2, Q_2 which both contain x and intersect $\{f > 0\}$ do not lie within Ω.

Example 2.13 (Lahti)**.** Let $\Omega = (-5, 5) \times (-10, 0) \cup (-1, 1) \times [0, 2)$ and $f(x) = \max\{0, -14 - x_1 - x_2\}$. Then Ω is open and $f \in W^{1,1}(\Omega)$, but neither the local maximal function with respect to axes-parallel cubes nor the local maximal function with respect to cubes with arbitrary orientation belong to $W^{1,1}(\Omega)$.

The reason is that both local maximal functions have a jump on the line $[-1, 1] \times \{0\}$. Every $x \in \Omega$ with $x_2 < 0$ is contained in a cube $Q_\varepsilon = (-5 + \varepsilon, 5 - \varepsilon) \times (-10 + \varepsilon, 0 - \varepsilon)$, which means the local maximal functions in such x attain at least the value $\frac{1}{100} \int_\Omega f > 0$. In $x \in \Omega$ with $x_2 \geq 0$ however both local maximal functions are zero. In order to show that let $x \in \Omega$ with $x_2 \geq 0$ and Q be a cube with $\overline{Q} \subset \Omega$ and $x \in Q$. Then Q must have a corner in $(-1, 1) \times (0, 2)$. In order for Q to intersect $\{f > 0\}$ which is the open triangle with endpoints $(-5, -10), (-5, -9), (-4, -10)$, it must also have a corner in this triangle. This is not possible for a cube which is contained in Ω, as is illustrated in Figure 1. The cubes Q_1 and Q_2 represent the case when the corners in questions are neighboring corners of Q. If the corners are opposing corners then Q is of the form Q_3. However, the leftmost corners of cubes like Q_1 and Q_3 and the lowest corner of Q_2 cannot be contained in Ω.

3 The case of a finite set of cubes

In this section we prove Theorem 2.3. Let $f \in L^1(\mathbb{R}^d)$ and $\mathcal{Q}$ be a finite set of cubes. For $\lambda \in \mathbb{R}$ set $\mathcal{Q}^\lambda = \{Q \in \mathcal{Q} : f_Q \geq \lambda\}$. We consider the following partition

$$\mathcal{Q}^\lambda = \mathcal{Q}_0^\lambda \cup \mathcal{Q}_1^\lambda \cup \mathcal{Q}_2^\lambda.$$

The set $\mathcal{Q}_0^\lambda$ consists of all cubes $Q \in \mathcal{Q}^\lambda$ with

$$\mathcal{L}(Q \cap \{f \geq \lambda\}) \geq 2^{-d-1} \mathcal{L}(Q),$$

the set $\mathcal{Q}_1^\lambda$ consists of all cubes $Q \in \mathcal{Q}^\lambda \setminus \mathcal{Q}_0^\lambda$ with

$$\mathcal{L}\Big(Q \cap \bigcup \mathcal{Q}_0^\lambda\Big) \geq 2^{-d-1}\mathcal{L}(Q),$$

and the set $\mathcal{Q}_2^\lambda$ consists of all remaining cubes in $\mathcal{Q}^\lambda$. We split the boundary of $\mathcal{Q}^\lambda$ as follows,

$$\mathcal{H}^{d-1}\Big(\partial\, \mathcal{Q}^\lambda \setminus \overline{\{f \geq \lambda\}}^*\Big) \leq \mathcal{H}^{d-1}\Big(\partial \left(\bigcup \mathcal{Q}_0^\lambda \cup \bigcup \mathcal{Q}_1^\lambda\right) \setminus \overline{\{f \geq \lambda\}}^*\Big) + \mathcal{H}^{d-1}\Big(\partial \bigcup \mathcal{Q}_2^\lambda\Big). \tag{3.1}$$

We bound the first summand in Section 3.1. In Sections 3.2 to 3.4 we collect the necessary ingredients to deal with the second summand. In Section 3.5 we combine these results to a proof of Theorem 2.3. The results in Sections 3.1 and 3.3 are taken from [Wei20b, Wei20a] with necessary modifications.

3.1 Cubes with large intersection

In this section we prove Proposition 3.6, a bound on the part of the boundary of the superlevelset of the maximal function which comes from cubes which intersect the superlevelset of the function significantly. In [Wei20b, Proposition 4.4] we already showed this result for balls instead of cubes. The proof of Proposition 3.6 works the same way, here we only need to ensure that the steps which we have done only for balls in [Wei20b] also work for cubes.

For two nonzero $x, y \in \mathbb{R}^d$ we denote by $\sphericalangle(x, y)$ the angle between x and y, i.e. the unique value $\sphericalangle(x, y) \in [0, \pi]$ with

$$\langle x, y\rangle = \|x\|\|y\| \cos(\sphericalangle(x, y)).$$

Lemma 3.1. Let Q be a cube centered in the origin and $x \in \partial Q$. Then for the outer normal vector e to ∂Q in x we have

$$\sphericalangle(x, e) \leq \pi/2 - \arcsin(1/\sqrt{d}).$$

The precise value $\arcsin(1/\sqrt{d})$ does not matter here, what is important is that the angle is bounded away from $\pi/2$.

Proof. It suffices to consider the cube $Q = (-1, 1)^d$ and the face with outer normal $e = (1, 0, \ldots, 0)$. Then $x_1 = 1$ and $x_i \in [-1, 1]$ which implies

$$\frac{\langle x, e\rangle}{\|x\|\|e\|} \geq \frac{1}{\sqrt{d}}.$$

□

Lemma 3.2 ([Wei20b, Lemma 4.2] with more general numerology)**.** For every $\varepsilon > 0$ there is a number N large enough such that the following holds. Let B be a ball centered in the origin. Then for any two points $y_1, y_2 \in B$ and $x_1, x_2 \in \mathbb{R}^d$ with $|x_1|, |x_2| \geq (N+1)\operatorname{diam}(B)/2$ and $\sphericalangle(x_1, x_2) \leq \varepsilon$ we have

$$\sphericalangle(y_1 - x_1, y_2 - x_2) \leq 2\varepsilon.$$

The proof works just like the proof of [Wei20b, Lemma 4.2].

Definition 3.3. For $L \in \mathbb{R}$ we call a set $S \subset \mathbb{R}^d$ a *Lipschitz surface with constant* L if there is a subset $U \subset \mathbb{R}^{d-1}$ and a function $f : U \to \mathbb{R}$ which is Lipschitz-continuous with constant L, such that S is a rotation and translation of the graph $\{(x, f(x)) : x \in U\}$.

Lemma 3.4. Let S be a bounded Lipschitz surface with constant L. Then $\mathcal{H}^{d-1}(S) \lesssim (1 + L)\operatorname{diam}(S)^{d-1}$.

Proof. It suffices to apply the area formula [EG15, Theorem 3.8] to the function $x \mapsto (x, f(x))$. $\square$

Lemma 3.5 ([Wei20b, Lemma 4.1] for cubes)**.** Let $K > 0$, let B be a ball and let $\mathcal{Q}$ be a finite set of cubes Q with $\mathrm{l}(Q) \geq K \operatorname{diam}(B)$. Then $\partial \bigcup \mathcal{Q} \cap B$ is a union of a dimensional constant times $K^{-d} + 1$ many Lipschitz surfaces with Lipschitz constant depending only on the dimension, and

$$\mathcal{H}^{d-1}\Big(\partial \bigcup \mathcal{Q} \cap B\Big) \lesssim (K^{-d} + 1)\mathcal{H}^{d-1}(\partial B).$$

The proof works the same way as in [Wei20b], except that we have to take into account Lemma 3.1, which for balls holds with angle 0 instead of $\pi/2 - \arcsin(1/\sqrt{d})$. The details are given below.

