Calc. Var. (2024) 63:205
https://doi.org/10.1007/s00526-024-02813-6

Calculus of Variations

Weighted fractional Poincaré inequalities via isoperimetric inequalities

Kim Myyryläinen[1] · Carlos Pérez[2] · Julian Weigt[3]

Received: 11 October 2023 / Accepted: 25 July 2024

Abstract

Our main result is a weighted fractional Poincaré–Sobolev inequality improving the celebrated estimate by Bourgain–Brezis–Mironescu. This also yields an improvement of the classical Meyers–Ziemer theorem in several ways. The proof is based on a fractional isoperimetric inequality and is new even in the non-weighted setting. We also extend the celebrated Poincaré–Sobolev estimate with A_p weights of Fabes–Kenig–Serapioni by means of a fractional type result in the spirit of Bourgain–Brezis–Mironescu. Examples are given to show that the corresponding L^p-versions of weighted Poincaré inequalities do not hold for $p > 1$.

Mathematics Subject Classification 46E35 · 42B25

Communicated by Y. Li.

The research was supported by the Academy of Finland. K. Myyryläinen and J. Weigt have also been supported by the Magnus Ehrnrooth Foundation. C. Pérez is supported by grant PID2020-113156GB-I00, Spanish Government; by the Basque Government through grant IT1615-22 and the BERC 2014-2017 program and by the BCAM Severo Ochoa accreditation CEX2021-001142-S, Spanish Government. J. Weigt is also supported by the European Union's Horizon 2020 research and innovation programme (Grant agreement No. 948021). We are very grateful to the Department of Mathematics at Aalto University for its support where the initiation of this research took place. In particular, we are very grateful to Juha Kinnunen for his support and for the discussions we had.

✉ Kim Myyryläinen
kim.myyrylainen@aalto.fi

Carlos Pérez
cperez@bcamath.org

Julian Weigt
julian.weigt@warwick.ac.uk

1 Department of Mathematics, Aalto University, P.O. Box 11100, 00076 Aalto, Finland

2 Department of Mathematics, University of the Basque Country, IKERBASQUE (Basque Foundation for Science) and BCAM – Basque Center for Applied Mathematics, Bilbao, Spain

3 Mathematics Institute, Zeeman Building, University of Warwick, Coventry CV4 7AL, UK

Published online: 22 August 2024

1 Introduction

The classical (q, p)-Poincaré–Sobolev inequality states that

$$\left(\fint_Q |f - f_Q|^q \mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\fint_Q |\nabla f|^p \mathrm{d}x\right)^{\frac{1}{p}}, \tag{1}$$

where $1 \le p < n$, $q = \frac{np}{n-p}$, $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$, $Q \subset \mathbb{R}^n$ is a cube and C is a dimensional constant. In 2002, Bourgain et al. [3] proved the following fractional (q, p)-Poincaré inequality

$$\left(\fint_Q |f - f_Q|^q \mathrm{d}x\right)^{\frac{1}{q}} \le C(1-\delta)^{\frac{1}{p}}\, l(Q)^\delta \left(\fint_Q \int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}} \mathrm{d}y\mathrm{d}x\right)^{\frac{1}{p}}, \tag{2}$$

where $\frac{1}{2} \le \delta < 1$, $1 \le p < \frac{n}{\delta}$, $q = \frac{np}{n-\delta p}$, $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $Q \subset \mathbb{R}^n$ is a cube and C is a dimensional constant. Note the factor $(1-\delta)^{\frac{1}{p}}$ in front of the fractional term which balances the limiting behaviour of the right-hand side when $\delta \to 1$. In particular, it was shown by Brezis [4] that without the factor the right-hand side of (2) is infinite for non-constant functions when $\delta = 1$. Moreover, Bourgain et al. [2] showed that with this factor the fractional term coincides with the L^p norm of the gradient when $\delta \to 1$. This means that in the limit (2) turns into the classical Poincaré inequality (1). Later, Maz'ya and Shaposhnikova [19] proved the corresponding inequality in $\mathbb{R}^n$. They showed in $\mathbb{R}^n$ that the fractional term multiplied with $\delta^{\frac{1}{p}}$ coincides with the L^p norm of the function when $\delta \to 0$. For other limiting behaviour results, we refer to Alberico et al. [1], Brezis et al. [5], Drelichman and Durán [9] and Karadzhov et al. [17].

The existing proofs of the fractional Poincaré inequality apply Fourier analysis techniques [3], Hardy type inequalities [19] or compactness arguments [23]. We give a new direct and transparent proof using a relative isoperimetric inequality as our main tool. We concentrate on the case $p = 1$ in (2). Our approach is based on a new fractional type isoperimetric inequality in Lemma 3.3 which can be seen as an improvement of the classical relative isoperimetric inequality, see Remark 3.4. To our knowledge this approach with isoperimetric inequalities has not been considered in the fractional case before. This allows further investigation of the theory of fractional Poincaré inequalities.

It is known that the classical $(1, 1)$-Poincaré inequality implies the classical (q, p)-Poincaré inequality. We investigate this in the fractional setting with A_p weights. The strategy is to first show that the fractional $(1, 1)$-Poincaré inequality implies the fractional $(1, p)$-Poincaré inequality in Corollary 5.2. Then we apply a self-improving property and a fractional truncation method to obtain the fractional (q, p)-Poincaré inequality with A_p weights, see Theorems 5.7 and 5.9. This extends the fractional Poincaré inequality in Hurri-Syrjänen et al. [15] from A_1 weights to A_p weights. Self-improving results are discussed in Canto and Pérez [6], Franchi et al. [13], Lerner et al. [18] and Pérez and Rela [22]. For fractional truncation methods, see Chua [8], Dyda et al. [10, 11] and Maz'ya [20].

Our proof for the fractional Poincaré inequality also works when we measure the oscillation against a Radon measure μ. Our main result Theorem 4.1 states that

$$\left(\int_Q |f - f_Q|^q \mathrm{d}\mu\right)^{\frac{1}{q}} \le C(1-\delta) \int_Q \int_Q \frac{|f(x)-f(y)|}{|x-y|^{n+\delta}} \mathrm{d}y\, (\mathrm{M}_\alpha \mu(x))^{\frac{1}{q}} \mathrm{d}x$$

for $0 \le \delta < 1$, $1 \le q \le \frac{n}{n-\delta}$ and where $\mathrm{M}_\alpha \mu$ is the fractional maximal function with $\alpha = n - q(n-\delta)$. This extends [15, Theorem 2.10] to all values $0 \le \delta < 1$ and exponents $1 \le q \le \frac{n}{n-\delta}$. Weighted classical Poincaré inequalities have been studied extensively starting from the classical result by Meyers and Ziemer [21] and generalized to

$$\left(\int_Q |f - f_Q|^q \, \mathrm{d}\mu\right)^{\frac{1}{q}} \le C \int_Q |\nabla f| \, (\mathrm{M}_\alpha \mu)^{\frac{1}{q}} \, \mathrm{d}x \tag{3}$$

for $1 \le q \le \frac{n}{n-1}$, $\alpha = n - q(n-1)$ by Franchi, Pérez and Wheeden in [14]. With Theorem 4.1 we extend their results to the fractional setting and are also able to deduce their original results from ours, see Corollaries 4.3 and 6.5. Moreover, in [14] they show (3) in two separate ranges of q and their constant C blows up when $q \to 1$. In our argument, C is uniformly bounded in q and depends only on the dimension. We also give an alternative proof by applying the relative isoperimetric inequality to highlight the differences between the classical and the fractional Poincaré inequalities.

It would be a natural question to ask if the weighted fractional or classical Poincaré inequality holds for $p > 1$ as in (1) and (2). However, this is not the case. We construct counterexamples in Sect. 7. This answers a question regarding the weighted classical Poincaré inequality posed in [14].

2 Preliminaries

Let $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Unless otherwise stated, constants are positive and dependent only on the dimension n. We denote the standard Euclidean norm of a point $x \in \mathbb{R}^n$ by $|x|$. The Lebesgue measure of a measurable subset A of $\mathbb{R}^n$ is denoted by $\mathcal{L}(A)$ and the s-dimensional Hausdorff measure is denoted by $\mathcal{H}^s(A)$. The absolute continuity of a measure μ with respect to another measure ν is denoted by $\mu \ll \nu$, that is, $\nu(A) = 0$ implies $\mu(A) = 0$.

Assume that $A \subset \mathbb{R}^n$ is a measurable set with $0 < \mathcal{L}(A) < \infty$ and that $f : A \to [-\infty, \infty]$ is a measurable function. The maximal median of f over A is defined by

$$m_f(A) = \inf\left\{a \in \mathbb{R} : \mathcal{L}(\{x \in A : f(x) > a\}) < \frac{1}{2}\mathcal{L}(A)\right\}.$$

The integral average of $f \in L^1(A)$ on A is denoted by

$$f_A = \fint_A f \, \mathrm{d}x = \frac{1}{\mathcal{L}(A)} \int_A f \, \mathrm{d}x.$$

We write

$$\{f > \lambda\} = \{x \in \mathbb{R}^n : f(x) > \lambda\}$$

for the superlevel set of a function $f : \mathbb{R}^n \to \mathbb{R}$. We define $\{f < \lambda\}$ similarly.

A cube $Q \subset \mathbb{R}^n$ is the product of n closed intervals of the same length, with sides parallel to the coordinate axes and equally long, that is, $Q = [a_1, a_1 + l] \times \cdots \times [a_n, a_n + l]$. In particular, we always consider a cube to be closed and axes-parallel. All our results hold for half open cubes as well. If we additionally assume that the measures in our results are absolutely continuous with respect to the Lebesgue measure then we can also use open cubes. We denote by $\mathrm{l}(Q) = l$ the side length of Q.

Let $Q_0 \subset \mathbb{R}^n$ be a cube. For each $k \in \mathbb{N}_0$ we denote by $\mathcal{D}_k(Q_0)$ the set of dyadic subcubes of Q_0 of generation k. Particularly, $\mathcal{D}_k(Q_0)$ consists of 2^{kn} cubes Q with pairwise disjoint interiors and with side length $\mathrm{l}(Q) = 2^{-k}\,\mathrm{l}(Q_0)$, such that Q_0 equals the union of all cubes in $\mathcal{D}_k$ up to a set of measure zero. If $k \geq 1$ and $Q \in \mathcal{D}_k(Q_0)$, there exists a unique cube $Q' \in \mathcal{D}_{k-1}(Q_0)$ with $Q \subset Q'$. The cube Q' is called the dyadic parent of Q, and Q is a dyadic child of Q'. The set of dyadic subcubes $\mathcal{D}(Q_0)$ of Q_0 is defined as $\mathcal{D}(Q_0) = \bigcup_{k=0}^{\infty} \mathcal{D}_k(Q_0)$.

The following lemma is a variant of the classical Calderón–Zygmund decomposition for sets.

Lemma 2.1 *Let $Q \subset \mathbb{R}^n$ be a cube and $E \subset \mathbb{R}^n$ a measurable set. Assume that*

$$\mathcal{L}(Q \cap E) \leq \lambda \mathcal{L}(Q)$$

holds for some $0 < \lambda < 1$. Then there exist countably many pairwise disjoint dyadic cubes $Q_i \in \mathcal{D}(Q)$, $i \in \mathbb{N}$, such that

(i) *$Q \cap E \subset \bigcup_i Q_i$ up to a set of Lebesgue measure zero,*
(ii) *$\mathcal{L}(Q_i \cap E) > 2^{-n}\lambda \mathcal{L}(Q_i)$,*
(iii) *$\mathcal{L}(Q_i \cap E) \leq \lambda \mathcal{L}(Q_i)$.*

If E is relatively open with respect to Q then $Q \cap E \subset \bigcup_i Q_i$ holds literally and not only up to a set of measure zero. The cubes in the collection $\{Q_i\}_{i\in\mathbb{N}}$ are called the Calderón–Zygmund cubes in Q at level λ.

Proof If

$$\mathcal{L}(Q \cap E) > 2^{-n}\lambda \mathcal{L}(Q),$$

we pick Q and observe that Q satisfies the required properties. Otherwise, if

$$\mathcal{L}(Q \cap E) \leq 2^{-n}\lambda \mathcal{L}(Q),$$

we decompose Q into dyadic subcubes that satisfy the required properties in the following way. Start by decomposing Q into 2^n dyadic subcubes $Q_1 \in \mathcal{D}_1(Q)$. We select those Q_1 for which $\mathcal{L}(Q_1 \cap E) > 2^{-n}\lambda \mathcal{L}(Q_1)$ and denote this collection by $\{Q_{1,j}\}_j$. If $\mathcal{L}(Q_1 \cap E) \leq 2^{-n}\lambda \mathcal{L}(Q_1)$, we subdivide Q_1 into 2^n dyadic subcubes $Q_2 \in \mathcal{D}_2(Q)$ and select Q_2 for which $\mathcal{L}(Q_2 \cap E) > 2^{-n}\lambda \mathcal{L}(Q_2)$. We denote so obtained collection by $\{Q_{2,j}\}_j$.

At the ith step, we partition unselected Q_{i-1} into dyadic subcubes $Q_i \in \mathcal{D}_i(Q)$ and select those Q_i for which $\mathcal{L}(Q_i \cap E) > 2^{-n}\lambda \mathcal{L}(Q_i)$. Denote the obtained collection by $\{Q_{i,j}\}_j$. If $\mathcal{L}(Q_i \cap E) \leq 2^{-n}\lambda \mathcal{L}(Q_i)$, we continue the selection process in Q_i. In this manner we obtain a collection $\{Q_{i,j}\}_{i,j}$ of pairwise disjoint dyadic subcubes of Q. Reindex $\{Q_i\}_i = \{Q_{i,j}\}_{i,j}$. We show that $\{Q_i\}_i$ satisfies the required properties.

Let $x \in Q \setminus \bigcup_i Q_i$. There exists a decreasing sequence $\{Q_k\}_k$ of dyadic subcubes of Q containing x such that $Q_{k+1} \subsetneq Q_k$ and $\mathcal{L}(Q_k \cap E) \leq 2^{-n}\lambda \mathcal{L}(Q_k)$ for every $k \in \mathbb{N}$. If E is relatively open then for k large enough we have $Q_k \subset E \cap Q$, a contradiction. If E is a general measurable set, then we have by the Lebesgue differentiation theorem that $1_E(x) \leq 2^{-n}\lambda$ for almost every $x \in Q \setminus \bigcup_i Q_i$ and thus $Q \cap E \subset \bigcup_i Q_i$ up to a set of Lebesgue measure zero. This proves (i). Property (ii) holds by the definition of Q_i. By the selection process, it holds that $\mathcal{L}(Q_i' \cap E) \leq 2^{-n}\lambda \mathcal{L}(Q_i')$ for every $i \in \mathbb{N}$, where Q_i' is the dyadic parent cube of Q_i. Hence, we have

$$\mathcal{L}(Q_i \cap E) \leq \mathcal{L}(Q_i' \cap E) \leq 2^{-n}\lambda \mathcal{L}(Q_i') = \lambda \mathcal{L}(Q_i).$$

This proves (iii). □

Let μ be a Radon measure. The fractional maximal function of μ is defined by

$$\mathrm{M}_\alpha \mu(x) = \sup_{Q \ni x} \mathrm{l}(Q)^\alpha \frac{\mu(Q)}{\mathcal{L}(Q)}.$$

For $\alpha = 0$, we have the classical Hardy–Littlewood maximal function $\mathrm{M} = \mathrm{M}_0$. Let $Q_0 \subset \mathbb{R}^n$. The dyadic local counterpart is defined by

$$\mathrm{M}^{\mathrm{d}}_{\alpha, Q_0} \mu(x) = \sup_{\substack{Q \ni x, \\ Q \in \mathcal{D}(Q_0)}} \mathrm{l}(Q)^\alpha \frac{\mu(Q)}{\mathcal{L}(Q)},$$

where we take the supremum only over the dyadic subcubes of Q_0.

For a measurable set $E \subset \mathbb{R}^n$ denote by $\mathring{E}$, $\overline{E}$ and ∂E the topological interior, closure and boundary of E, respectively. The measure theoretic closure and the measure theoretic boundary of E are defined by

$$\overline{E}^* = \left\{ x : \limsup_{r \to 0} \frac{\mathcal{L}(B(x,r) \cap E)}{r^n} > 0 \right\} \quad \text{and} \quad \partial_* E = \overline{E}^* \cap \overline{\mathbb{R}^n \setminus E}^*.$$

The measure theoretic versions are robust against changes with measure zero. Note that $\overline{E}^* \subset \overline{E}$ and thus $\partial_* E \subset \partial E$. For a cube, its measure theoretic boundary and its closure agree with the respective topological quantities.

We will need the following relative isoperimetric inequality [12, Theorem 5.11].

Lemma 2.2 *Let $Q \subset \mathbb{R}^n$ be a cube and E a set of finite perimeter. Then there exists a dimensional constant C such that*

$$\min\{\mathcal{L}(Q \cap E), \mathcal{L}(Q \setminus E)\}^{\frac{n-1}{n}} \le C \mathcal{H}^{n-1}(Q \cap \partial_* E).$$

3 Fractional type isoperimetric inequality

This section discusses a rougher fractional type isoperimetric inequality, Lemma 3.3, which is used later to prove the weighted fractional Poincaré inequality. To prove this fractional isoperimetric inequality, we need first some auxiliary results.

Lemma 3.1 *Let $Q_0 \subset \mathbb{R}^n$ be a cube, $a \le \mathrm{l}(Q_0)/2$ and $0 < \varepsilon < \frac{1}{2}$. Let $Q \subset Q_0$ be a cube with $\mathrm{l}(Q) \le a \frac{\sqrt{\pi}}{2^{n+4} n}$. Then for any measurable set $E \subset \mathbb{R}^n$ with*

$$\varepsilon \le \frac{\mathcal{L}(Q \cap E)}{\mathcal{L}(Q)} \le 1 - \varepsilon$$

we have

$$\mathcal{L}(Q) \le \frac{4}{\varepsilon} \int_Q \left| 1_E(x) - \frac{\mathcal{L}(A(x) \cap E)}{\mathcal{L}(A(x))} \right| \mathrm{d}x,$$

where $A(x) = Q_0 \cap B(x,a) \setminus B(x, a/2)$.

Proof Denote the center of Q by x_0. Let $x \in Q$. Then we have

$$|x - x_0| \le \frac{\sqrt{n}}{2} \mathrm{l}(Q) \le a \frac{\sqrt{\pi}}{2^{n+5} \sqrt{n}}. \tag{4}$$

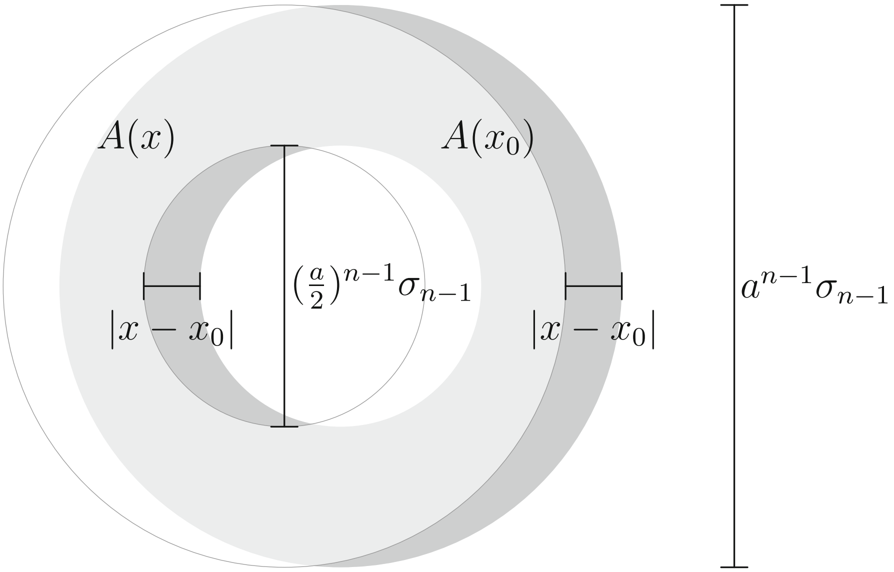

Fig. 1 Difference of two shifted annuli

Our first step is to show that (4) implies

$$\left| \frac{\mathcal{L}(A(x) \cap E)}{\mathcal{L}(A(x))} - \frac{\mathcal{L}(A(x_0) \cap E)}{\mathcal{L}(A(x_0))} \right| \leq \frac{1}{4}. \tag{5}$$

Denote by σ_n the n-dimensional Lebesgue measure of the unit ball in n dimensions. Then

$$\begin{aligned}
\left|\mathcal{L}(A(x) \cap E) - \mathcal{L}(A(x_0) \cap E)\right| &= \left|\mathcal{L}(A(x) \cap E \setminus A(x_0)) - \mathcal{L}(A(x_0) \cap E \setminus A(x))\right| \\
&\leq \max\left\{\mathcal{L}(A(x_0) \setminus A(x)), \mathcal{L}(A(x) \setminus A(x_0))\right\} \\
&\leq (a^{n-1}\sigma_{n-1} + (a/2)^{n-1}\sigma_{n-1})|x - x_0| \\
&= (1 + 2^{-n+1})a^{n-1}\sigma_{n-1}|x - x_0|,
\end{aligned}$$

where the second inequality follows from the fact that we can estimate the difference of shifted annuli by two differences of shifted balls as illustrated in Fig. 1. This implies

$$\begin{aligned}
&\left|\mathcal{L}(A(x) \cap E)\mathcal{L}(A(x_0)) - \mathcal{L}(A(x_0) \cap E)\mathcal{L}(A(x))\right| \\
&\quad \leq \left|\mathcal{L}(A(x) \cap E)\mathcal{L}(A(x_0)) - \mathcal{L}(A(x_0) \cap E)\mathcal{L}(A(x_0))\right| \\
&\qquad + \left|\mathcal{L}(A(x_0) \cap E)\mathcal{L}(A(x_0)) - \mathcal{L}(A(x_0) \cap E)\mathcal{L}(A(x))\right| \\
&\quad = \left|\mathcal{L}(A(x) \cap E) - \mathcal{L}(A(x_0) \cap E)\right|\mathcal{L}(A(x_0)) \\
&\qquad + \left|\mathcal{L}(A(x_0)) - \mathcal{L}(A(x))\right|\mathcal{L}(A(x_0) \cap E) \\
&\quad \leq 2(1 + 2^{-n+1})a^{n-1}\sigma_{n-1}|x - x_0|\mathcal{L}(A(x_0)).
\end{aligned} \tag{6}$$

By the formula $\sigma_n = \pi^{\frac{n}{2}} / \Gamma(\frac{n}{2} + 1)$ and [25], we have

$$\frac{\sigma_{n-1}}{\sigma_n} \leq \sqrt{\frac{n+1}{2\pi}}.$$

The inequality

$$\frac{(1+2^{-n+1})\sqrt{n+1}}{(1-2^{-n})\sqrt{n}} \leq 4\sqrt{2}$$

clearly holds for $n = 1$, and thus for all $n \in \mathbb{N}$, as the left-hand side is decreasing in n. Combining the two previous inequalities with (4) and

$$\mathcal{L}(A(x)) \geq \frac{1-2^{-n}}{2^n}\sigma_n a^n, \tag{7}$$

we obtain

$$(1+2^{-n+1})a^{n-1}\sigma_{n-1}|x-x_0| \leq \frac{1}{8}\mathcal{L}(A(x)).$$

Thus, (6) implies

$$\left|\mathcal{L}(A(x)\cap E)\mathcal{L}(A(x_0)) - \mathcal{L}(A(x_0)\cap E)\mathcal{L}(A(x))\right| \leq \frac{1}{4}\mathcal{L}(A(x))\mathcal{L}(A(x_0)).$$

Dividing the previous inequality by $\mathcal{L}(A(x))\mathcal{L}(A(x_0))$, we conclude (5).

If

$$\frac{\mathcal{L}(A(x_0)\cap E)}{\mathcal{L}(A(x_0))} \geq \frac{1}{2},$$

then it holds that

$$\frac{\mathcal{L}(A(x)\cap E)}{\mathcal{L}(A(x))} \geq \frac{\mathcal{L}(A(x_0)\cap E)}{\mathcal{L}(A(x_0))} - \left|\frac{\mathcal{L}(A(x)\cap E)}{\mathcal{L}(A(x))} - \frac{\mathcal{L}(A(x_0)\cap E)}{\mathcal{L}(A(x_0))}\right| \geq \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

On the other hand, if

$$\frac{\mathcal{L}(A(x_0)\cap E)}{\mathcal{L}(A(x_0))} < \frac{1}{2},$$

then

$$\begin{aligned}\left|1 - \frac{\mathcal{L}(A(x)\cap E)}{\mathcal{L}(A(x))}\right| &\geq 1 - \frac{\mathcal{L}(A(x_0)\cap E)}{\mathcal{L}(A(x_0))} - \left|\frac{\mathcal{L}(A(x)\cap E)}{\mathcal{L}(A(x))} - \frac{\mathcal{L}(A(x_0)\cap E)}{\mathcal{L}(A(x_0))}\right| \\ &\geq 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.\end{aligned}$$

As a consequence there is $i \in \{0, 1\}$ such that

$$\left|i - \frac{\mathcal{L}(A(x)\cap E)}{\mathcal{L}(A(x))}\right| \geq \frac{1}{4}$$

for every $x \in Q$. Denote $F = E$ if $i = 1$ and $F = \mathbb{R}^n \setminus E$ if $i = 0$. By the assumption, we have

$$\mathcal{L}(Q \cap F) \geq \varepsilon \mathcal{L}(Q).$$

Then we can conclude that

$$\mathcal{L}(Q) \leq \frac{1}{\varepsilon}\int_{Q\cap F} 1\,\mathrm{d}x \leq \frac{4}{\varepsilon}\int_{Q\cap F}\left|i - \frac{\mathcal{L}(A(x)\cap E)}{\mathcal{L}(A(x))}\right|\mathrm{d}x \leq \frac{4}{\varepsilon}\int_Q \left|1_E(x) - \frac{\mathcal{L}(A(x)\cap E)}{\mathcal{L}(A(x))}\right|\mathrm{d}x.$$

The proof is complete. □

The next lemma is a decomposition of a set near its boundary into cubes.