Proof. It suffices to consider the case that B is centered in the origin. Take N from Lemma 3.2 with $\varepsilon = \arcsin(1/\sqrt{d})/2$. First consider the case $K \geq N$. Denote by c_Q the center of a cube Q. Let $Q_1, Q_2 \in \mathcal{Q}$ be cubes whose centers have an angle of at most $\arcsin(1/\sqrt{d})/2$. For $i = 1, 2$ let $x_i \in \partial Q_i \cap B$. Then by Lemma 3.2 the angle between $x_1 - c_{Q_1}$ and $x_2 - c_{Q_2}$ is at most $\arcsin(1/\sqrt{d})$. For $i = 1, 2$ let e_i be the outer surface normal to Q_i in x_i. Then by Lemma 3.1 we have $\sphericalangle(e_i, x_i - c_{Q_i}) \leq \pi/2 - \arcsin(1/\sqrt{d})$, and by the subadditivity of angles we can conclude

$$\sphericalangle(e_1, e_2) \leq 2\big[\pi/2 - \arcsin(1/\sqrt{d})\big] + \arcsin(1/\sqrt{d}) = \pi - \arcsin(1/\sqrt{d}). \tag{3.2}$$

Take a maximal set A of unit vectors which are separated by an angle of at least $\arcsin(1/\sqrt{d})/4$. Then $|A| \lesssim d^{(d-1)/2}$, and for every $Q \in \mathcal{Q}$ theres is an $e \in A$ with $\sphericalangle(e, c_Q) \leq \arcsin(1/\sqrt{d})$. That means we can write $\partial \bigcup \mathcal{Q} \cap B$ as the union

$$\partial \bigcup \mathcal{Q} \cap B = \bigcup_{e \in A} \partial \bigcup \Big\{Q \in \mathcal{Q} : \sphericalangle(e, c_Q) \leq \arcsin(1/\sqrt{d})/4\Big\} \cap B.$$

By (3.2) for each $e \in A$ the set in the union on the right hand side of the previous display is a Lipschitz surface with constant $1/\tan(\arcsin(1/\sqrt{d})) \sim \sqrt{d}$, whose perimeter is thus bounded by a fixed multiple of $\sqrt{d}\mathcal{H}^{d-1}(\partial B)$ due to Lemma 3.4. We can conclude

$$\begin{aligned}\mathcal{H}^{d-1}\Big(\partial \bigcup \mathcal{Q} \cap B\Big) &\leq \sum_{e \in A} \mathcal{H}^{d-1}\Big(\partial \bigcup \Big\{Q \in \mathcal{Q} : \sphericalangle(e, c_Q) \leq \arcsin(1/\sqrt{d})/4\Big\} \cap B\Big) \\ &\lesssim |A|\sqrt{d}\mathcal{H}^{d-1}(\partial B) \lesssim d^{d/2}\mathcal{H}^{d-1}(\partial B).\end{aligned}$$

The case $K \leq N$ can be concluded from the case $K \geq N$ just as in [Wei20b] by covering B by a dimensional constant times $(N/K)^d$ many balls B' with $\operatorname{diam}(B') = \frac{K}{N} \operatorname{diam}(B)$ and applying the above argument to each B'. $\square$

Proposition 3.6 ([Wei20b, Proposition 4.4] for cubes)**.** Let $\varepsilon \in (0, 1)$. Let $E \subset \mathbb{R}^d$ be a set of locally finite perimeter and let $\mathcal{Q}$ be a finite set of cubes such that for each $Q \in \mathcal{Q}$ we have $\mathcal{L}(E \cap Q) > \varepsilon \mathcal{L}(Q)$. Then

$$\mathcal{H}^{d-1}\Big(\partial \bigcup \mathcal{Q} \setminus \overline{E}^*\Big) \lesssim_\varepsilon \mathcal{H}^{d-1}\Big(\partial_* E \cap \bigcup \mathcal{Q}\Big).$$

We explain below how the proof of [Wei20b, Proposition 4.4] also proves Proposition 3.6. The only addition we need from this paper is Lemma 3.5.

Proof. [Wei20b, Proposition 4.4] is proved using [Wei20b, Lemma 4.1 and Lemma 4.3]. The proof of [Wei20b, Lemma 4.3] verbatim also works for cubes, as its dependency [Wei20b, Lemma 2.5] is already proven for cubes as well in [Wei20b]. Using [Wei20b, Lemma 4.3] for cubes, and Lemma 3.5 instead of [Wei20b, Lemma 4.1], one can take the proof of [Wei20b, Proposition 4.4] verbatim to prove Proposition 3.6. Note that [Wei20b, Proposition 4.4] is stated with the measure theoretic boundary instead where in Proposition 3.6 is the topological boundary, but since we are dealing with finitely many cubes, the arguments work the same for both notions of the boundary. □

We use Proposition 3.6 and the following boundary decomposition to deal with the first summand in (3.1).

Lemma 3.7 ([Wei20b, Lemma 1.7])**.** Let $A, B \subset \mathbb{R}^d$ be measurable. Then

$$\partial_* (A \cup B) \subset (\partial_* A \setminus \overline{B}^*) \cup \partial_* B. \tag{3.3}$$

Formula (3.3) also holds with the topological instead of the measure theoretic quantities. Because $\overline{E}^* \subset \overline{E}$, it continues to hold with the topological boundary and the measure theoretic closure.

Corollary 3.8. Define $\mathcal{Q}^\lambda$, $\mathcal{Q}_0^\lambda$ and $\mathcal{Q}_1^\lambda$ from f and $\mathcal{Q}$ as in the beginning of Section 3. Then

$$\mathcal{H}^{d-1}\Big(\partial\Big(\bigcup \mathcal{Q}_0^\lambda \cup \bigcup \mathcal{Q}_1^\lambda\Big) \setminus \overline{\{f \geq \lambda\}}^*\Big) \lesssim \mathcal{H}^{d-1}\Big(\partial_* \{f \geq \lambda\} \cap \bigcup \mathcal{Q}^\lambda\Big).$$

Proof. By Lemma 3.7 with the topological boundary we have

$$\partial\Big(\bigcup \mathcal{Q}_0^\lambda \cup \bigcup \mathcal{Q}_1^\lambda\Big) \subset \partial\Big(\bigcup \mathcal{Q}_1^\lambda\Big) \setminus \overline{\bigcup \mathcal{Q}_0^\lambda}^* \cup \partial\Big(\bigcup \mathcal{Q}_0^\lambda\Big),$$

by Proposition 3.6 we have

$$\mathcal{H}^{d-1}\Big(\partial \bigcup \mathcal{Q}_0^\lambda \setminus \overline{\{f \geq \lambda\}}^*\Big) \lesssim \mathcal{H}^{d-1}\Big(\partial_* \{f \geq \lambda\} \cap \bigcup \mathcal{Q}_0^\lambda\Big)$$

and by Proposition 3.6 and Lemma 3.7 we obtain

$$\begin{aligned}\mathcal{H}^{d-1}\Big(\partial \bigcup \mathcal{Q}_1^\lambda \setminus \overline{\bigcup \mathcal{Q}_0^\lambda \cup \{f \geq \lambda\}}^*\Big) &\lesssim \mathcal{H}^{d-1}\Big(\partial_* \Big(\bigcup \mathcal{Q}_0^\lambda \cup \{f \geq \lambda\}\Big) \cap \bigcup \mathcal{Q}_1^\lambda\Big) \\ &\leq \mathcal{H}^{d-1}\Big(\partial_* \bigcup \mathcal{Q}_0^\lambda \setminus \overline{\{f \geq \lambda\}}^* \cap \bigcup \mathcal{Q}_1^\lambda\Big) + \mathcal{H}^{d-1}\Big(\partial_* \{f \geq \lambda\} \cap \bigcup \mathcal{Q}_1^\lambda\Big).\end{aligned}$$

We use that the measure theoretic boundary is contained in the topological boundary and combine these three displays to finish the proof. □

3.2 Reducing to almost dyadically structured cubes

Proposition 3.9. Let $f, \mathcal{Q}, \mathcal{Q}_2^\lambda$ be as in the beginning of Section 3. Then there exists a subset $\mathcal{S} \subset \bigcup_\lambda \mathcal{Q}_2^\lambda$ with the following properties. Let $Q, R \in \mathcal{S}$ with $\mathrm{l}(R) \leq \mathrm{l}(Q)$. Then $\mathcal{L}(R \cap Q) < 2^{-1}\mathcal{L}(R)$, or R has a strictly smaller scale than Q and $f_R > f_Q$. Furthermore

$$\int_{-\infty}^{\infty} \mathcal{H}^{d-1}\Big(\partial \bigcup \mathcal{Q}_2^\lambda\Big)\,\mathrm{d}\lambda \lesssim \sum_{Q \in \mathcal{S}} (f_Q - \lambda_Q)\mathcal{H}^{d-1}(\partial Q) \tag{3.4}$$

where

$$\lambda_Q = \inf\{\lambda : \mathcal{L}(Q \cap \{f \geq \lambda\}) \leq 2^{-d-1}\mathcal{L}(Q)\}.$$

Proof. We construct $\mathcal{S}$ using the following algorithm.