Lemma 3.2 *Let $Q_0 \subset \mathbb{R}^n$ be a cube and $E \subset \mathbb{R}^n$ a measurable set such that*

$$\frac{1}{2^{n+1}} \le \frac{\mathcal{L}(Q_0 \cap E)}{\mathcal{L}(Q_0)} \le \frac{1}{2}.$$

Let $k \in \mathbb{N}_0$. Then there exist a dimensional constant C and pairwise disjoint cubes $Q_1, \dots, Q_N \subset Q_0$ such that $\mathrm{l}(Q_i) = 2^{-k}\,\mathrm{l}(Q_0)$ and

$$\frac{1}{2^{n+2}} \le \frac{\mathcal{L}(Q_i \cap E)}{\mathcal{L}(Q_i)} \le \frac{3}{4}$$

for every $i = 1, \dots, N$ and

$$2^{-k}\mathcal{L}(Q_0) \le C \sum_{i=1}^{N} \mathcal{L}(Q_i).$$

Proof Recall that $\mathcal{D}_k(Q_0)$ is the set of dyadic subcubes of Q_0 of generation k. In particular, $\mathcal{D}_k(Q_0)$ consists of 2^{nk} many pairwise disjoint cubes with side length $2^{-k}\,\mathrm{l}(Q_0)$ which decompose Q_0. Denote by $\mathcal{Q}$ the collection of those dyadic subcubes $Q \in \mathcal{D}_k(Q_0)$ with

$$\mathcal{L}(Q \cap E) \ge \frac{1}{2^{n+2}} \mathcal{L}(Q)$$

and let $A = \bigcup_{Q \in \mathcal{Q}} Q$. We have

$$\mathcal{L}(Q_0 \cap E \setminus A) = \sum_{Q \in \mathcal{D}_k(Q_0) \setminus \mathcal{Q}} \mathcal{L}(Q \cap E) \le \frac{1}{2^{n+2}} \sum_{Q \in \mathcal{D}_k(Q_0) \setminus \mathcal{Q}} \mathcal{L}(Q) \le \frac{1}{2^{n+2}} \mathcal{L}(Q_0),$$

and thus

$$\begin{aligned} \mathcal{L}(A) \ge \mathcal{L}(A \cap E) &= \mathcal{L}(Q_0 \cap E) - \mathcal{L}(Q_0 \cap E \setminus A \cap E) \\ &\ge \frac{1}{2^{n+1}} \mathcal{L}(Q_0) - \frac{1}{2^{n+2}} \mathcal{L}(Q_0) = \frac{1}{2^{n+1}} \mathcal{L}(Q_0). \end{aligned} \tag{8}$$

Denote

$$\mathcal{A} = \left\{ Q \in \mathcal{Q} : \mathcal{H}^{n-1}(\mathring{Q}_0 \cap \partial_* A \cap \partial Q) \ge \mathrm{l}(Q)^{n-1} \right\},$$

where $\mathring{Q}_0$ is the interior of Q_0, so $\mathcal{A}$ is the set of those cubes in $\mathcal{Q}$ that have at least one of their faces contained in $\mathring{Q}_0 \cap \partial_* A$. Note that $\mathring{Q}_0 \cap \partial_* A \subset \bigcup_{Q \in \mathcal{A}} \partial Q$. For every cube $Q \in \mathcal{A}$, there exists a neighbouring dyadic cube $P \in \mathcal{D}_k(Q_0) \setminus \mathcal{Q}$. Thus, the cube $\widetilde{Q}_\lambda = (1 - \lambda) Q + \lambda P$ with side length $2^{-k}\,\mathrm{l}(Q_0)$ is contained in $Q \cup P$ for every $0 \le \lambda \le 1$. By the definition of $\mathcal{Q}$ and $\mathcal{D}_k(Q_0) \setminus \mathcal{Q}$, there exists $0 \le \lambda \le 1$ such that

$$\mathcal{L}(\widetilde{Q}_\lambda \cap E) = \frac{1}{2^{n+2}} \mathcal{L}(\widetilde{Q}_\lambda).$$

We denote $\widetilde{Q} = \widetilde{Q}_\lambda$ for this λ. The collection $\{\widetilde{Q} : Q \in \mathcal{A}\}$ is not necessarily disjoint. Observe that every cube in $\mathcal{D}_k$ has $2n$ faces and thus at most $2n$ neighbouring cubes in $\mathcal{D}_k$. Hence, for every $x \in Q_0$ there are at most $2n$ many cubes $\widetilde{Q}$ with $x \in \widetilde{Q}$. Let $|\mathcal{A}|$ denote the number of cubes in $\mathcal{A}$. Thus, we may extract a maximal disjoint subcollection $\widetilde{\mathcal{A}} \subset \{\widetilde{Q} : Q \in \mathcal{A}\}$ such that $|\mathcal{A}| \le 2n|\widetilde{\mathcal{A}}|$.

If $\mathcal{L}(A) \leq \frac{3}{4}\mathcal{L}(Q_0)$, then by (8) and Lemma 2.2, we have

$$\begin{aligned}
\left(\frac{\mathcal{L}(Q_0)}{2^{n+1}}\right)^{\frac{n-1}{n}} &= \min\left\{\frac{\mathcal{L}(Q_0)}{2^{n+1}}, \frac{\mathcal{L}(Q_0)}{4}\right\}^{\frac{n-1}{n}} \\
&\leq \min\{\mathcal{L}(Q_0 \cap A), \mathcal{L}(Q_0 \setminus A)\}^{\frac{n-1}{n}} \\
&\leq C_1 \mathcal{H}^{n-1}(Q_0 \cap \partial_* A) \leq C_1 \sum_{Q\in\mathcal{A}} \mathcal{H}^{n-1}(\partial Q) \\
&= C_1 |\mathcal{A}| 2^{-k(n-1)} \mathcal{H}^{n-1}(\partial Q_0) \\
&\leq C_1 4n^2 |\widetilde{\mathcal{A}}| 2^{-k(n-1)} \, \mathrm{l}(Q_0)^{n-1} \\
&= \frac{C_1 4n^2 2^k}{\mathrm{l}(Q_0)} \sum_{Q\in\widetilde{\mathcal{A}}} \mathcal{L}(Q),
\end{aligned}$$

where C_1 is the constant in Lemma 2.2. Thus, it holds that

$$2^{-k}\mathcal{L}(Q_0) \leq C \sum_{Q\in\widetilde{\mathcal{A}}} \mathcal{L}(Q),$$

where $C = 2^{n+2}n^2C_1$. Hence, the cubes $\{Q_1, \ldots, Q_N\} = \widetilde{A}$ satisfy the conclusion of the lemma.

It remains to consider the case $\mathcal{L}(A) > \frac{3}{4}\mathcal{L}(Q_0)$. We define

$$\{Q_1, \ldots, Q_N\} = \left\{Q \in \mathcal{Q} : \mathcal{L}(Q \cap E) \leq \frac{3}{4}\mathcal{L}(Q)\right\} = \left\{Q \in \mathcal{D}_k : \frac{1}{2^{n+2}} \leq \frac{\mathcal{L}(Q \cap E)}{\mathcal{L}(Q)} \leq \frac{3}{4}\right\}.$$

Then we have

$$\sum_{Q\in\mathcal{Q}\setminus\{Q_1,\ldots,Q_N\}} \mathcal{L}(Q) \leq \frac{4}{3} \sum_{Q\in\mathcal{Q}\setminus\{Q_1,\ldots,Q_N\}} \mathcal{L}(Q \cap E) \leq \frac{4}{3}\mathcal{L}(Q_0 \cap E) \leq \frac{2}{3}\mathcal{L}(Q_0).$$

We conclude that

$$\begin{aligned}
2^{-k}\mathcal{L}(Q_0) \leq \mathcal{L}(Q_0) &= 12\left(\frac{3}{4} - \frac{2}{3}\right)\mathcal{L}(Q_0) \\
&\leq 12 \sum_{Q\in\mathcal{Q}} \mathcal{L}(Q) - 12 \sum_{Q\in\mathcal{Q}\setminus\{Q_1,\ldots,Q_N\}} \mathcal{L}(Q) \\
&= 12 \sum_{i=1}^{N} \mathcal{L}(Q_i).
\end{aligned}$$

This completes the proof. □

We are ready to prove the following rougher version of the isoperimetric inequality. Observe that the difference compared to the relative isoperimetric inequality (Lemma 2.2) is that the right-hand side in Lemma 3.3 measures the area around the boundary by annuli of certain size.

Lemma 3.3 *Let $Q \subset \mathbb{R}^n$ be a cube, $E \subset \mathbb{R}^n$ a measurable set, $k \in \mathbb{N}$ and $s \geq 0$ such that*

$$\frac{1}{2^{(k+s)n}} \leq \frac{\mathcal{L}(Q \cap E)}{\mathcal{L}(Q)} \leq \frac{1}{2}.$$

Then there exists a dimensional constant C such that

$$\left(\frac{\mathcal{L}(Q\cap E)}{\mathcal{L}(Q)}\right)^{\frac{n-1}{n}} \le C2^{k+s} \fint_Q \fint_{Q\cap B(x,2^{-k}\,\mathrm{l}(Q))\setminus B(x,2^{-k-1}\,\mathrm{l}(Q))} |1_E(x)-1_E(y)|\,\mathrm{d}y\mathrm{d}x.$$

Proof Both sides of the claim are invariant under the dilation of Q and E by the same factor. Hence, it suffices to consider the case $\mathrm{l}(Q)=1$.

By the assumption of the lemma, we may apply Lemma 2.1 for E on Q at level $\frac{1}{2}$. Thus, we obtain a collection $\{Q_i\}_i$ of Calderón–Zygmund cubes such that $Q\cap E \subset \bigcup_i Q_i$ up to a set of Lebesgue measure zero and

$$\frac{1}{2^{n+1}} < \frac{\mathcal{L}(Q_i\cap E)}{\mathcal{L}(Q_i)} \le \frac{1}{2}$$

for every $i\in\mathbb{N}$. Note that $\mathrm{l}(Q_i)=2^{-M_i}$ for some $M_i\in\mathbb{N}_0$. Denote by $K\in\mathbb{N}$ the smallest integer with $2^K \ge \frac{2^{n+4}n}{\sqrt{\pi}}$. We apply Lemma 3.2 with $\max\{k+K-M_i,0\}$ for E on each Q_i. Then for every $i\in\mathbb{N}$ we obtain a collection $\{Q_{i,1},\dots,Q_{i,N_i}\}$ of pairwise disjoint subcubes with

$$\mathrm{l}(Q_{i,j}) = 2^{-\max\{k+K-M_i,0\}}\,\mathrm{l}(Q_i) = \min\{2^{-k-K},\mathrm{l}(Q_i)\}$$

such that

$$\frac{1}{2^{n+2}} \le \frac{\mathcal{L}(Q_{i,j}\cap E)}{\mathcal{L}(Q_{i,j})} \le \frac{3}{4}$$

for every $j=1,\dots,N_i$ and

$$\min\left\{2^{-k-K}\mathcal{L}(Q_i)^{\frac{n-1}{n}},\mathcal{L}(Q_i)\right\} = 2^{-\max\{k+K-M_i,0\}}\mathcal{L}(Q_i) \le C_1\sum_{j=1}^{N_i}\mathcal{L}(Q_{i,j}),$$

where C_1 is the constant in Lemma 3.2. By the properties of Q_i, the assumption $2^{-k-c} \le \mathcal{L}(Q\cap E)^{\frac{1}{n}}$ and the previous inequality, we get

$$\begin{aligned}
\mathcal{L}(Q\cap E)^{\frac{n-1}{n}} &\le \mathcal{L}(Q\cap E)^{-\frac{1}{n}}\sum_i \mathcal{L}(Q_i) \\
&= \min\left\{\mathcal{L}(Q\cap E)^{-\frac{1}{n}},2^{k+s}\right\}\sum_i\mathcal{L}(Q_i) \\
&\le \sum_i \min\left\{\mathcal{L}(Q_i\cap E)^{-\frac{1}{n}},2^{k+s}\right\}\mathcal{L}(Q_i) \\
&\le \sum_i \min\left\{2^{1+\frac{1}{n}}\mathcal{L}(Q_i)^{-\frac{1}{n}},2^{k+s}\right\}\mathcal{L}(Q_i) \\
&= 2^{k+K}\sum_i \min\left\{2^{1+\frac{1}{n}-k-K}\mathcal{L}(Q_i)^{\frac{n-1}{n}},2^{s-K}\mathcal{L}(Q_i)\right\} \\
&\le 2^{k+s+K+1+\frac{1}{n}}\sum_i \min\left\{2^{-k-K}\mathcal{L}(Q_i)^{\frac{n-1}{n}},\mathcal{L}(Q_i)\right\} \\
&\le 2^{k+s+K+1+\frac{1}{n}}C_1\sum_{i,j}\mathcal{L}(Q_{i,j}).
\end{aligned}$$

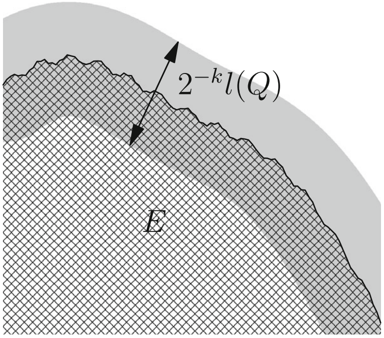

Fig. 2 For a very regular set the inner integral in Lemma 3.3 behaves approximately like the characteristic function of a neighborhood of the boundary

Using Lemma 3.1 with $\varepsilon = 1/2^{n+2}$, we obtain

$$\mathcal{L}(Q_{i,j}) \le 2^{n+4} \int_{Q_{i,j}} \left| 1_E(x) - \frac{\mathcal{L}(A(x) \cap E)}{\mathcal{L}(A(x))} \right| \mathrm{d}x$$

for every $i, j \in \mathbb{N}$, where $A(x) = Q \cap B(x, 2^{-k}) \setminus B(x, 2^{-k-1})$. Thus, we may estimate

$$\begin{aligned}\sum_{i,j} \mathcal{L}(Q_{i,j}) &\le 2^{n+4} \sum_{i,j} \int_{Q_{i,j}} \left| 1_E(x) - \frac{\mathcal{L}(A(x) \cap E)}{\mathcal{L}(A(x))} \right| \mathrm{d}x \\ &= 2^{n+4} \sum_{i,j} \int_{Q_{i,j}} \left| 1_E(x) - \fint_{A(x)} 1_E(y) \mathrm{d}y \right| \mathrm{d}x \\ &\le 2^{n+4} \sum_{i,j} \int_{Q_{i,j}} \fint_{A(x)} |1_E(x) - 1_E(y)| \mathrm{d}y \mathrm{d}x \\ &\le 2^{n+4} \int_Q \fint_{A(x)} |1_E(x) - 1_E(y)| \mathrm{d}y \mathrm{d}x.\end{aligned}$$

Combining the obtained estimates, we conclude that

$$\mathcal{L}(Q \cap E)^{\frac{n-1}{n}} \le C 2^{k+s} \int_Q \fint_{Q \cap B(x,2^{-k}) \setminus B(x,2^{-k-1})} |1_E(x) - 1_E(y)| \mathrm{d}y \mathrm{d}x,$$

where $C = \frac{n}{\sqrt{\pi}} 2^{2n+11} C_1$. This completes the proof. □

Remark 3.4 As mentioned in the introduction, the fractional Poincaré inequality (2) is an improvement of the classical Poincaré inequality (1) in the sense that the fractional integral of f on right hand side of (2) can be bounded by the integral of the gradient on the right hand side of (1). Likewise, Lemma 3.3 is an improvement of the relative isoperimetric inequality (Lemma 2.2), as

$$2^k \fint_Q \fint_{Q \cap B(x, 2^{-k} \mathrm{l}(Q)) \setminus B(x, 2^{-k-1} \mathrm{l}(Q))} |1_E(x) - 1_E(y)| \mathrm{d}y \mathrm{d}x \le C \frac{\mathcal{H}^{n-1}(Q \cap \partial_* E)}{\mathcal{L}(Q)^{\frac{n-1}{n}}} \tag{9}$$

for some dimensional constant C. Here we do not need to assume any bound on $\mathcal{L}(Q \cap E)/\mathcal{L}(Q)$. Note that Lemma 3.3 still holds if we integrate over $Q \cap B(x, 2^{-k} \mathrm{l}(Q))$ instead of $Q \cap B(x, 2^{-k} \mathrm{l}(Q)) \setminus B(x, 2^{-k-1} \mathrm{l}(Q))$, and so does (9).

Define the following averaged out version of the inner integral by

$$f(z) = \frac{1}{\mathcal{L}(B(0, 2^{-k}\,\mathrm{l}(Q)))} \int_{Q \cap B(z, 2^{-k}\,\mathrm{l}(Q))} \fint_{Q \cap B(x, 2^{-k}\,\mathrm{l}(Q)) \setminus B(x, 2^{-k-1}\,\mathrm{l}(Q))} |1_E(x) - 1_E(y)| \mathrm{d}y \mathrm{d}x.$$

Then

$$\int_{\mathbb{R}^n} f(z) \mathrm{d}z = \int_Q \fint_{Q \cap B(x, 2^{-k}\,\mathrm{l}(Q)) \setminus B(x, 2^{-k-1}\,\mathrm{l}(Q))} |1_E(x) - 1_E(y)| \mathrm{d}y \mathrm{d}x.$$

If the boundary of E is very regular then f behaves roughly like the characteristic function of the $2^{-k}\,\mathrm{l}(Q)$-neighborhood of $Q \cap \partial_* E$ as shown in Fig. 2, and its integral evaluates to approximately $2^{-k}\,\mathrm{l}(Q)\mathcal{H}^{n-1}(Q \cap \partial_* E)$. This means the two sides in (9) are comparable and Lemma 3.3 becomes the classical relative isoperimetric inequality. If the boundary of E is rougher then f instead resembles the characteristic function of the neighborhood of a straightened out boundary of E. Morally this means that (9) represents the removal of small wiggles in the boundary of E and Lemma 3.3 holds by the relative isoperimetric inequality since we can replace E on the left hand side by a straightened out set with similar volume.

For a formal proof of (9) define

$$E_i = \{z \in \mathbb{R}^n : f(z) \geq 2^{-i}\} \setminus \bigcup_{z \in E_1 \cup \ldots \cup E_{i-1}} B(z, 2^{-k+2}\,\mathrm{l}(Q))$$

recursively for all $i \in \mathbb{N}$. By the Vitali covering theorem, for each i there exists a collection $\mathcal{B}_i$ of pairwise disjoint balls $B(z, 2^{-k+1}\,\mathrm{l}(Q))$ with $z \in E_i$ such that

$$\bigcup_{z \in E_i} B(z, 2^{-k+2}\,\mathrm{l}(Q)) \subset \bigcup_{B \in \mathcal{B}_i} 8B.$$

Furthermore, by the definition of E_j for any $j > i$ the balls in $\mathcal{B}_j$ do not intersect the balls in $\mathcal{B}_i$. For any $z \in E_i$ we have

$$\begin{aligned}
2^{-i} \leq f(z) &\leq \frac{2^{n+1}}{\mathcal{L}(B(0, 2^{-k}\,\mathrm{l}(Q)))^2} \int_{Q \cap B(z, 2^{-k+1}\,\mathrm{l}(Q))} \int_{Q \cap B(z, 2^{-k+1}\,\mathrm{l}(Q))} |1_E(x) - 1_E(y)| \mathrm{d}y \mathrm{d}x \\
&= \frac{2^{n+2}}{\mathcal{L}(B(0, 2^{-k}\,\mathrm{l}(Q)))^2} \mathcal{L}(Q \cap B(z, 2^{-k+1}\,\mathrm{l}(Q)) \cap E) \mathcal{L}(Q \cap B(z, 2^{-k+1}\,\mathrm{l}(Q)) \setminus E) \\
&\leq \frac{2^{3n+2} \min\{\mathcal{L}(Q \cap B(z, 2^{-k+1}\,\mathrm{l}(Q)) \cap E), \mathcal{L}(Q \cap B(z, 2^{-k+1}\,\mathrm{l}(Q)) \setminus E)\}}{\mathcal{L}(B(0, 2^{-k+1}\,\mathrm{l}(Q)))},
\end{aligned}$$

and thus by the relative isoperimetric inequality (Lemma 2.2) for $Q \cap B(z, 2^{-k}\,\mathrm{l}(Q))$ there exists a constant C_1 such that

$$\begin{aligned}
&2^{-i} \mathcal{L}(B(z, 2^{-k+1}\,\mathrm{l}(Q))) \\
&\quad \leq 2^{3n+2} \min\{\mathcal{L}(Q \cap B(z, 2^{-k+1}\,\mathrm{l}(Q)) \cap E), \mathcal{L}(Q \cap B(z, 2^{-k+1}\,\mathrm{l}(Q)) \setminus E)\} \\
&\quad \leq 2^{3n+3} \sigma_n^{\frac{1}{n}} 2^{-k}\,\mathrm{l}(Q) \min\{\mathcal{L}(Q \cap B(z, 2^{-k+1}\,\mathrm{l}(Q)) \cap E), \mathcal{L}(Q \cap B(z, 2^{-k+1}\,\mathrm{l}(Q)) \setminus E)\}^{\frac{n-1}{n}} \\
&\quad \leq 2^{3n+3} \sigma_n^{\frac{1}{n}} C_1 2^{-k}\,\mathrm{l}(Q) \mathcal{H}^{n-1}(Q \cap B(z, 2^{-k+1}\,\mathrm{l}(Q)) \cap \partial_* E).
\end{aligned}$$

We can conclude that

$$\int_{\mathbb{R}^n} f(z) \mathrm{d}z \leq \sum_{i=1}^{\infty} 2^{-i+1} \mathcal{L}\Big(\bigcup_{z \in E_i} B(z, 2^{-k+2}\,\mathrm{l}(Q))\Big)$$

$$\leq 8^n \sum_{i=1}^{\infty} 2^{-i+1} \sum_{B \in \mathcal{B}_i} \mathcal{L}(B)$$
$$\leq 2^{6n+4} \sigma_n^{\frac{1}{n}} C_1 2^{-k} \,\mathrm{l}(Q) \sum_{i=1}^{\infty} \sum_{B \in \mathcal{B}_i} \mathcal{H}^{n-1}(Q \cap B(z, 2^{-k+1}\,\mathrm{l}(Q)) \cap \partial_* E)$$
$$\leq 2^{6n+4} \sigma_n^{\frac{1}{n}} C_1 2^{-k} \,\mathrm{l}(Q) \mathcal{H}^{n-1}(Q \cap \partial_* E),$$

finishing the proof of (9).

4 Weighted fractional $(q, 1)$-Poincaré inequality

In this section, we prove our main result Theorem 4.1, the weighted fractional Poincaré inequality in the case $p = 1$. This improves Theorem 2.10 in [15]. Observe that by choosing $\mu = \mathcal{L}$, we obtain the non-weighted fractional Poincaré inequality. Recall that $\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu$ is the local fractional dyadic maximal function. Since it is pointwise bounded by the fractional maximal function, Theorem 4.1 also holds with $\mathrm{M}_\alpha \mu$ in place of $\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu$.

Theorem 4.1 *Let $0 \leq \delta < 1$, $1 \leq q \leq \frac{n}{n-\delta}$, $\alpha = n - q(n-\delta)$, $f \in L^1_{loc}(\mathbb{R}^n)$ and let μ be a Radon measure with $\mu \ll \mathcal{L}$. Then there exists a dimensional constant C such that*

$$\left(\int_Q |f - f_Q|^q \,\mathrm{d}\mu \right)^{\frac{1}{q}} \leq C(1-\delta) \int_Q \int_Q \frac{|f(x) - f(y)|}{|x-y|^{n+\delta}} \mathrm{d}y \, (\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu(x))^{\frac{1}{q}} \,\mathrm{d}x$$

for every cube $Q \subset \mathbb{R}^n$.

Alternatively, we can assume that μ is a general Radon measure and the claim holds for any continuous function f.

Note that the conditions on the parameters in Theorem 4.1 can also be written as $0 \leq \alpha \leq \delta < 1$, $q = \frac{n-\alpha}{n-\delta}$.

Remark 4.2 The proof of the theorem given next is for the Lorentz norm, namely the L^q norm in the left of the claim of the theorem can be replaced by the $\|\cdot\|_{L^{q,1}(\mu)}$ norm, namely

$$||f - f_Q||_{L^{q,1}(\mu)} = q \int_0^\infty \mu(\Omega_\lambda)^{\frac{1}{q}} \,\mathrm{d}\lambda,$$

where $\Omega_\lambda = \{x \in Q : |f - f_Q| > \lambda\}$.

Proof of Theorem 4.1 Fix $Q \subset \mathbb{R}^n$ and denote $\Omega_\lambda = \{x \in Q : |f - f_Q| > \lambda\}$. It holds that

$$\lambda^{q-1} \mu(\Omega_\lambda)^{\frac{q-1}{q}} \leq \left(\int_0^\lambda \mu(\Omega_t)^{\frac{1}{q}} \,\mathrm{d}t \right)^{q-1},$$

since $\Omega_\lambda \subset \Omega_t$ for $0 < t \le \lambda$. By Cavalieri's principle, this implies

$$\begin{aligned}\left(\int_Q |f - f_Q|^q \mathrm{d}\mu\right)^{\frac{1}{q}} &= \left(q \int_0^\infty \lambda^{q-1} \mu(Q \cap \{|f - f_Q| > \lambda\}) \mathrm{d}\lambda\right)^{\frac{1}{q}} \\ &= \left(q \int_0^\infty \lambda^{q-1} \mu(\Omega_\lambda)^{\frac{q-1}{q}} \mu(\Omega_\lambda)^{\frac{1}{q}} \mathrm{d}\lambda\right)^{\frac{1}{q}} \\ &\le \left(q \int_0^\infty \left(\int_0^\lambda \mu(\Omega_t)^{\frac{1}{q}} \mathrm{d}t\right)^{q-1} \mu(\Omega_\lambda)^{\frac{1}{q}} \mathrm{d}\lambda\right)^{\frac{1}{q}} \\ &\le q^{\frac{1}{q}} \left(\int_0^\infty \mu(\Omega_t)^{\frac{1}{q}} \mathrm{d}t\right)^{\frac{q-1}{q}} \left(\int_0^\infty \mu(\Omega_\lambda)^{\frac{1}{q}} \mathrm{d}\lambda\right)^{\frac{1}{q}} \\ &\le 2 \int_0^\infty \mu(\Omega_\lambda)^{\frac{1}{q}} \mathrm{d}\lambda \\ &= 2 \int_{-\infty}^{f_Q} \mu(Q \cap \{f < \lambda\})^{\frac{1}{q}} \mathrm{d}\lambda + 2 \int_{f_Q}^\infty \mu(Q \cap \{f > \lambda\})^{\frac{1}{q}} \mathrm{d}\lambda.\end{aligned} \tag{10}$$

The previous two terms swap when replacing f by $-f$. Thus it suffices to bound the second term. We split it into two parts

$$\begin{aligned}\int_{f_Q}^\infty \mu(Q \cap \{f > \lambda\})^{\frac{1}{q}} \mathrm{d}\lambda = &\int_{f_Q}^{\max\{m_f, f_Q\}} \mu(Q \cap \{f > \lambda\})^{\frac{1}{q}} \mathrm{d}\lambda \\ &+ \int_{\max\{m_f, f_Q\}}^\infty \mu(Q \cap \{f > \lambda\})^{\frac{1}{q}} \mathrm{d}\lambda.\end{aligned} \tag{11}$$

We abbreviate the maximal median of f over Q by $m_f = m_f(Q)$.