Algorithm. Initiate $\mathcal{R} = \bigcup_\lambda \mathcal{Q}_2^\lambda$ and $\mathcal{S} = \emptyset$. We iterate the following procedure.

If $\mathcal{R}$ is empty then output $\mathcal{S}$ and stop. If $\mathcal{R}$ is nonempty let 2^n be the scale of the largest cube in $\mathcal{R}$, i.e. n is the largest integer for which there is a $Q \in \mathcal{R}$ with $\mathrm{l}(Q) \geq 2^n$. Take a cube $S \in \mathcal{R}$ scale 2^n which attains

$$\max\{f_Q : Q \in \mathcal{R},\ \mathrm{l}(Q) \geq 2^n\}$$

and add it to $\mathcal{S}$. Then remove all cubes Q with $f_Q \leq f_S$ and $\mathcal{L}(Q \cap S) > 2^{-1}\min\{\mathcal{L}(Q), \mathcal{L}(S)\}$ from $\mathcal{R}$ and repeat.

In each iteration in the above loop we remove at least the cube S from $\mathcal{R}$ that we added to $\mathcal{S}$. Since $\bigcup_\lambda \mathcal{Q}_2^\lambda$ is finite this means the algorithm will terminate and return a set $\mathcal{S}$ of cubes. Let $S, T \in \mathcal{S}$ be distinct cubes with $\mathrm{l}(S) \leq \mathrm{l}(T)$. If $\mathcal{L}(S \cap T) < 2^{-1}\mathcal{L}(S)$ then there is nothing to show, so assume $\mathcal{L}(S \cap T) \geq 2^{-1}\mathcal{L}(S)$. Then S and T cannot be of the same scale, because otherwise one of them would have been removed while the other was added to $\mathcal{S}$. That means S is of strictly smaller scale than T, which means T has been added to $\mathcal{S}$ in an earlier step than S, and thus $f_S > f_T$ because otherwise S would have been removed in that step. It remains to prove (3.4). Let $\lambda \in \mathbb{R}$ and $Q \in \mathcal{Q}_2^\lambda$. Then there is a cube S which has been added to $\mathcal{S}$ when Q was removed from $\mathcal{R}$. This means S has at least the same scale as Q and

$$\mathcal{L}(Q \cap S) \geq 2^{-1}\min\{\mathcal{L}(Q), \mathcal{L}(S)\} \geq 2^{-d-1}\mathcal{L}(Q)$$

and $f_S \geq f_Q \geq \lambda$. This implies $\lambda \geq \lambda_S$ because otherwise $S \in \mathcal{Q}_0^\lambda$ which would contradict $Q \in \mathcal{Q}_2^\lambda$. So for each $\lambda \in \mathbb{R}$ we have

$$\mathcal{Q}_2^\lambda = \bigcup_{S \in \mathcal{S} : \lambda_S \leq \lambda \leq f_S} \{Q \in \mathcal{Q}_2^\lambda : \mathcal{L}(Q \cap S) \geq 2^{-d-1}\mathcal{L}(Q)\},$$

and we can conclude from Proposition 3.6 that

$$\begin{aligned}\mathcal{H}^{d-1}\Big(\partial \bigcup \mathcal{Q}_2^\lambda\Big) &\leq \sum_{S \in \mathcal{S} : \lambda_S \leq \lambda \leq f_S} \mathcal{H}^{d-1}\Big(\partial \bigcup \{Q \in \mathcal{Q}_2^\lambda : \mathcal{L}(Q \cap S) \geq 2^{-d-1}\mathcal{L}(Q)\}\Big) \\ &\lesssim \sum_{S \in \mathcal{S} : \lambda_S \leq \lambda \leq f_S} \mathcal{H}^{d-1}(\partial S).\end{aligned}$$

Integrating both sides over λ implies (3.4) and finishes the proof. $\square$

For a cube Q denote by c_Q the center of Q and by v_Q the orientation of Q.

Lemma 3.10. For every $\varepsilon > 0$ there is a $\delta > 0$ such that for any cubes Q, P with $\mathrm{l}(P) \leq (1+\delta)\,\mathrm{l}(Q)$ and $|c_Q - c_P| \leq \delta\,\mathrm{l}(Q)$ and $|v_Q - v_P| \leq \delta$ we have $P \subset (1+\varepsilon)Q$.

Proof. For $\varepsilon > 0$ set $\delta = \varepsilon/(2 + 2\sqrt{d})$. Let P and Q be cubes as above. Denote

$$D(P, Q) = \sup_{x \in P} \inf_{y \in Q} |x - y|.$$

For an orientation v denote by $r_v(Q)$ the cube Q rotated by v. Then $P = \frac{\mathrm{l}(P)}{\mathrm{l}(Q)} r_{v_P - v_Q}(Q) + c_P - c_Q$ and thus

$$\begin{aligned} D(P,Q) &\le D\Big(P, \frac{\mathrm{l}(P)}{\mathrm{l}(Q)} r_{v_P-v_Q}(Q)\Big) + D\Big(\frac{\mathrm{l}(P)}{\mathrm{l}(Q)} r_{v_P-v_Q}(Q), r_{v_P-v_Q}(Q)\Big) + D(r_{v_P-v_Q}(Q), Q) \\ &\le \delta\,\mathrm{l}(Q) + \sqrt{d}\delta\,\mathrm{l}(Q) + \sqrt{d}\delta\,\mathrm{l}(Q) < \varepsilon\,\mathrm{l}(Q). \end{aligned}$$

Since every point x with $\inf_{y\in Q}|x-y| < \varepsilon\,\mathrm{l}(Q)$ is contained in $(1+\varepsilon)Q$ this concludes the proof. □

Lemma 3.11. Let $\mathcal{S}$ be a set of cubes with the same scale such that for any two distinct $Q, R \in \mathcal{S}$ we have $\mathcal{L}(Q \cap R) \le 2^{-1}\min\{\mathcal{L}(Q), \mathcal{L}(R)\}$. Then for any constant $C \ge 1$ and any point $x \in \mathbb{R}^d$, the number of cubes in $\{CQ : Q \in \mathcal{S}\}$ which contain x is bounded by a constant depending only on C and the dimension.

Proof. It suffices to consider the case that all cubes Q in $\mathcal{S}$ satisfy $2^{-1} \le \mathrm{l}(Q) < 1$. Then for every cube $Q \in \mathcal{S}$ with $x \in CQ$ we have $c_Q \in B(x, \sqrt{d}C)$. Let $R \in \mathcal{S}$ with $x \in CR$ be a cube with $\mathrm{l}(R) \le \mathrm{l}(Q)$ that is distinct from Q. Then by Lemma 3.10 there is a $\delta > 0$ such that $|c_Q - c_R| > \delta$ or $|v_Q - v_R| > \delta$ or $R \subset (1 + 2^{-d-1})^{\frac{1}{d}} Q$. Because the last alternative implies

$$\mathcal{L}(Q \cap R) \ge \mathcal{L}(R) - 2^{-d-1}\mathcal{L}(Q) > 2^{-1}\mathcal{L}(R) = 2^{-1}\min\{\mathcal{L}(Q), \mathcal{L}(R)\},$$

one of the first two must hold. We can conclude that the number of cubes Q in $\mathcal{S}$ with $x \in CQ$ is uniformly bounded because the set of coordinates $\{(c_Q, v_Q) : Q \in \mathcal{S},\ x \in CQ\}$ is $\delta-$separated and contained in a bounded subset of $\mathbb{R}^{d+1}$. □

3.3 A sparse mass estimate

Recall that $\mathfrak{D}(Q_0)$ is the set of dyadic cubes with respect to a base cube Q_0. We have the following proposition at our disposal.

Proposition 3.12 ([Wei20a, Corollary 3.3])**.** Let Q_0 be a cube, $\lambda_0 \in \mathbb{R}$ and $f \in L^1(Q_0)$ with $\mathcal{L}(\{f \ge \lambda_0\} \cap Q_0) \le 2^{-d-1}\mathcal{L}(Q_0)$. Then

$$\mathcal{L}(Q_0)(f_{Q_0} - \lambda_0) \le 2^{d+1}\int_{f_{Q_0}}^{\infty} \mathcal{L}\Big(\{f \ge \lambda\} \cap \bigcup \{Q \in \mathfrak{D}(Q_0) : f_Q \ge \lambda,\ \mathcal{L}(Q \cap \{f \ge \lambda\}) < \mathcal{L}(Q)/2\}\Big)\,\mathrm{d}\lambda.$$

In [Wei20a, Corollary 3.3] the union on the right hand side is only over maximal cubes Q, and the condition $f \ge \lambda$ is replaced by $f > \lambda$. Proposition 3.12 still holds in this form because dropping the maximality assumption only makes the statement weaker and the sets $\{f > \lambda\}$ and $\{f \ge \lambda\}$ agree up to Lebesgue-measure zero for almost every λ.

Remark 3.13. In order to prove Theorem 1.1 for the Hardy-Littlewood maximal operator over balls, a variant of Proposition 3.12 for balls instead of cubes would be useful. One way to formulate it for balls is to replace the condition $Q \in \mathfrak{D}(Q_0)$ by the condition $B \subset 2B_0$. However the proof of Proposition 3.12 in [Wei20a] relies strongly on dyadic cubes; a proof for balls requires a new idea. Note that in its current form Proposition 3.12 only takes into account the parts of f that are contained within Q_0, and lie above f_{Q_0}. This is not strictly necessary, maybe for balls a variant that takes f into account also below f_{B_0} and within CB_0 is easier to prove.

Lemma 3.14. Let Q_0 be a cube and let $E \subset Q$ a be measurable set with $\mathcal{L}(E \cap Q_0) < \mathcal{L}(Q_0)/2$. Then the cubes $Q \in \mathfrak{D}(Q_0)$ with

$$\frac{1}{2^{d+1}} \leq \frac{\mathcal{L}(E \cap Q)}{\mathcal{L}(Q)} < \frac{1}{2} \tag{3.5}$$

cover almost all of $E \cap Q_0$.

Proof. Let x be a Lebesgue point of $E \cap Q_0$ outside of $\bigcup\{\partial Q : Q \in \mathfrak{D}(Q_0)\}$. Denote by Q_n the cube in $\mathfrak{D}(Q_0)$ with $x \in Q_n$ and $\mathrm{l}(Q_n) = 2^{-n}\,\mathrm{l}(Q_0)$. By the Lebesgue differentiation theorem we have $\mathcal{L}(Q_n \cap E)/\mathcal{L}(Q_n) \to 1$ as $n \to \infty$. Let n be the smallest index such that

$$\mathcal{L}(Q_n \cap E) \geq 2^{-d-1}\mathcal{L}(Q_n).$$

If $n = 0$ then

$$\mathcal{L}(Q_n \cap E) < 2^{-1}\mathcal{L}(Q_n) \tag{3.6}$$

holds by assumption on Q_0. If $n > 0$ then $\mathcal{L}(Q_{n-1} \cap E) < 2^{-d-1}\mathcal{L}(Q_{n-1})$ which implies (3.6). $\square$

Lemma 3.15. There exists an $\varepsilon > 0$ which depends only on the dimension such that for each cube Q and for each measurable set $E \subset Q$ which satisfy (3.5) we have

$$\frac{1}{2^{d+2}} < \frac{\mathcal{L}((1-\varepsilon)^2 Q \cap E)}{\mathcal{L}((1-\varepsilon)^2 Q)} < \frac{1}{2} + \frac{1}{2^{d+2}}.$$

Proof. It is straightforward to see that it suffices to take ε with $\mathcal{L}(Q \setminus (1-\varepsilon)^2 Q) \leq 2^{-d-2}\mathcal{L}(Q)$, for example, we may choose $\varepsilon = 2^{-d-3}/d$. $\square$

3.4 Organizing mass

Lemma 3.16. Let $\mathcal{S}$ be a finite set of cubes and for each $Q_0 \in \mathcal{S}$ let $\mathcal{D}(Q_0)$ be any set of cubes which are contained in Q_0. Denote $\mathcal{D} = \bigcup_{Q_0 \in \mathcal{S}} \mathcal{D}(Q_0)$ and assume that no cube in $\mathcal{S}$ is strictly contained in a cube in $\mathcal{D}$. Then for every $\varepsilon > 0$ the set $\mathcal{D}$ has a subset $\mathcal{F}$ with the following properties.

(1) For any $x \in \mathbb{R}^d$, there are at most C many cubes $Q \in \mathcal{F}$ with $x \in (1-\varepsilon)^2 Q$.

(2) For each $Q_0 \in \mathcal{S}$ and $Q \in \mathcal{D}(Q_0)$ there exists a cube $P \in \mathcal{F}$ with $Q \subset C_1 P$ and $P \subset C_2 Q_0$.

The constants C, C_1 and C_2 depend only on ε and on the dimension d.

Proof. Denote by $\widetilde{\mathcal{D}}$ the set of cubes $Q \in \mathcal{D}$ for which there is no $P \in \mathcal{D}$ with $Q \subset (1-\varepsilon)P$. For each $n \in \mathbb{Z}$ denote by $\widetilde{\mathcal{D}}_n$ the cubes in $\widetilde{\mathcal{D}}$ of scale 2^n. Take a maximal set of cubes $\mathcal{F}_n \subset \widetilde{\mathcal{D}}_n$ such that for any two distinct cubes $Q, P \in \mathcal{F}_n$ their dilated cubes $(1-\varepsilon)^2 Q$ and $(1-\varepsilon)^2 P$ are disjoint. Set $\mathcal{F} = \bigcup_{n \in \mathbb{Z}} \mathcal{F}_n$.

First we prove (1). Let $x \in \mathbb{R}^d$. Let Q, P be cubes with so different scales such that $\mathrm{diam}(P) \leq \varepsilon(1-\varepsilon)2^{-1}\,\mathrm{l}(Q)$. If $x \in (1-\varepsilon)^2 Q$ and $x \in (1-\varepsilon)^2 P$ then $P \subset (1-\varepsilon)Q$. This means it is not possible that both Q and P belong to $\mathcal{F} \subset \widetilde{\mathcal{D}}$. Thus the set of integers $n \in \mathbb{Z}$, for which there is a cube $Q \in \mathcal{F}_n$ with $x \in (1-\varepsilon)^2 Q$, is bounded. Since by definition of $\mathcal{F}_n$ for each $n \in \mathbb{Z}$ there is at most one such cube $Q \in \mathcal{F}_n$ we can conclude (1).