For the first term in (11) it suffices to consider $f_Q < m_f$. Recall that $\mathcal{L}(Q \cap \{f > \lambda\}) \ge \mathcal{L}(Q)/2$ for $\lambda < m_f$. By the definition of f_Q, it holds that

$$\int_{f_Q}^\infty \mathcal{L}(Q \cap \{f > \lambda\}) \mathrm{d}\lambda = \int_{-\infty}^{f_Q} \mathcal{L}(Q \cap \{f < \lambda\}) \mathrm{d}\lambda.$$

Using these facts, we get

$$\begin{aligned}\int_{f_Q}^{\max\{m_f, f_Q\}} \mu(Q \cap \{f > \lambda\})^{\frac{1}{q}} \mathrm{d}\lambda &\le (m_f - f_Q) \mu(Q)^{\frac{1}{q}} \\ &\le 2 \frac{\mu(Q)^{\frac{1}{q}}}{\mathcal{L}(Q)} \int_{f_Q}^{m_f} \mathcal{L}(Q \cap \{f > \lambda\}) \mathrm{d}\lambda \\ &\le 2 \frac{\mu(Q)^{\frac{1}{q}}}{\mathcal{L}(Q)} \int_{f_Q}^{\infty} \mathcal{L}(Q \cap \{f > \lambda\}) \mathrm{d}\lambda \\ &= 2 \frac{\mu(Q)^{\frac{1}{q}}}{\mathcal{L}(Q)} \int_{-\infty}^{f_Q} \mathcal{L}(Q \cap \{f < \lambda\}) \mathrm{d}\lambda \\ &\le 2^{\frac{n-\delta}{n}} \frac{\mu(Q)^{\frac{1}{q}}}{\mathcal{L}(Q)^{\frac{n-\delta}{n}}} \int_{-\infty}^{m_f} \mathcal{L}(Q \cap \{f < \lambda\})^{\frac{n-\delta}{n}} \mathrm{d}\lambda,\end{aligned}$$

where in the last inequality we used $\mathcal{L}(Q \cap \{f < \lambda\}) \leq \mathcal{L}(Q)/2$ for $\lambda < m_f$. Denote

$$A_k(x) = Q \cap B(x, 2^{-k}\,\mathrm{l}(Q)) \setminus B(x, 2^{-k-1}\,\mathrm{l}(Q))$$

for $k \in \mathbb{N}$ and

$$K_\lambda = \left\lceil \log_2\!\left(\mathrm{l}(Q)/\mathcal{L}(Q \cap \{f < \lambda\})^{\frac{1}{n}}\right) \right\rceil$$

for $\lambda < m_f$. Then we have

$$\frac{1}{2^{kn}} \leq \frac{\mathcal{L}(Q \cap \{f < \lambda\})}{\mathcal{L}(Q)} \leq \frac{1}{2}$$

for every $k \geq K_\lambda$, $\lambda < m_f$. Thus for each $k \geq K_\lambda$, $\lambda < m_f$, we may apply Lemma 3.3 with $s = 0$ for $E = \{f < \lambda\}$ on Q to obtain

$$\mathcal{L}(Q \cap \{f < \lambda\})^{\frac{n-1}{n}} \leq C_1 \frac{2^k}{\mathrm{l}(Q)} \int_Q \fint_{A_k(x)} |1_{\{f<\lambda\}}(x) - 1_{\{f<\lambda\}}(y)|\,\mathrm{d}y\mathrm{d}x.$$

We multiply both sides of the previous estimate by $2^{-k(1-\delta)}$ and sum over $k \geq K_\lambda$ to get

$$\sum_{k=K_\lambda}^{\infty} 2^{-k(1-\delta)} \mathcal{L}(Q \cap \{f < \lambda\})^{\frac{n-1}{n}} \leq C_1 \sum_{k=K_\lambda}^{\infty} \frac{2^{k\delta}}{\mathrm{l}(Q)} \int_Q \fint_{A_k(x)} |1_{\{f<\lambda\}}(x) - 1_{\{f<\lambda\}}(y)|\,\mathrm{d}y\mathrm{d}x.$$

Furthermore,

$$\sum_{k=K_\lambda}^{\infty} 2^{-k(1-\delta)} = \frac{2^{-K_\lambda(1-\delta)}}{1 - 2^{-(1-\delta)}} \geq \frac{2^{-(1-\delta)}}{1 - 2^{\delta-1}} \frac{\mathcal{L}(Q \cap \{f < \lambda\})^{\frac{1-\delta}{n}}}{\mathrm{l}(Q)^{1-\delta}} \geq \frac{1}{2}\frac{1}{1-\delta} \frac{\mathcal{L}(Q \cap \{f < \lambda\})^{\frac{1-\delta}{n}}}{\mathrm{l}(Q)^{1-\delta}}.$$

By combining the two previous estimates with (7), we conclude that

$$\mathcal{L}(Q \cap \{f < \lambda\})^{\frac{n-\delta}{n}} \leq C_2(1-\delta) \sum_{k \in \mathbb{N}} \frac{2^{k(n+\delta)}}{\mathrm{l}(Q)^{n+\delta}} \int_Q \int_{A_k(x)} |1_{\{f<\lambda\}}(x) - 1_{\{f<\lambda\}}(y)|\,\mathrm{d}y\mathrm{d}x,$$

where $C_2 = 2^{n+2} C_1/\sigma_n$. It follows that

$$
\begin{aligned}
&\int_{f_Q}^{\max\{m_f, f_Q\}} \mu(Q \cap \{f > \lambda\})^{\frac{1}{q}} \,\mathrm{d}\lambda \\
&\leq 2^{\frac{n-\delta}{n}} \frac{\mu(Q)^{\frac{1}{q}}}{\mathcal{L}(Q)^{\frac{n-\delta}{n}}} \int_{-\infty}^{m_f} \mathcal{L}(Q \cap \{f < \lambda\})^{\frac{n-\delta}{n}} \,\mathrm{d}\lambda \\
&\leq 2C_2(1-\delta) \frac{\mu(Q)^{\frac{1}{q}}}{\mathcal{L}(Q)^{\frac{n-\delta}{n}}} \int_{-\infty}^{m_f} \sum_{k\in\mathbb{N}} \frac{2^{k(n+\delta)}}{\mathrm{l}(Q)^{n+\delta}} \int_Q \int_{A_k(x)} |1_{\{f<\lambda\}}(x) - 1_{\{f<\lambda\}}(y)| \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\lambda \\
&= 2C_2(1-\delta) \frac{\mu(Q)^{\frac{1}{q}}}{\mathcal{L}(Q)^{\frac{n-\delta}{n}}} \sum_{k\in\mathbb{N}} \frac{2^{k(n+\delta)}}{\mathrm{l}(Q)^{n+\delta}} \int_Q \int_{A_k(x)} \int_{-\infty}^{m_f} |1_{\{f<\lambda\}}(x) - 1_{\{f<\lambda\}}(y)| \,\mathrm{d}\lambda\,\mathrm{d}y\,\mathrm{d}x \qquad (12)\\
&\leq 2C_2(1-\delta) \frac{\mu(Q)^{\frac{1}{q}}}{\mathcal{L}(Q)^{\frac{n-\delta}{n}}} \sum_{k\in\mathbb{N}} \frac{2^{k(n+\delta)}}{\mathrm{l}(Q)^{n+\delta}} \int_Q \int_{A_k(x)} |f(x) - f(y)| \,\mathrm{d}y\,\mathrm{d}x \\
&\leq 2C_2(1-\delta) \frac{\mu(Q)^{\frac{1}{q}}}{\mathcal{L}(Q)^{\frac{n-\delta}{n}}} \sum_{k\in\mathbb{N}} \int_Q \int_{A_k(x)} \frac{|f(x) - f(y)|}{|x-y|^{n+\delta}} \,\mathrm{d}y\,\mathrm{d}x \\
&\leq 2C_2(1-\delta) \frac{\mu(Q)^{\frac{1}{q}}}{\mathcal{L}(Q)^{\frac{n-\delta}{n}}} \int_Q \int_{Q\cap B(x,\mathrm{l}(Q)/2)} \frac{|f(x) - f(y)|}{|x-y|^{n+\delta}} \,\mathrm{d}y\,\mathrm{d}x \\
&\leq 2C_2(1-\delta) \int_Q \int_Q \frac{|f(x) - f(y)|}{|x-y|^{n+\delta}} \,\mathrm{d}y \,(\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu)^{\frac{1}{q}} \,\mathrm{d}x,
\end{aligned}
$$

where in the last inequality we used

$$
\frac{\mu(Q)}{\mathcal{L}(Q)^{\frac{n-\delta}{n}q}} \leq \mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu(x)
$$

for every $x \in Q$.

It is left to estimate the second term in (11). In that case, we have $\mathcal{L}(Q \cap \{f > \lambda\}) \leq \mathcal{L}(Q)/2$ since $\lambda > m_f$. We apply Lemma 2.1 for $E = \{f > \lambda\}$ on Q at level $\frac{1}{2}$ to obtain a collection $\{Q_i\}_i$ of Calderón–Zygmund cubes with $\mathrm{l}(Q_i) = 2^{-N_i}\,\mathrm{l}(Q)$ for some $N_i \in \mathbb{N}_0$ such that $Q \cap \{f > \lambda\} \subset \bigcup_i Q_i$ up to a set of Lebesgue measure zero and

$$
\frac{1}{2^{n+1}} < \frac{\mathcal{L}(Q_i \cap \{f > \lambda\})}{\mathcal{L}(Q_i)} \leq \frac{1}{2}.
$$

Fix $i \in \mathbb{N}$ and let $k \geq N_i + 1$. We apply Lemma 3.3 with $k - N_i$ instead of k and $s = 1$ for $E = \{f > \lambda\}$ on Q_i. Observe that $2^{-(k-N_i)}\,\mathrm{l}(Q_i) = 2^{-k}\,\mathrm{l}(Q)$. For every $k \geq N_i + 1$ we obtain

$$
\begin{aligned}
\mathcal{L}(Q_i)^{\frac{n-1}{n}} &\leq 2^{(n+1)\frac{n-1}{n}} \mathcal{L}(Q_i \cap \{f > \lambda\})^{\frac{n-1}{n}} \\
&\leq 2^n C_1 \frac{2^k}{\mathrm{l}(Q)} \int_{Q_i} \fint_{Q_i \cap A_k(x)} |1_{\{f>\lambda\}}(x) - 1_{\{f>\lambda\}}(y)| \,\mathrm{d}y\,\mathrm{d}x,
\end{aligned}
$$

where $A_k(x) = Q \cap B(x, 2^{-k}\,\mathrm{l}(Q)) \setminus B(x, 2^{-k-1}\,\mathrm{l}(Q))$ as above. Multiplying both sides by $2^{-k(1-\delta)}$ and summing over $k \geq N_i + 1$, we get

$$
\sum_{k\geq N_i+1} 2^{-k(1-\delta)} \mathcal{L}(Q_i)^{\frac{n-1}{n}} \leq 2^n C_1 \sum_{k\geq N_i+1} \frac{2^{k\delta}}{\mathrm{l}(Q)} \int_{Q_i} \fint_{Q_i \cap A_k(x)} |1_{\{f>\lambda\}}(x) - 1_{\{f>\lambda\}}(y)| \,\mathrm{d}y\,\mathrm{d}x.
$$

We note that

$$\sum_{k\geq N_i+1} 2^{-k(1-\delta)} = \frac{2^{-(N_i+1)(1-\delta)}}{1-2^{-(1-\delta)}} = \frac{2^{-(1-\delta)}}{1-2^{-(1-\delta)}}\frac{\mathrm{l}(Q_i)^{1-\delta}}{\mathrm{l}(Q)^{1-\delta}} \geq \frac{1}{2(1-\delta)}\frac{\mathrm{l}(Q_i)^{1-\delta}}{\mathrm{l}(Q)^{1-\delta}}.$$

By combining the two previous estimates with (7), we conclude that

$$\mathcal{L}(Q_i)^{\frac{n-\delta}{n}} \leq C_3(1-\delta)\sum_{k\in\mathbb{N}} \frac{2^{k(n+\delta)}}{\mathrm{l}(Q)^{n+\delta}} \int_{Q_i}\int_{A_k(x)} |1_{\{f>\lambda\}}(x) - 1_{\{f>\lambda\}}(y)|\,\mathrm{d}y\mathrm{d}x,$$

where $C_3 = 2^{2n+2}C_1/\sigma_n$. Since

$$\mu(Q\cap\{f>\lambda\}) \leq \mu\Big(\bigcup_i Q_i\Big)$$

by Lemma 2.1 and

$$\frac{\mu(Q_i)}{\mathcal{L}(Q_i)^{\frac{n-\delta}{n}q}} \leq \mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu(x)$$

for every $x\in Q_i$, it follows that

$$\begin{aligned}
\mu(Q\cap\{f>\lambda\})^{\frac{1}{q}} &\leq \sum_i \mu(Q_i)^{\frac{1}{q}}\\
&\leq C_3(1-\delta)\sum_i \frac{\mu(Q_i)^{\frac{1}{q}}}{\mathcal{L}(Q_i)^{\frac{n-\delta}{n}}}\sum_{k\in\mathbb{N}}\frac{2^{k(n+\delta)}}{\mathrm{l}(Q)^{n+\delta}}\int_{Q_i}\int_{A_k(x)} |1_{\{f>\lambda\}}(x) - 1_{\{f>\lambda\}}(y)|\,\mathrm{d}y\mathrm{d}x\\
&\leq C_3(1-\delta)\sum_{k\in\mathbb{N}}\frac{2^{k(n+\delta)}}{\mathrm{l}(Q)^{n+\delta}}\sum_i\int_{Q_i}\int_{A_k(x)} |1_{\{f>\lambda\}}(x) - 1_{\{f>\lambda\}}(y)|\,\mathrm{d}y\,(\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu(x))^{\frac{1}{q}}\,\mathrm{d}x\\
&\leq C_3(1-\delta)\sum_{k\in\mathbb{N}}\frac{2^{k(n+\delta)}}{\mathrm{l}(Q)^{n+\delta}}\int_{Q}\int_{A_k(x)} |1_{\{f>\lambda\}}(x) - 1_{\{f>\lambda\}}(y)|\,\mathrm{d}y\,(\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu(x))^{\frac{1}{q}}\,\mathrm{d}x.
\end{aligned}$$