Now we prove (2). Let $Q_0 \in \mathcal{S}$ and $Q \in \mathcal{D}(Q_0)$. If $Q \in \mathcal{F}$ then we can take $P = Q$. If $Q \in \widetilde{\mathcal{D}} \setminus \mathcal{F}$ then there is a cube $P \in \mathcal{F}$ with $\mathrm{l}(Q) \le 2\,\mathrm{l}(P)$ and $\mathrm{l}(P) \le 2\,\mathrm{l}(Q) \le 2\,\mathrm{l}(Q_0)$ which intersects Q and thus also Q_0. This implies $Q \subset (2\sqrt{d}+1)P$ and $P \subset (2\sqrt{d}+1)Q_0$. If $Q \notin \widetilde{\mathcal{D}}$, there exists a cube $R \in \widetilde{\mathcal{D}}$ with $Q \subset (1-\varepsilon)R$. As in the previous case, this means there is cube $P \in \mathcal{F}$ with $\mathrm{l}(R) \le 2\,\mathrm{l}(P) \le 4\,\mathrm{l}(R)$ which intersects R, and hence $Q \subset R \subset (2\sqrt{d}+1)P$. We observe that Q_0 intersects $(1-\varepsilon)R$. Thus if $\operatorname{diam}(Q_0) < \varepsilon(1-\varepsilon)2^{-1}\,\mathrm{l}(R)$ then $Q_0 \subsetneq R$. But this contradicts our assumption on $\mathcal{S}$ and $\mathcal{D}$, so we must have $\mathrm{l}(R) \le 2\sqrt{d}[\varepsilon(1-\varepsilon)]^{-1}\,\mathrm{l}(Q_0)$ and hence $P \subset (2\sqrt{d}+1)R \subset C_2 Q_0$. □

3.5 Combining the results

We shall apply the following Poincaré inequality.

Lemma 3.17 ([EG15, Theorem 5.10(ii)])**.** Let $Q \subset \mathbb{R}^d$ be a cube and $f \in L^1(Q)$ with $\operatorname{var}_Q f < \infty$. Then

$$\|f - f_Q\|_{L^{d/(d-1)}(Q)} \lesssim \operatorname{var}_Q f.$$

In [EG15] it is formulated for balls, but it also holds for cubes. There it is furthermore assumed that $f \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$, but it is not necessary to make this assumption a priori, because $f \in L^1(Q)$ and $\operatorname{var}_Q f < \infty$ imply that f extended by 0 outside of Q is indeed a function in $\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$. And in fact, the proof of [EG15, Theorem 5.10(ii)] in [EG15] also works verbatim for $f \in L^1(Q)$ with $\operatorname{var}_Q f < \infty$. Moreover, $f \in L^1_{\mathrm{loc}}(Q)$ and $\operatorname{var}_Q f < \infty$ already imply $f \in L^1(Q)$. We conclude this from Proposition 4.1 in the beginning of the proof of Theorem 2.4. For a characteristic function, Lemma 3.17 reduces to the isoperimetric inequality. It has a straightforward consequence.

Corollary 3.18. Let $\delta > 0$. For any cube $Q \subset \mathbb{R}^d$ and any measurable set $E \subset \mathbb{R}^d$ with $\mathcal{L}(Q \cap E) \le (1-\delta)\mathcal{L}(Q)$ we have

$$\mathcal{H}^{d-1}(\partial_* E \cap Q)^d \gtrsim_\delta \mathcal{L}(E \cap Q)^{d-1}.$$

Proof. We apply Lemma 3.17 with $f = 1_E$ and obtain

$$\mathcal{L}(E \cap Q)^{\frac{d-1}{d}} \lesssim_\delta \|f - f_Q\|_{L^{d/(d-1)}(Q)} \lesssim \operatorname{var}_Q f = \mathcal{H}^{d-1}(\partial_* E \cap Q).$$

□

Lemma 3.19. For a finite set of cubes $\mathcal{Q}$ let $\widetilde{\mathcal{Q}}$ be the set of cubes $Q \in \mathcal{Q}$ such that for all $P \in \mathcal{Q}$ with $P \supsetneq Q$ we have $f_Q > f_P$. Then for every $\lambda \in \mathbb{R}$ we have $\bigcup\{Q \in \widetilde{\mathcal{Q}} : f_Q \ge \lambda\} = \bigcup\{Q \in \mathcal{Q} : f_Q \ge \lambda\}$.

Proof. For $\lambda \in \mathbb{R}$ and $R \in \mathcal{Q}$ with $f_R \ge \lambda$ let Q be the largest cube $Q \in \mathcal{Q}$ with $R \subset Q$ and $f_R \ge \lambda$. Since $\mathcal{Q}$ is finite such a cube Q exists. Then we have $f_Q > f_P$ for any cube $P \in \mathcal{Q}$ with $Q \subsetneq P$, which means $Q \in \widetilde{\mathcal{Q}}$. This finishes the proof. □

Proof of Theorem 2.3. By Lemma 3.19 it suffices to consider the set of cubes $\widetilde{\mathcal{Q}}$ instead of $\mathcal{Q}$. Define $\mathcal{Q}_0$, $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}^\lambda$ as in the beginning of Section 3, but from $\widetilde{\mathcal{Q}}$ instead of from $\mathcal{Q}$. We integrate (3.1) and Corollary 3.8 over $\lambda \in \mathbb{R}$ and obtain

$$\int \mathcal{H}^{d-1}\Big(\partial \bigcup \widetilde{\mathcal{Q}}^\lambda \setminus \overline{\{f \ge \lambda\}}^*\Big)\,\mathrm{d}\lambda \le \int \mathcal{H}^{d-1}\Big(\partial \bigcup \mathcal{Q}_2^\lambda\Big)\,\mathrm{d}\lambda + C \int \mathcal{H}^{d-1}\Big(\partial_* \{f \ge \lambda\} \cap \bigcup \widetilde{\mathcal{Q}}^\lambda\Big)\,\mathrm{d}\lambda.$$

It remains to estimate the first term. We can bound it using Proposition 3.9,

$$\int_{-\infty}^{\infty} \mathcal{H}^{d-1}\Big(\partial \bigcup \mathcal{Q}_2^\lambda\Big)\,\mathrm{d}\lambda \lesssim \sum_{Q\in\mathcal{S}} (f_Q-\lambda_Q)\mathcal{H}^{d-1}(\partial Q). \tag{3.7}$$

For every $Q_0 \in \mathcal{S}$ and $\lambda \in \mathbb{R}$ set

$$\mathcal{D}^\lambda(Q_0) = \big\{Q \in \mathfrak{D}(Q_0) : \exists P \in \mathfrak{D}(Q_0)\ Q \subset P,\ f_P \geq \lambda,\ 2^{-d-1}\mathcal{L}(Q) \leq \mathcal{L}(Q \cap \{f \geq \lambda\}) < 2^{-1}\mathcal{L}(Q)\big\}.$$

By Proposition 3.12 and Lemma 3.14 we have

$$(f_{Q_0}-\lambda_{Q_0})\mathcal{H}^{d-1}(\partial Q_0) \lesssim \mathrm{l}(Q_0)^{-1}\int_{f_{Q_0}}^{\infty} \mathcal{L}\Big(\bigcup \mathcal{D}^\lambda(Q_0)\Big)\,\mathrm{d}\lambda \tag{3.8}$$

for every $Q_0 \in \mathcal{S}$. Now we show that for each $\lambda \in \mathbb{R}$ the premise of Lemma 3.16 holds for the sets $\{Q_0 \in \mathcal{S} : f_{Q_0} \leq \lambda\}$ and $\mathcal{D}^\lambda(Q_0)$. So let $\lambda \in \mathbb{R}$ and $Q, Q_0 \in \mathcal{S}$ with $f_Q, f_{Q_0} \leq \lambda$ and $P \in \mathcal{D}^\lambda(Q_0)$. We need to show that Q is not strictly contained in P, so assume for a contradiction that it is. By the definition of $\mathcal{D}^\lambda(Q_0)$ there is a cube $R \in \mathfrak{D}(Q_0)$ with $R \supset P$ and $f_R \geq \lambda$. But because $\mathcal{Q}$ is dyadically complete we have $R \in \mathcal{Q}$, and so from $Q \subsetneq P \subset R$ and $f_R \geq \lambda \geq f_Q$ we obtain a contradiction to our definition of $\widetilde{\mathcal{Q}}$ in Lemma 3.19.

That means for every $\lambda \in \mathbb{R}$ we can apply Lemma 3.16 to $\{Q_0 \in \mathcal{S} : f_{Q_0} \leq \lambda\}$ and $\mathcal{D}^\lambda(Q_0)$, with $\varepsilon > 0$ from Lemma 3.15. We denote the resulting set of cubes by $\mathcal{F}^\lambda$. By Lemma 3.16(2) we have

$$\mathcal{L}\Big(\bigcup \mathcal{D}^\lambda(Q_0)\Big) \leq \mathcal{L}\Big(\bigcup\{C_1 Q : Q \in \mathcal{F}^\lambda,\ Q \subset C_2 Q_0\}\Big) \leq C_1^d \sum_{Q\in\mathcal{F}^\lambda : Q \subset C_2 Q_0} \mathcal{L}(Q). \tag{3.9}$$

By (3.7) to (3.9) and Fubini's theorem we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} \mathcal{H}^{d-1}\Big(\partial \bigcup \mathcal{Q}_2^\lambda\Big)\,\mathrm{d}\lambda &\lesssim \sum_{Q_0\in\mathcal{S}} \mathrm{l}(Q_0)^{-1}\int_{f_{Q_0}}^{\infty} \sum_{Q\in\mathcal{F}^\lambda : Q\subset C_2 Q_0} \mathcal{L}(Q)\,\mathrm{d}\lambda \\ &\leq \int_{-\infty}^{\infty} \sum_{Q\in\mathcal{F}^\lambda} \mathcal{L}(Q) \sum_{Q_0\in\mathcal{S}, Q\subset C_2 Q_0} \mathrm{l}(Q_0)^{-1}\,\mathrm{d}\lambda. \end{aligned}\tag{3.10}$$

For $Q \in \mathcal{F}^\lambda$ let $n \in \mathbb{Z}$ with $2^{n-1} < \mathrm{l}(Q) \leq 2^n$. By Proposition 3.9 and Lemma 3.11 for each $k \in \mathbb{Z}$ the number of cubes $Q_0 \in \mathcal{S}$ with $2^{k-1} < \mathrm{l}(Q_0) \leq 2^k$ and $Q \subset C_2 Q_0$ is uniformly bounded. Thus

$$\sum_{Q_0\in\mathcal{S}, Q\subset C_2 Q_0} \mathrm{l}(Q_0)^{-1} = \sum_{k\geq n-2-\log C_2} \sum_{\substack{Q_0\in\mathcal{S},\\ Q\subset C_2 Q_0,\\ 2^{k-1}<\mathrm{l}(Q_0)\leq 2^k}} \mathrm{l}(Q_0)^{-1} \lesssim \sum_{k\geq n-2-\log C_2} 2^{-k} \lesssim 2^{-n} \leq 2\,\mathrm{l}(Q)^{-1}. \tag{3.11}$$

Because $Q \in \mathcal{D}^\lambda(Q_0)$ we further have by Lemma 3.15 and Corollary 3.18 that

$$\begin{aligned}\mathrm{l}(Q)^{d-1} &\lesssim \mathcal{L}((1-\varepsilon)^2 Q)^{(d-1)/d} \lesssim \mathcal{L}(\{f \geq \lambda\} \cap (1-\varepsilon)^2 Q)^{(d-1)/d} \\ &\lesssim \mathcal{H}^{d-1}(\partial_* \{f \geq \lambda\} \cap (1-\varepsilon)^2 Q). \end{aligned}\tag{3.12}$$

Recall that by Lemma 3.16(1) for every $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^d$ there are at most C different cubes $Q \in \mathcal{F}^\lambda$ with $x \in (1-\varepsilon)^2 Q$. Thus

$$\begin{aligned}\sum_{Q\in\mathcal{F}^\lambda} \mathcal{H}^{d-1}(\partial_* \{f \geq \lambda\} \cap (1-\varepsilon)^2 Q) &= \int_{\bigcup \mathcal{F}^\lambda \cap \partial_* \{f\geq\lambda\}} \sum_{Q\in\mathcal{F}^\lambda} 1_{(1-\varepsilon)^2 Q}(x) \,\mathrm{d}\mathcal{H}^{d-1}(x) \\ &\leq C \int_{\bigcup \mathcal{S}\cap\partial_* \{f\geq\lambda\}} \mathrm{d}\mathcal{H}^{d-1}(x) \\ &= C\mathcal{H}^{d-1}\Big(\bigcup \mathcal{S} \cap \partial_* \{f \geq \lambda\}\Big) \qquad (3.13)\end{aligned}$$

Combining (3.11) to (3.13) we obtain

$$\sum_{Q\in\mathcal{F}^\lambda} \mathcal{L}(Q) \sum_{Q_0\in\mathcal{S}, Q\subset C_2 Q_0} \mathrm{l}(Q_0)^{-1} \lesssim \mathcal{H}^{d-1}\Big(\bigcup \mathcal{S} \cap \partial_* \{f \geq \lambda\}\Big). \qquad (3.14)$$

We integrate (3.14) over $\lambda \in \mathbb{R}$ and apply (3.10) to finish the proof. □

Remark 3.20. Except from Proposition 3.12, the above proof also works for balls instead of cubes.

4 Local integrability and approximation

4.1 Uncountable sets of cubes

In this section we prove the local integrability of the local maximal function, and use it to deduce Theorem 2.4 from Theorem 2.3.

Proposition 4.1. Let $\Omega \subset \mathbb{R}^d$ be an open set and assume that $f \in L^1_{\mathrm{loc}}(\Omega)$ with $f \geq 0$ and $\mathrm{var}_\Omega f < \infty$. Denote

$$\widetilde{\mathrm{M}}_\Omega f(x) = \sup_{x\in\overline{Q},\ Q\subset\Omega} \frac{1}{\mathcal{L}(Q)} \int_Q f(y) \,\mathrm{d}y.$$

Then $\widetilde{\mathrm{M}}_\Omega f \in L^{d/(d-1)}_{\mathrm{loc}}(\Omega)$.

Remark 4.2. In the global setting, Proposition 4.1 directly follows from general theory: Assume $\Omega = \mathbb{R}^d$ and let $f \in L^1(\mathbb{R}^d)$ with $\mathrm{var}\, f < \infty$. Then by the Sobolev embedding theorem we have $f \in L^{d/(d-1)}(\mathbb{R}^d)$, and thus $\widetilde{\mathrm{M}}_{\mathbb{R}^d} f \in L^{d/(d-1)}(\mathbb{R}^d)$ follows from the Hardy-Littlewood maximal function theorem.

Proof of Proposition 4.1. It suffices to prove that for every cube Q_0 with $3Q_0 \subset \Omega$ we have

$$\int_{Q_0} \mathrm{M} f(x)^{d/(d-1)} \,\mathrm{d}x < \infty.$$

So let Q_0 be a cube with $3Q_0 \subset \Omega$. Then $\overline{Q_0} \subset \Omega$ so that $f_{Q_0} < \infty$, and therefore we have $\int_{2Q_0} f^{d/(d-1)} < \infty$ by Lemma 3.17. We can conclude

$$\int_{Q_0} \widetilde{\mathrm{M}}_{2Q_0} f(x)^{d/(d-1)} \,\mathrm{d}x < \infty$$

by the Hardy-Littlewood maximal function theorem. Denote

$$K = \sup\Big\{\frac{1}{\mathcal{L}(Q)}\int_Q f(x)\,\mathrm{d}x \;\Big|\; Q \subset \Omega,\ Q \cap (2\overline{Q_0})^{\complement} \neq \emptyset,\ \overline{Q} \cap Q_0 \neq \emptyset\Big\}.$$