Integrating both sides in λ, we obtain

$$\begin{aligned}
&\int_{\max\{m_f,f_Q\}}^{\infty}\mu(Q\cap\{f>\lambda\})^{\frac{1}{q}}\,\mathrm{d}\lambda\\
&\quad\leq C_3(1-\delta)\sum_{k\in\mathbb{N}}\frac{2^{k(n+\delta)}}{\mathrm{l}(Q)^{n+\delta}}\int_{Q}\int_{A_k(x)}\int_{m_f}^{\infty} |1_{\{f>\lambda\}}(x) - 1_{\{f>\lambda\}}(y)|\,\mathrm{d}\lambda\mathrm{d}y\,(\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu(x))^{\frac{1}{q}}\,\mathrm{d}x\\
&\quad\leq C_3(1-\delta)\sum_{k\in\mathbb{N}}\frac{2^{k(n+\delta)}}{\mathrm{l}(Q)^{n+\delta}}\int_{Q}\int_{A_k(x)} |f(x)-f(y)|\,\mathrm{d}y\,(\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu(x))^{\frac{1}{q}}\,\mathrm{d}x\\
&\quad\leq C_3(1-\delta)\sum_{k\in\mathbb{N}}\int_{Q}\int_{A_k(x)} \frac{|f(x)-f(y)|}{|x-y|^{n+\delta}}\,\mathrm{d}y\,(\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu(x))^{\frac{1}{q}}\,\mathrm{d}x\\
&\quad\leq C_3(1-\delta)\int_{Q}\int_{Q\cap B(x,\mathrm{l}(Q)/2)} \frac{|f(x)-f(y)|}{|x-y|^{n+\delta}}\,\mathrm{d}y\,(\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu(x))^{\frac{1}{q}}\,\mathrm{d}x\\
&\quad\leq C_3(1-\delta)\int_{Q}\int_{Q} \frac{|f(x)-f(y)|}{|x-y|^{n+\delta}}\,\mathrm{d}y\,(\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu(x))^{\frac{1}{q}}\,\mathrm{d}x.
\end{aligned} \tag{13}$$

By combining the obtained estimates (10) to (13), we conclude that

$$\left(\int_Q |f - f_Q|^q \mathrm{d}\mu\right)^{\frac{1}{q}} \le C(1-\delta)\int_Q \int_Q \frac{|f(x)-f(y)|}{|x-y|^{n+\delta}}\mathrm{d}y\, (\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu(x))^{\frac{1}{q}}\mathrm{d}x,$$

where $C = 4(2C_2 + C_3) = (2^{2n+4} + 2^{n+5})C_1/\sigma_n$. □

For a Radon measure satisfying a polynomial growth condition, the following fractional Poincaré inequality holds.

Corollary 4.3 *Let $0 \le \delta < 1$, $1 \le q \le \frac{n}{n-\delta}$, $\alpha = n - q(n-\delta)$, $f \in L^1_{loc}(\mathbb{R}^n)$ and μ be a Radon measure. Assume that there exists a constant C_μ such that*

$$\mu(Q) \le C_\mu\, \mathrm{l}(Q)^{n-\alpha}$$

for every cube $Q \subset \mathbb{R}^n$. Then there exists a dimensional constant C such that

$$\left(\int_Q |f - f_Q|^q \mathrm{d}\mu\right)^{\frac{1}{q}} \le C_\mu^{\frac{1}{q}} C(1-\delta)\int_Q \int_Q \frac{|f(x)-f(y)|}{|x-y|^{n+\delta}}\mathrm{d}y\mathrm{d}x$$

for every cube $Q \subset \mathbb{R}^n$.

Remark 4.4 We remark that this result combined with Theorem 6.2 with $p = 1$ yields the classical Meyers–Ziemer theorem [21].

Proof of Corollary 4.3 Fix a cube $Q \subset \mathbb{R}^n$. By the assumption, we have

$$\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu(x) = \sup_{\substack{Q' \ni x,\\ Q' \in \mathcal{D}(Q)}} \mathrm{l}(Q')^\alpha \frac{\mu(Q')}{\mathcal{L}(Q')} \le C_\mu$$

for every $x \in Q$. Thus, by Theorem 4.1, the claim follows. □

5 From fractional (1, 1)-Poincaré inequality to fractional (q, p)-Poincaré inequality with A_p weights

In this section, we show that the fractional (1, 1)-Poincaré inequality implies the fractional (q, p)-Poincaré inequality. Moreover, we are able to obtain the result with A_p weights as conjectured in [15].

We recall briefly some concepts about the classes of Muckenhoupt weights. A weight is a function $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfying $w(x) > 0$ for almost every point $x \in \mathbb{R}^n$.

Definition 5.1 Let w be a weight.

(i) We say that $w \in A_1$ if there is a constant C such that

$$\mathrm{M}w(x) \le Cw(x)$$

for almost every $x \in \mathbb{R}^n$. The A_1 constant $[w]_{A_1}$ is defined as the smallest C for which the condition above holds.

(ii) For $1 < p < \infty$ we say that $w \in A_p$ if

$$[w]_{A_p} = \sup_Q \fint_Q w\mathrm{d}x \left(\fint_Q w^{1-p'}\mathrm{d}x\right)^{p-1} < \infty.$$

(iii) The A_∞ class is defined as the union of all the A_p classes, that is,

$$A_\infty = \bigcup_{1\le p<\infty} A_p,$$

and the A_∞ constant is defined as

$$[w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q \mathrm{M}(1_Q w)\mathrm{d}x.$$

Recall that for $1 \le r \le p < \infty$ we have

$$c[w]_{A_\infty} \le [w]_{A_p} \le [w]_{A_r} \le [w]_{A_1} \tag{14}$$

for some $c > 0$ depending only on the dimension.

We observe that the fractional (1, 1)-Poincaré inequality implies the fractional $(1, p)$-Poincaré inequality with A_p weights on the right-hand side. However, note the extra factor $\delta^{\frac{1}{p}-1}$ that appears in front.

Corollary 5.2 *Let $0 < \delta < 1$, $1 \le p < \infty$, $f \in L^1_{loc}(\mathbb{R}^n)$ and $w \in A_p$. Then there exists a dimensional constant C such that*

$$\fint_Q |f - f_Q|\mathrm{d}x \le C[w]_{A_p}^{\frac{1}{p}} \frac{(1-\delta)^{\frac{1}{p}}}{\delta^{1-\frac{1}{p}}} \mathrm{l}(Q)^\delta \left(\frac{1}{w(Q)} \int_Q \int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}} \mathrm{d}y\, w(x)\mathrm{d}x \right)^{\frac{1}{p}}$$

for every cube $Q \subset \mathbb{R}^n$.

Proof Let $0 \le \varepsilon \le \delta$. By Theorem 4.1 with $\mu = \mathcal{L}$, there exists a constant C_1 such that

$$\fint_Q |f - f_Q|\mathrm{d}x \le C_1(1-\delta+\varepsilon)\,\mathrm{l}(Q)^{\delta-\varepsilon} \fint_Q \int_Q \frac{|f(x)-f(y)|}{|x-y|^{n+\delta-\varepsilon}} \mathrm{d}y\mathrm{d}x. \tag{15}$$

If $p = 1$, the claim of the corollary follows from (15) with $\varepsilon = 0$ combined with the definition of A_1 weights. It remains to consider $p > 1$. Assume $0 < \varepsilon \le \delta$ and fix $x \in Q$. Then by Hölder's inequality we have

$$\begin{aligned}\int_Q \frac{|f(x)-f(y)|}{|x-y|^{n+\delta-\varepsilon}}\mathrm{d}y &\le \left(\int_Q \frac{1}{|x-y|^{n-\varepsilon p'}}\mathrm{d}y \right)^{\frac{1}{p'}} \left(\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\mathrm{d}y \right)^{\frac{1}{p}} \\ &\le (n^{\frac{3}{2}}\sigma_n)^{\frac{1}{p'}} \frac{\mathrm{l}(Q)^\varepsilon}{\varepsilon^{\frac{1}{p'}}} \left(\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\mathrm{d}y \right)^{\frac{1}{p}}.\end{aligned}$$

We plug this into (15) and apply Hölder's inequality once more with the definition of A_p weights to get

$$\begin{aligned}\fint_Q |f - f_Q|\mathrm{d}x &\le C_2\,\mathrm{l}(Q)^\delta \frac{1-\delta+\varepsilon}{\varepsilon^{\frac{1}{p'}}} \fint_Q \left(\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\mathrm{d}y \right)^{\frac{1}{p}} \frac{w(x)^{\frac{1}{p}}}{w(x)^{\frac{1}{p}}}\mathrm{d}x \\ &\le C_2\,\mathrm{l}(Q)^\delta \frac{1-\delta+\varepsilon}{\varepsilon^{\frac{1}{p'}}} \left(\fint_Q \int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\mathrm{d}y\, w(x)\mathrm{d}x \right)^{\frac{1}{p}} \left(\fint_Q w(x)^{1-p'}\mathrm{d}x \right)^{\frac{p-1}{p}} \\ &\le C_2\,\mathrm{l}(Q)^\delta \frac{1-\delta+\varepsilon}{\varepsilon^{\frac{1}{p'}}} [w]_{A_p}^{\frac{1}{p}} \left(\frac{1}{w(Q)} \int_Q \int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\mathrm{d}y\, w(x)\mathrm{d}x \right)^{\frac{1}{p}}\end{aligned}$$

with $C_2 = C_1 \max\{1, n^{\frac{3}{2}}\sigma_n\} \le 92C_1$. Setting $\varepsilon = \min\{\delta, 1-\delta\}$ finishes the proof. □

We recall the definitions of the weighted $D_p(w)$ and $SD_p^s(w)$ conditions.

Definition 5.3 Let $0 < p < \infty$, $0 < s < \infty$, w be a weight and $a : \mathcal{Q} \to [0, \infty)$ be a general functional defined over the collection of all cubes in $\mathbb{R}^n$.

(i) The functional a belongs to $D_p(w)$ if there is a constant c such that

$$\left(\sum_i a(Q_i)^p \frac{w(Q_i)}{w(Q)} \right)^{\frac{1}{p}} \le Ca(Q)$$

for any family of disjoint dyadic subcubes $\{Q_i\}_i$ of any given cube $Q \subset \mathbb{R}^n$. The smallest constant C above is denoted by $\|a\|_{D_p(w)}$.

(ii) The functional a belongs to $SD_p^s(w)$ if there is a constant C such that

$$\left(\sum_i a(Q_i)^p \frac{w(Q_i)}{w(Q)} \right)^{\frac{1}{p}} \le C \left(\frac{\mathcal{L}(\bigcup_i Q_i)}{\mathcal{L}(Q)} \right)^{\frac{1}{s}} a(Q)$$

for any family of disjoint dyadic subcubes $\{Q_i\}_i$ of any given cube $Q \subset \mathbb{R}^n$. The smallest constant C above is denoted by $\|a\|_{SD_p^s(w)}$.

The following self-improving property from [6, Theorem 1.6] is relevant for us.

Theorem 5.4 *Let $1 < p < \infty$, $w \in A_\infty$ and $a \in D_p(w)$. Assume that $f \in L^1_{loc}(\mathbb{R}^n)$ such that*

$$\fint_Q |f - f_Q| \mathrm{d}x \le a(Q)$$

for every cube $Q \subset \mathbb{R}^n$. Then there exists a dimensional constant C such that

$$\|f - f_Q\|_{L^{p,\infty}(Q, \frac{w\mathrm{d}x}{w(Q)})} \le Cp[w]_{A_\infty} \|a\|_{D_p(w)} a(Q).$$

for every cube $Q \subset \mathbb{R}^n$.

For the stronger $SD_p^s(w)$ condition, we have a better self-improvement, see [18, Theorem 5.3].

Theorem 5.5 *Let $1 \le p < \infty$, $1 < s < \infty$, w be a weight and $a \in SD_p^s(w)$. Assume that $f \in L^1_{loc}(\mathbb{R}^n)$ such that*

$$\fint_Q |f - f_Q| \mathrm{d}x \le a(Q)$$

for every cube $Q \subset \mathbb{R}^n$. Then there exists a dimensional constant C such that

$$\|f - f_Q\|_{L^p(Q, \frac{w\mathrm{d}x}{w(Q)})} \le Cs\|a\|_{SD_p^s(w)} a(Q)$$

for every cube $Q \subset \mathbb{R}^n$.

Another important tool that we need is the following fractional truncation method which can be shown by adapting the proof of [10, Theorem 4.1].

Theorem 5.6 *Let $0 < \delta < 1$, $1 \le p \le q < \infty$, $f \in L^1_{loc}(\mathbb{R}^n)$ and w be a weight. Then the following conditions are equivalent.*

(i) *There is a constant* C_1 *such that*

$$\inf_{c\in\mathbb{R}}\|f-c\|_{L^{q,\infty}(Q,\frac{w\mathrm{d}x}{w(Q)})}\le C_1\left(\frac{1}{w(Q)}\int_Q\int_Q\frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\mathrm{d}y\,w(x)\mathrm{d}x\right)^{\frac{1}{p}}$$

for every cube $Q\subset\mathbb{R}^n$.

(ii) *There is a constant* C_2 *such that*

$$\inf_{c\in\mathbb{R}}\|f-c\|_{L^{q}(Q,\frac{w\mathrm{d}x}{w(Q)})}\le C_2\left(\frac{1}{w(Q)}\int_Q\int_Q\frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\mathrm{d}y\,w(x)\mathrm{d}x\right)^{\frac{1}{p}}$$

for every cube $Q\subset\mathbb{R}^n$.

Moreover, in the implication from (i) to (ii) the constant C_2 *is of the form* CC_1*, where* C *only depends on the dimension, and in the implication from (ii) to (i) we have* $C_1=C_2$.

We are ready to state and prove the main results of this section, which are the fractional (q,p)-Poincaré inequalities with A_p weights. These results extend Theorems 2.1 and 2.3 in [15]. We emphasize that the factor $(1-\delta)^{\frac{1}{p}}$ remains despite the singularity introduced by the weight.

Theorem 5.7 *Let* $0<\delta<1$, $1\le r\le p<\frac{n}{\delta}$, $f\in L^1_{loc}(\mathbb{R}^n)$ *and* $w\in A_r$. *Let* q *be defined by*

$$\frac{1}{p}-\frac{1}{q}=\frac{\delta}{nr}.$$

Then there exists a dimensional constant C *such that*

$$\inf_{c\in\mathbb{R}}\left(\frac{1}{w(Q)}\int_Q|f-c|^q\,w\mathrm{d}x\right)^{\frac{1}{q}}$$
$$\le Cq[w]_{A_p}^{\frac{1}{p}}[w]_{A_r}^{\frac{\delta}{nr}}[w]_{A_\infty}\frac{(1-\delta)^{\frac{1}{p}}}{\delta^{1-\frac{1}{p}}}\,l(Q)^\delta\left(\frac{1}{w(Q)}\int_Q\int_Q\frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\mathrm{d}y\,w(x)\mathrm{d}x\right)^{\frac{1}{p}}$$

for every cube $Q\subset\mathbb{R}^n$.

Remark 5.8 We remark that we could replace $[w]_{A_p}^{\frac{1}{p}}[w]_{A_r}^{\frac{\delta}{nr}}[w]_{A_\infty}$ by a multiple of $[w]_{A_r}^{\frac{1}{p}+\frac{\delta}{nr}+1}$.

Proof of Theorem 5.7 Denote

$$a_f(Q)=C_1[w]_{A_p}^{\frac{1}{p}}\frac{(1-\delta)^{\frac{1}{p}}}{\delta^{1-\frac{1}{p}}}\,l(Q)^\delta\left(\frac{1}{w(Q)}\int_Q\int_Q\frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\mathrm{d}y\,w(x)\mathrm{d}x\right)^{\frac{1}{p}},$$

where C_1 is the dimensional constant in Corollary 5.2. By Corollary 5.2, it holds that

$$⨍_Q|f-f_Q|\mathrm{d}x\le a_f(Q).$$

In addition, by [7, Lemma 3.3] (which also holds for $M=1$ corresponding to our case), we have $a_f\in D_q(w)$ such that

$$\|a_f\|_{D_q(w)}\le[w]_{A_r}^{\frac{\delta}{nr}},$$

uniformly in f. Hence, we may apply Theorem 5.4 to obtain

$$\|f - f_Q\|_{L^{q,\infty}(Q,\frac{w\mathrm{d}x}{w(Q)})} \le C_2 q[w]_{A_\infty}[w]_{A_r}^{\frac{\delta}{nr}} a_f(Q)$$
$$= Cq[w]_{A_\infty}[w]_{A_r}^{\frac{\delta}{nr}}[w]_{A_p}^{\frac{1}{p}} \frac{(1-\delta)^{\frac{1}{p}}}{\delta^{1-\frac{1}{p}1}} l(Q)^{\delta}\left(\frac{1}{w(Q)}\int_Q\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\mathrm{d}y\, w(x)\mathrm{d}x\right)^{\frac{1}{p}},$$

where C_2 is the constant in Theorem 5.4 and $C = C_1C_2$. An application of Theorem 5.6 finishes the proof. □

A better dependency on the A_p constants in front can be attained at the expense of having a smaller borderline exponent.

Theorem 5.9 *Let $0 < \delta < 1$, $1 \le r \le p < \frac{n}{\delta}$, $f \in L^1_{loc}(\mathbb{R}^n)$ and $w \in A_r$. Let q be defined by*

$$\frac{1}{p} - \frac{1}{q} = \frac{\delta}{n}\frac{1}{r + \log[w]_{A_r}}.$$

Then there exists a dimensional constant C such that

$$\inf_{c\in\mathbb{R}}\left(\frac{1}{w(Q)}\int_Q |f-c|^q\, w\mathrm{d}x\right)^{\frac{1}{q}}$$
$$\le C\frac{npr}{nr-\delta p}[w]_{A_p}^{\frac{1}{p}}[w]_{A_\infty}\frac{(1-\delta)^{\frac{1}{p}}}{\delta^{1-\frac{1}{p}}} l(Q)^{\delta}\left(\frac{1}{w(Q)}\int_Q\int_Q\frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\mathrm{d}y\, w(x)\mathrm{d}x\right)^{\frac{1}{p}}$$

for every cube $Q \subset \mathbb{R}^n$.

Proof Denote

$$a_f(Q) = C_1[w]_{A_p}^{\frac{1}{p}}\frac{(1-\delta)^{\frac{1}{p}}}{\delta^{1-\frac{1}{p}}} l(Q)^{\delta}\left(\frac{1}{w(Q)}\int_Q\int_Q\frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\mathrm{d}y\, w(x)\mathrm{d}x\right)^{\frac{1}{p}},$$

where C_1 is the constant in Corollary 5.2. By Corollary 5.2, it holds that

$$⨍_Q |f - f_Q|\mathrm{d}x \le a_f(Q).$$

We distinguish between the cases $[w]_{A_r} > e^{\frac{1}{\delta}}$ and the opposite. Assume first that $[w]_{A_r} > e^{\frac{1}{\delta}}$. By [7, Lemma 6.2], we have $a \in SD^s_q(w)$ with $M = 1 + \frac{1}{r}\log[w]_{A_r}$ and $s = \frac{nM'}{\delta} > 1$, such that

$$\|a_f\|_{SD^s_q(w)} \le [w]_{A_r}^{\frac{\delta}{nrM}},$$

uniformly in f. Hence, applying Theorem 5.5 with q, we obtain

$$\|f - f_Q\|_{L^q(Q,\frac{w\mathrm{d}x}{w(Q)})} \le C_2\frac{nM'}{\delta}[w]_{A_r}^{\frac{\delta}{nrM}} a_f(Q) \le C_2\frac{nM'}{\delta}[w]_{A_r}^{\frac{1}{r+\log[w]_{A_r}}} a_f(Q)$$
$$\le C_2\frac{n}{\delta}\frac{r+\log[w]_{A_r}}{\log[w]_{A_r}} e^1 a_f(Q),$$

where C_2 is the constant in Theorem 5.5. By the assumption $[w]_{A_r} > e^{\frac{1}{\delta}}$, we have

$$\|f - f_Q\|_{L^q(Q,\frac{w\mathrm{d}x}{w(Q)})} \le C(r + \log[w]_{A_r})a_f(Q) \le Cr[w]_{A_r}a_f(Q),$$

where $C = C_1C_2ne^1$. This gives the claim when $[w]_{A_r} > e^{\frac{1}{\delta}}$.

Assume now that $[w]_{A_r} \le e^{\frac{1}{\delta}}$. By [7, Lemma 6.2], we have $a_f \in D_m(w)$ such that

$$\|a_f\|_{D_m(w)} \le [w]_{A_r}^{\frac{\delta}{nr}} \le e^{\frac{1}{nr}},$$

where the exponent m is defined by $\frac{1}{p} - \frac{1}{m} = \frac{\delta}{nr}$. Applying Theorem 5.4, we get

$$\|f - f_Q\|_{L^{m,\infty}(Q, \frac{w\mathrm{d}x}{w(Q)})} \le C_3 m[w]_{A_\infty} e^{\frac{1}{nr}} a_f(Q) \le Cm[w]_{A_r} a_f(Q),$$

where C_3 is the constant in Theorem 5.4, C_4 is the constant in (14) and $C = C_3C_4e^1$. Since $q \le m$, Jensen's inequality implies

$$\|f - f_Q\|_{L^{q,\infty}(Q, \frac{w\mathrm{d}x}{w(Q)})} \le \|f - f_Q\|_{L^{m,\infty}(Q, \frac{w\mathrm{d}x}{w(Q)})} \le Cm[w]_{A_r} a_f(Q).$$

An application of Theorem 5.6 finishes the proof. □

6 From weighted fractional to weighted classical Poincaré inequality

This section shows that Theorem 4.1 implies the corresponding weighted classical Poincaré inequality Corollary 6.5. For any $0 < \alpha \le n$ the Riesz potential I_α of a Radon measure μ is

$$\mathrm{I}_\alpha \mu(x) = \int_{\mathbb{R}^n} \frac{\mathrm{d}\mu(y)}{|x-y|^{n-\alpha}}$$

for every $x \in \mathbb{R}^n$. The following lemma is an improved version of the well-known result that the Riesz potential is bounded by the maximal function.

Lemma 6.1 *Let $Q \subset \mathbb{R}^n$ be a cube, μ be a Radon measure and $0 < \alpha < n$. Then*

$$\mathrm{I}_\alpha(1_Q\mu)(x) \le \frac{2^{n-\alpha}n}{\alpha} \mu(Q)^{\frac{\alpha}{n}} (\mathrm{M}\mu(x))^{1-\frac{\alpha}{n}}$$

for every $x \in Q$.

Proof Let $Q \subset \mathbb{R}^n$ be a fixed cube and $x \in Q$. For $t > 0$ let $Q_{x,t}$ be the cube with center at x and side lenght $2t^{-\frac{1}{n-\alpha}}$. Then using Cavalieri's principle, we obtain

$$\begin{aligned}
\int_Q \frac{\mathrm{d}\mu(y)}{|x-y|^{n-\alpha}} &= \int_0^\infty \mu\Big(\Big\{y \in Q : \frac{1}{|x-y|^{n-\alpha}} > t\Big\}\Big)\mathrm{d}t \\
&= \int_0^\infty \mu\big(\big\{y \in Q : |x-y| < t^{-\frac{1}{n-\alpha}}\big\}\big)\mathrm{d}t \\
&\le \int_0^\infty \min\Big\{\mu(Q), \frac{\mu(Q_{x,t})}{\mathcal{L}(Q_{x,t})}\mathcal{L}(Q_{x,t})\Big\}\mathrm{d}t. \\
&\le \int_0^\infty \min\big\{\mu(Q), \mathrm{M}\mu(x)2^n t^{-\frac{n}{n-\alpha}}\big\}\mathrm{d}t \\
&= \int_0^{2^{n-\alpha}(\mathrm{M}\mu(x)/\mu(Q))^{\frac{n-\alpha}{n}}} \mu(Q)\mathrm{d}t + 2^n \int_{2^{n-\alpha}(\mathrm{M}\mu(x)/\mu(Q))^{\frac{n-\alpha}{n}}}^\infty \mathrm{M}\mu(x)t^{-\frac{n}{n-\alpha}}\mathrm{d}t \\
&= \frac{2^{n-\alpha}n}{\alpha}\mu(Q)^{\frac{\alpha}{n}}(\mathrm{M}\mu(x))^{\frac{n-\alpha}{n}}.
\end{aligned}$$

Thus, the claim holds. □

The next theorem states that the weighted fractional term can be bounded by the weighted gradient term, but we have the maximal function of the measure on the right hand side. We are mainly interested in the case $p = 1$. This theorem improves Theorems 2.1 from [16].

Theorem 6.2 *Let $1 \le p < \infty$, $\frac{p-1}{p} < \delta < 1$, $f \in W^{1,p}_{loc}(\mathbb{R}^n)$ and $\mu \ll \mathcal{L}$ be a Radon measure. Then*

$$\int_Q \int_Q \frac{|f(x) - f(y)|^p}{|x-y|^{n+\delta p}} \mathrm{d}y \mathrm{d}\mu(x) \le \frac{2^{n-(1-\delta)p} n}{(1-\delta)p} \frac{\mu(Q)^{\frac{(1-\delta)p}{n}}}{1-(1-\delta)p} \int_Q |\nabla f|^p \, (\mathrm{M}(1_Q \mu))^{1-\frac{(1-\delta)p}{n}} \mathrm{d}x$$

for every cube $Q \subset \mathbb{R}^n$. As a direct consequence,

$$\int_Q \int_Q \frac{|f(x) - f(y)|^p}{|x-y|^{n+\delta p}} \mathrm{d}y \mathrm{d}\mu(x) \le \frac{2^{n-(1-\delta)p} n}{(1-\delta)p} \frac{\mathrm{l}(Q)^{(1-\delta)p}}{1-(1-\delta)p} \int_Q |\nabla f|^p \, \mathrm{M}(1_Q \mu) \mathrm{d}x.$$

Alternatively, we can assume that μ is a general Radon measure and the claim holds for any continuous function $f \in W^{1,p}_{loc}(\mathbb{R}^n)$.