Then for every $x \in Q_0$ we have $\widetilde{\mathrm{M}}_\Omega f(x) \leq \max\{\widetilde{\mathrm{M}}_{2Q_0} f(x), K\}$. Thus it remains to show $K < \infty$. Let $Q \subset \Omega$ be a cube with $Q \cap (2\overline{Q_0})^{\complement} \neq \emptyset$ and $\overline{Q} \cap Q_0 \neq \emptyset$. Then for ε small enough, the dilated cube $(1-\varepsilon)Q$ is admissible in the supremum and $(1-\varepsilon)\overline{Q} \subset \Omega$. That means there is a sequence of cubes $Q_1, Q_2, \dots$ with $f_{Q_n} \to K$ as $n \to \infty$ such that for every $n \in \mathbb{N}$ we have $\overline{Q_n} \subset \Omega$ and that $\overline{Q_n}$ intersects Q_0 and $\Omega \setminus 2Q_0$. That means every cube Q_n has side length at least $\mathrm{l}(Q_0)/\sqrt{d}$, and $Q_n \cap 2Q_0$ contains a cube P_n with side length uniformly bounded from below. By a compactness argument there is a subsequence $(n_k)_k$ such that the cubes $(P_{n_k})_k$ converge in $L^1(2Q_0)$ to a cube $P \subset 2Q_0$ with positive side length. That means for $Q = 2^{-1}P$ there is a k_0 such that for all $k \geq k_0$ we have $Q \subset P_{n_k} \subset Q_{n_k}$. Since Q and Q_{n_k} are compactly contained in Ω we have $f_Q, f_{Q_{n_k}} < \infty$. Then by Hölder's inequality and Lemma 3.17 we obtain for $k \geq k_0$ that

$$\begin{aligned}
|f_{Q_{n_k}} - f_Q| &\leq \mathcal{L}(Q)^{-1}\|f - f_{Q_{n_k}}\|_{L^1(Q)} \\
&\leq \mathcal{L}(Q)^{-(d-1)/d}\|f - f_{Q_{n_k}}\|_{L^{d/(d-1)}(Q)} \\
&\leq \mathcal{L}(Q)^{-(d-1)/d}\|f - f_{Q_{n_k}}\|_{L^{d/(d-1)}(Q_{n_k})} \\
&\lesssim \mathcal{L}(Q)^{-(d-1)/d}\operatorname{var}_{Q_{n_k}} f \\
&\leq \mathcal{L}(Q)^{-(d-1)/d}\operatorname{var}_\Omega f.
\end{aligned}$$

Thus we can conclude $K \lesssim |f_Q| + \mathrm{l}(Q)^{-(d-1)/d}\operatorname{var}_\Omega f < \infty$. This finishes the proof. □

Lemma 4.3 ([EG15, Theorem 5.2])**.** Let $\Omega \subset \mathbb{R}^d$ be an open set and assume that $f_1, f_2, \dots \in L^1_{\mathrm{loc}}(\Omega)$ are functions with $\operatorname{var}_\Omega f_n < \infty$ which converge in $L^1_{\mathrm{loc}}(\Omega)$ to a function f as $n \to \infty$. Then

$$\operatorname{var}_\Omega f \leq \liminf_{n\to\infty} \operatorname{var}_\Omega f_n.$$

In [EG15, Theorem 5.2] they assume $f_n \in \mathrm{BV}(\Omega)$, which is not necessary. Now we are ready to prove Theorem 2.4.

Proof of Theorem 2.4. Let $Q \in \mathcal{Q}$. Then for every $\varepsilon > 0$ we have $\overline{(1-\varepsilon)Q} \subset \Omega$ and thus

$$\int_Q |f(x)|\,\mathrm{d}x = \lim_{\varepsilon\to 0}\int_{(1-\varepsilon)Q} |f(x)|\,\mathrm{d}x \leq \mathcal{L}(Q)\inf_{x \in \frac{1}{2}Q} \widetilde{\mathrm{M}}_\Omega |f|(x),$$

which is finite by Proposition 4.1. For every $q \in \mathbb{Q}$, the set $\{\mathrm{M}_{\mathcal{Q}} f > q\}$ is the union of all cubes Q with $f_Q > q$. By Lindelöf's lemma or Proposition 4.4 it has a countable subcover $\mathcal{Q}^q$. Define

$$\mathcal{P} = \mathcal{Q} \cap \bigcup_{q\in\mathbb{Q}}\bigcup_{Q\in\mathcal{Q}^q} \mathfrak{D}(Q).$$

Then $\mathcal{P}$ is countable and dyadically complete because $\mathcal{Q}$ is dyadically complete. Let $x \in \Omega$. Then for every $\lambda \in \mathbb{R}$ with $\mathrm{M}_{\mathcal{Q}} f(x) > \lambda$ there exists a $q \in \mathbb{Q}$ with $\mathrm{M}_{\mathcal{Q}} f(x) > q \geq \lambda$, and thus there is a $Q \in \mathcal{Q}^q \subset \mathcal{P}$ with $f_Q > q \geq \lambda$. We can conclude that

$$\mathrm{M}_{\mathcal{Q}} f(x) = \max\Big\{f(x), \sup_{Q\in\mathcal{P},\ x\in Q}\frac{1}{\mathcal{L}(Q)}\int_Q f(y)\,\mathrm{d}y\Big\}.$$

Consider an increasing sequence $\mathcal{Q}_1, \mathcal{Q}_2, \ldots$ of finite subsets of $\mathcal{P}$ which are dyadically complete and with $\bigcup_n \mathcal{Q}_n = \mathcal{P}$, and define

$$\mathrm{M}_n f(x) = \max\left\{ f(x), \max_{Q \in \mathcal{Q}_n,\ x \in Q} \frac{1}{\mathcal{L}(Q)} \int_Q f(y)\,\mathrm{d}y \right\}.$$

Then for every $x \in \Omega$ we have that $f(x) \leq \mathrm{M}_n f(x) \leq \mathrm{M}_{\mathcal{Q}} f(x)$ and $\mathrm{M}_n f(x)$ monotonously tends to $\mathrm{M}_{\mathcal{Q}} f(x)$ from below. Let B be a ball with $\overline{B} \subset \Omega$. Since $f \in L^1_{\mathrm{loc}}(\Omega)$ we have $\int_B |f| < \infty$ and by Proposition 4.1 we have $\int_B |\mathrm{M}_{\mathcal{Q}} f| < \infty$. So we can conclude by monotone convergence that

$$\int_B |\mathrm{M}_n f(x) - \mathrm{M}_{\mathcal{Q}} f(x)|\,\mathrm{d}x \to 0.$$

It follows from Lemma 4.3 that

$$\operatorname{var}_\Omega \mathrm{M}_{\mathcal{Q}} f \leq \liminf_{n\to\infty} \operatorname{var}_\Omega \mathrm{M}_n f, \tag{4.1}$$

and it suffices to bound $\operatorname{var}_\Omega \mathrm{M}_n f$ uniformly. We have

$$\{\mathrm{M}_n f \geq \lambda\} = \{f \geq \lambda\} \cup \bigcup \{Q \in \mathcal{Q}_n : f_Q \geq \lambda\}$$

and thus by Lemma 3.7 we have

$$\mathcal{H}^{d-1}(\partial_* \{\mathrm{M}_n f \geq \lambda\} \cap \Omega) \leq \mathcal{H}^{d-1}\Big(\partial_* \bigcup \{Q \in \mathcal{Q}_n : f_Q \geq \lambda\} \setminus \overline{\{f \geq \lambda\}}^* \cap \Omega\Big) + \mathcal{H}^{d-1}(\partial_* \{f \geq \lambda\} \cap \Omega).$$

Using Lemma 2.2 we can conclude from Theorem 2.3 that

$$\begin{aligned}
\operatorname{var}_\Omega \mathrm{M}_n f &\leq \int_{-\infty}^{\infty} \mathcal{H}^{d-1}\Big(\partial_* \bigcup \{Q \in \mathcal{Q}_n : f_Q \geq \lambda\} \cap \Omega \setminus \overline{\{f \geq \lambda\}}^*\Big) + \mathcal{H}^{d-1}(\partial_* \{f \geq \lambda\} \cap \Omega)\,\mathrm{d}\lambda \\
&\leq (C_d + 1) \int_{-\infty}^{\infty} \mathcal{H}^{d-1}(\partial_* \{f \geq \lambda\} \cap \Omega)\,\mathrm{d}\lambda \\
&= (C_d + 1) \operatorname{var}_\Omega f.
\end{aligned}$$

By (4.1) this finishes the proof. □

4.2 Open and closed cubes

In this section we prove Proposition 2.8, which states that it makes essentially no difference if the maximal operator is defined using open or closed cubes.

Proposition 4.4. Let $\mathcal{Q}$ be a set of cubes Q with $\mathrm{l}(Q) > 0$. Then there is a sequence of cubes $Q_1, Q_2, \ldots \in \mathcal{Q}$ with

$$\bigcup \mathcal{Q} = Q_1 \cup Q_2 \cup \ldots.$$

Furthermore, $\bigcup \mathcal{Q}$ and $\bigcup \{\overline{Q} \in \mathcal{Q}\}$ are measurable and

$$\mathcal{L}\Big(\bigcup \{\overline{Q} : Q \in \mathcal{Q}\} \setminus \bigcup \mathcal{Q}\Big) = 0.$$

In order to prove Proposition 4.4 we need a result on the volume of neighborhoods of Lipschitz surfaces, recall Definition 3.3.

Lemma 4.5. Let $\varepsilon > 0$ and let S be a Lipschitz surface with constant L. Then

$$\mathcal{L}(\{x \in \mathbb{R}^d : \exists y \in S \ |x-y| < \varepsilon\}) \lesssim (\operatorname{diam}(S) + \varepsilon)^{d-1}(1+L)\varepsilon.$$

Proof. After a translation and rotation of S, by the Kirszbraun theorem there is a Lipschitz function $f : \mathbb{R}^{d-1} \to \mathbb{R}$ with constant L such that $S \subset \{(z, f(z)) : z \in \mathbb{R}^{d-1},\ |z| \leq \operatorname{diam}(S)\}$. Then for every $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $y \in S$ with $|x-y| < \varepsilon$ we have $|(x_1, \ldots, x_{d-1})| \leq \operatorname{diam}(S) + \varepsilon$ and

$$|f(x_1, \ldots, x_{d-1}) - x_d| \leq |f(x_1, \ldots, x_{d-1}) - f(y_1, \ldots, y_{d-1})| + |y_d - x_d| \leq L\varepsilon + \varepsilon.$$

We can conclude that

$$\begin{aligned}
&\mathcal{L}(\{x \in \mathbb{R}^d : \exists y \in S \ |x-y| < \varepsilon\}) \\
&\quad \leq \mathcal{L}(\{x \in \mathbb{R}^d : |(x_1, \ldots, x_{d-1})| \leq \operatorname{diam}(S) + \varepsilon,\ |x_d - f(x_1, \ldots, x_{d-1})| \leq (1+L)\varepsilon\}) \\
&\quad \lesssim (\operatorname{diam}(S) + \varepsilon)^{d-1}(1+L)\varepsilon.
\end{aligned}$$

$\square$

Proof of Proposition 4.4. Because we may write $\mathcal{Q}$ as the countable union

$$\mathcal{Q} = \bigcup_{z \in \mathbb{Z}} \bigcup_{k \in \mathbb{N}} \{Q \in \mathcal{Q} : 2^z \leq \mathrm{l}(Q) < 2^{z+1},\ Q \subset B(0, 2^k)\}$$

it is enough to prove the result for each set in the union separately. Then after rescaling it suffices to consider the case that $\mathcal{Q}$ is a set of cubes Q with $2^{-1} \leq \mathrm{l}(Q) < 1$ which are all contained in a fixed ball B_0.

For each $n \in \mathbb{N}$ let δ_n be the δ for $\varepsilon = 2^{-n}$ from Lemma 3.10. We inductively define a finite sequence of cubes $(Q_k)_k$ as follows. For each k select a cube $Q_k \in \mathcal{Q}$ such that for all $i = 1, \ldots, k-1$ we have $|c_{Q_k} - c_{Q_i}| \geq \delta_n$ or $|v_{Q_k} - v_{Q_i}| \geq \delta_n$, if such a cube exists, otherwise stop. We furthermore select this cube Q_k so that it maximizes $\mathrm{l}(Q_k)$ up to a factor $(1+\delta_n)^{-1}$ among all cubes in $\mathcal{Q}$ eligible for selection. Since $\{(c_Q, v_Q) : Q \in \mathcal{Q}\}$ is a bounded set, this sequence terminates after a finite number K_n of steps, and we denote $\mathcal{Q}_n = \{Q_1, \ldots, Q_{K_n}\}$. Then for each $P \in \mathcal{Q}$ there is a $Q \in \mathcal{Q}_n$ such that $\mathrm{l}(P) \leq (1+\delta_n)\,\mathrm{l}(Q)$ and $|c_P - c_Q| \leq \delta_n\,\mathrm{l}(Q)$ and $|v_P - v_Q| \leq \delta_n$. By Lemma 3.10 this implies $P \subset (1+2^{-n})Q$ and thus

$$\bigcup\{\overline{Q} : Q \in \mathcal{Q}\} \subset \bigcup\{(1+2^{-n+1})Q : Q \in \mathcal{Q}_n\}. \tag{4.2}$$

Let $P \in \mathcal{Q}$ and $x \in P$. Then there is an $\varepsilon > 0$ with $B(x, \varepsilon) \subset P$. Take n with $2^{-n}\sqrt{d} < \varepsilon$ and $Q \in \mathcal{Q}_n$ with $P \subset (1+2^{-n})Q$. Then the Hausdorff distance between Q and $(1+2^{-n})Q$ is less than ε which implies $x \in Q$. We can conclude

$$\bigcup \mathcal{Q} = \bigcup(\mathcal{Q}_1 \cup \mathcal{Q}_2 \cup \ldots).$$

We can cover B_0 by a set of unit balls $B_1, \ldots, B_K$ with $K \lesssim \mathcal{L}(B_0)$. Since for all $Q \in \mathcal{Q}_n$ we have $\mathrm{l}(Q) > 1/2$, it follows from Lemma 3.5 that for every $1 \leq k \leq K$ we can write $\partial \bigcup \mathcal{Q}_n \cap B_k$

as a union of boundedly many Lipschitz surfaces with uniformly bounded constants. We apply Lemma 4.5 to each of these Lipschitz surfaces and obtain

$$\begin{aligned}\mathcal{L}\Big(\bigcup\{(1+2^{-n+1})Q : Q \in \mathcal{Q}_n\} \setminus \bigcup \mathcal{Q}_n\Big) &\leq \mathcal{L}\Big(\big\{x \in \mathbb{R}^d :\ \exists y \in \partial \bigcup \mathcal{Q}_n,\ |x-y| \leq 2^{-n+1}\sqrt{d}\big\}\Big)\\ &\lesssim 2^{-n}\mathcal{L}(B_0).\end{aligned}$$

Now we can conclude from (4.2) that

$$\mathcal{L}\Big(\bigcup\{\overline{Q} : Q \in \mathcal{Q}\} \setminus \bigcup \mathcal{Q}\Big) \leq \mathcal{L}\Big(\bigcup\{(1+2^{-n})Q : Q \in \mathcal{Q}_n\} \setminus \bigcup \mathcal{Q}_n\Big) \to 0$$

for $n \to \infty$. □

Proof of Proposition 2.8. By Proposition 4.4 we have for each $q \in \mathbb{Q}$ that the superlevelsets $\{\mathrm{M}_{\mathcal{Q}} f > q\} = \bigcup\{Q \in \mathcal{Q} : f_Q > q\}$ and $\{\overline{\mathrm{M}}_{\mathcal{Q}} f > q\} = \bigcup\{\overline{Q} \in \mathcal{Q} : f_Q > q\}$ only differ up to a set of measure zero. Therefore the set

$$\{\overline{\mathrm{M}}_{\mathcal{Q}} f > \mathrm{M}_{\mathcal{Q}} f\} = \bigcup_{q \in \mathbb{Q}} \{x \in \Omega : \mathrm{M}_{\mathcal{Q}} f(x) > q > \overline{\mathrm{M}}_{\mathcal{Q}} f(x)\}$$

has measure zero. Since $\overline{\mathrm{M}}_{\mathcal{Q}} f(x) \geq \mathrm{M}_{\mathcal{Q}} f(x)$ for every $x \in \Omega$, this finishes the proof. □

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