Proof The second inequality follows from the first inequality due to the fact that $\mu(Q)/\mathrm{l}(Q)^n \le \mathrm{M}\mu(x)$ for any $x \in Q$. It remains to prove the first inequality. If f is continuously differentiable then by the Fundamental Theorem of Calculus we have

$$f(y) - f(x) = \int_0^1 \nabla f(x + t(y-x)) \cdot (y-x) \mathrm{d}t$$

for every $(x, y) \in Q \times Q$. If f is a Sobolev function then the previous equality still holds for almost every $(x, y) \in Q \times Q$. Then by Hölder's inequality, it holds that

$$|f(x) - f(y)|^p \le \int_0^1 |\nabla f(x + t(y-x))|^p |x-y|^p \mathrm{d}t.$$

Applying this with Fubini's theorem and doing the change of variables $y \mapsto z = x + t(y-x)$, we get

$$\begin{aligned}
&\int_Q \int_Q \frac{|f(x) - f(y)|^p}{|x-y|^{n+\delta p}} \mathrm{d}y \mathrm{d}\mu(x) \\
&\le \int_Q \int_0^1 \int_Q \frac{|\nabla f(x + t(y-x))|^p}{|x-y|^{n-(1-\delta)p}} \mathrm{d}y \mathrm{d}t \mathrm{d}\mu(x) \\
&= \int_Q \int_0^1 \int_{(1-t)x+tQ} \frac{|\nabla f(z)|^p}{|x-z|^{n-(1-\delta)p}} \frac{t^{n-(1-\delta)p}}{t^n} \mathrm{d}z \mathrm{d}t \mathrm{d}\mu(x) \\
&\le \int_Q \int_Q \frac{|\nabla f(z)|^p}{|x-z|^{n-(1-\delta)p}} \int_0^1 \frac{1}{t^{(1-\delta)p}} \mathrm{d}t \mathrm{d}z \mathrm{d}\mu(x) \\
&= \frac{1}{1-(1-\delta)p} \int_Q |\nabla f(z)|^p \int_Q \frac{\mu(x)}{|x-z|^{n-(1-\delta)p}} \mathrm{d}x \mathrm{d}z.
\end{aligned}$$

Here we used $(1-t)x + tQ \subset Q$ for $x \in Q$ and $(1-\delta)p < 1$. By applying Lemma 6.1, we obtain

$$\int_Q \frac{1}{|x-z|^{n-(1-\delta)p}} \mathrm{d}\mu(x) = \mathrm{I}_{(1-\delta)p}(1_Q \mu)(z) \le \frac{2^{n-(1-\delta)p} n}{(1-\delta)p} \mu(Q)^{\frac{(1-\delta)p}{n}} (\mathrm{M}\mu(z))^{1-\frac{(1-\delta)p}{n}}$$

for every $z \in Q$. Hence, we conclude that

$$\int_Q \int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}} \mathrm{d}y \mathrm{d}\mu(x) \le \frac{2^{n-(1-\delta)p} n}{(1-\delta)p} \frac{\mu(Q)^{\frac{(1-\delta)p}{n}}}{1-(1-\delta)p} \int_Q |\nabla f(z)|^p \, (\mathrm{M}\mu(z))^{1-\frac{(1-\delta)p}{n}} \mathrm{d}z.$$

This completes the proof. □

For A_1 weights, we can replace the maximal function in Theorem 6.2 by the weight itself.

Corollary 6.3 *Let $\frac{p-1}{p} < \delta < 1$, $f \in W^{1,1}_{loc}(\mathbb{R}^n)$ and $w \in A_1$. Then*

$$\int_Q \int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}} \mathrm{d}y \, w(x) \mathrm{d}x \le \frac{2^{n-(1-\delta)p} n}{(1-\delta)p} \frac{w(Q)^{\frac{(1-\delta)p}{n}}}{1-(1-\delta)p} [w]_{A_1}^{1-\frac{(1-\delta)p}{n}} \int_Q |\nabla f|^p \, w^{1-\frac{(1-\delta)p}{n}} \mathrm{d}x$$

for every cube $Q \subset \mathbb{R}^n$. As a direct consequence,

$$\int_Q \int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}} \mathrm{d}y \, w(x) \mathrm{d}x \le \frac{2^{n-(1-\delta)p} n}{(1-\delta)p} \frac{1(Q)^{(1-\delta)p}}{1-(1-\delta)p} [w]_{A_1} \int_Q |\nabla f|^p \, w \mathrm{d}x.$$

Proof The second inequality follows from the first inequality due to the fact that $w(Q)/\mathcal{L}(Q) \le [w]_{A_1} w(x)$ for any $x \in Q$. It remains to prove the first inequality. By Theorem 6.2 and the definition of A_1 weights, we get

$$\begin{aligned}\int_Q \int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}} \mathrm{d}y \, w(x) \mathrm{d}x &\le \frac{2^{n-(1-\delta)p} n}{(1-\delta)p} \frac{w(Q)^{\frac{(1-\delta)p}{n}}}{1-(1-\delta)p} \int_Q |\nabla f|^p \, (\mathrm{M}w)^{1-\frac{(1-\delta)p}{n}} \mathrm{d}x \\ &\le \frac{2^{n-(1-\delta)p} n}{(1-\delta)p} \frac{w(Q)^{\frac{(1-\delta)p}{n}}}{1-(1-\delta)p} [w]_{A_1}^{1-\frac{(1-\delta)p}{n}} \int_Q |\nabla f|^p \, w^{1-\frac{(1-\delta)p}{n}} \mathrm{d}x.\end{aligned}$$

□

The next lemma is the coarea formula for Sobolev functions [24, Proposition 3.2].

Lemma 6.4 *Let $f \in W^{1,1}_{loc}(\mathbb{R}^n)$ and let $g : \mathbb{R}^n \to \mathbb{R}_+$ be a measurable function. Then*

$$\int_E |\nabla f(x)| \, g(x) \mathrm{d}x = \int_{-\infty}^{\infty} \int_{E \cap \partial_* \{f > \lambda\}} g(x) \mathrm{d}\mathcal{H}^{n-1}(x) \mathrm{d}\lambda$$

for every Lebesgue measurable set $E \subset \mathbb{R}^n$.

Combining Theorem 4.1 with Corollary 6.3 we obtain the corresponding weighted classical Poincaré inequality. For thoroughness, we also give another, direct proof for Corollary 6.5 by applying the coarea formula and the relative isoperimetric inequality (Lemma 2.2) instead of Lemma 3.3.

Corollary 6.5 *Let $1 \le q \le \frac{n}{n-1}$, $\alpha = n - q(n-1)$, $f \in W^{1,1}_{loc}(\mathbb{R}^n)$ and let μ be a Radon measure with $\mu \ll \mathcal{L}$. There exists a dimensional constant C such that*

$$\left(\int_Q |f - f_Q|^q \mathrm{d}\mu \right)^{\frac{1}{q}} \le C \int_Q |\nabla f| \, (\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu)^{\frac{1}{q}} \mathrm{d}x$$

for every cube $Q \subset \mathbb{R}^n$.

Alternatively, we can assume that μ is a general Radon measure and the claim holds for any continuous function $f \in W^{1,1}(\mathbb{R}^n)$.

Proof 1 We prove the claim first for $1 < q \le \frac{n}{n-1}$, which means $0 \le \alpha < 1$. For any $\alpha < \delta < 1$ let $q_\delta = \frac{n-\alpha}{n-\delta}$. Note that $q_\delta \to q$ for $\delta \to 1$. The function $(\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu)^{\frac{1}{q_\delta}}$ is an A_1 weight with

$$\big[(\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu)^{\frac{1}{q_\delta}}\big]_{A_1} = \big[(\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu)^{\frac{n}{n-\alpha}-\frac{\delta}{n-\alpha}}\big]_{A_1} \le \frac{15^n 4n}{\delta}$$

by for example [15, Lemma 3.6], and we may apply Corollary 6.3 with $p = 1$ to get

$$\int_Q\int_Q \frac{|f(x)-f(y)|}{|x-y|^{n+\delta}}\mathrm{d}y\,(\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu(x))^{\frac{1}{q_\delta}}\,\mathrm{d}x \le \frac{30^n n^2 2^{1+\delta}}{\delta^2(1-\delta)}\,\mathrm{l}(Q)^{1-\delta}\int_Q |\nabla f|\,(\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu)^{\frac{1}{q_\delta}}\,\mathrm{d}x.$$

Then Theorem 4.1 further implies that there exists a constant C_1 such that

$$\begin{aligned}\left(\int_Q |f-f_Q|^{q_\delta}\mathrm{d}\mu\right)^{\frac{1}{q_\delta}} &\le C_1(1-\delta)\int_Q\int_Q \frac{|f(x)-f(y)|}{|x-y|^{n+\delta}}\mathrm{d}y\,(\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu(x))^{\frac{1}{q_\delta}}\,\mathrm{d}x\\ &\le \frac{C}{\delta^2}\,\mathrm{l}(Q)^{1-\delta}\int_Q |\nabla f|\,(\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu)^{\frac{1}{q_\delta}}\,\mathrm{d}x,\end{aligned}$$

where $C = 30^n 4n^2 C_1$. For any function $g:\mathbb{R}^n \to \mathbb{R}$ the restriction $g^{q_\delta}1_{g\le 1}$ is bounded by 1, and $g^{q_\delta}1_{g>1}$ is pointwise increasing in δ. Thus, by the monotone and the dominated convergence theorem both sides of the previous display converge for $\delta \to 1$ to the desired limit, concluding the proof for $q > 1$.

In order to prove the claim for $q = 1$ we have to differentiate between the two alternatives in the assumptions of the corollary. We first consider the case that f is a Sobolev function and μ is absolutely continuous. Then μ has a density function $v \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by the Radon–Nikodym theorem. Let $k \in \mathbb{N}$ and denote by μ_k the truncated measure that has the bounded density $\min\{v,k\}$. Then $(\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu_k(x))^{\frac{1}{q}}$ is uniformly bounded in α, q and x and converges pointwise to $\mathrm{M}^{\mathrm{d}}_{1,Q}\mu_k(x)$ for $q \to 1$. Thus by Fatou's lemma and the dominated convergence theorem we have

$$\begin{aligned}\int_Q |f-f_Q|\mathrm{d}\mu_k &\le \liminf_{q\to 1}\left(\int_Q |f-f_Q|^q\mathrm{d}\mu_k\right)^{\frac{1}{q}}\\ &\le C\liminf_{q\to 1}\int_Q |\nabla f|\,(\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu_k)^{\frac{1}{q}}\,\mathrm{d}x\\ &= C\int_Q |\nabla f|\,\mathrm{M}^{\mathrm{d}}_{1,Q}\mu_k\mathrm{d}x.\end{aligned}\tag{16}$$

Because μ_k converges to μ and $\mathrm{M}^{\mathrm{d}}_{1,Q}\mu_k$ converges to $\mathrm{M}^{\mathrm{d}}_{1,Q}\mu$ pointwise monotonously from below we can use the monotone convergence theorem to conclude from the previous display that

$$\begin{aligned}\int_Q |f-f_Q|\mathrm{d}\mu &= \lim_{k\to\infty}\int_Q |f-f_Q|\mathrm{d}\mu_k\\ &\le C\lim_{k\to\infty}\int_Q |\nabla f|\,\mathrm{M}^{\mathrm{d}}_{1,Q}\mu_k\mathrm{d}x\\ &= C\int_Q |\nabla f|\,\mathrm{M}^{\mathrm{d}}_{1,Q}\mu\mathrm{d}x,\end{aligned}\tag{17}$$

finishing the proof for $q = 1$ in the case that $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ and $\mu \ll \mathcal{L}$.

In the case that f is continuous and μ is a general Radon measure the proof goes the same, except we let μ_k not be a truncation, but instead the measure that averages μ over dyadic cubes of scale $2^{-k}\,\mathrm{l}(Q)$, i.e.

$$\mu_k(E) = \sum_{P\in\mathcal{D}_k(Q)} \mathcal{L}(E\cap P)\frac{\mu(P)}{\mathcal{L}(P)}.$$

Then $(\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu_k(x))^{\frac{1}{q}}$ is uniformly bounded in α, q and x and converges pointwise to $\mathrm{M}^{\mathrm{d}}_{1,Q}\mu_k(x)$ for $q\to 1$, which means we can conclude (16) also in this case. Also $\mathrm{M}^{\mathrm{d}}_{1,Q}\mu_k$ converges to $\mathrm{M}^{\mathrm{d}}_{1,Q}\mu$ pointwise monotonously from below. Furthermore, μ_k converges to μ weakly, see [12, Theorem 1.40]. Thus, we can conclude (17) also in this case, finishing the proof for $q=1$ also in the case that $f\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ is continuous and μ is a general Radon measure. □

Proof 2 Fix $Q\subset\mathbb{R}^n$ and denote $\Omega_\lambda=\{x\in Q:|f-f_Q|>\lambda\}$. As in the proof of Theorem 4.1, we reduce the problem to bounding the sum

$$\int_{f_Q}^{\max\{m_f,f_Q\}} \mu(Q\cap\{f>\lambda\})^{\frac{1}{q}}\,\mathrm{d}\lambda + \int_{\max\{m_f,f_Q\}}^{\infty} \mu(Q\cap\{f>\lambda\})^{\frac{1}{q}}\,\mathrm{d}\lambda, \tag{18}$$

and estimate the first summand by

$$\begin{aligned}
2^{\frac{n-1}{n}}\frac{\mu(Q)^{\frac{1}{q}}}{\mathcal{L}(Q)^{\frac{n-1}{n}}}\int_{-\infty}^{f_Q}\mathcal{L}(Q\cap\{f<\lambda\})^{\frac{n-1}{n}}\mathrm{d}\lambda &\le 2C_1\frac{\mu(Q)^{\frac{1}{q}}}{\mathcal{L}(Q)^{\frac{n-1}{n}}}\int_{-\infty}^{f_Q}\mathcal{H}^{n-1}(Q\cap\partial_*\{f<\lambda\})\mathrm{d}\lambda\\
&\le 2C_1\frac{\mu(Q)^{\frac{1}{q}}}{\mathcal{L}(Q)^{\frac{n-1}{n}}}\int_Q|\nabla f|\mathrm{d}x\\
&\le 2C_1\int_Q|\nabla f|\,(\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu)^{\frac{1}{q}}\,\mathrm{d}x,
\end{aligned}$$

where we used $\mathcal{L}(Q\cap\{f<\lambda\})\le\mathcal{L}(Q)/2$, Lemmas 2.2, 6.4 and

$$\frac{\mu(Q)^{\frac{1}{q}}}{\mathcal{L}(Q)^{\frac{n-1}{n}}}\le(\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu(x))^{\frac{1}{q}}$$

for every $x\in Q$.

It is left to estimate the second term in (18). In that case, we have $\mathcal{L}(Q\cap\{f>\lambda\})\le\mathcal{L}(Q)/2$ since $\lambda>m_f$. We apply Lemma 2.1 for $E=\{f>\lambda\}$ on Q at level $\frac{1}{2}$ to obtain a collection $\{Q_i\}_i$ of Calderón–Zygmund cubes such that $Q\cap\{f>\lambda\}\subset\bigcup_i Q_i$ up to a set of Lebesgue measure zero and

$$\frac{1}{2^{n+1}}<\frac{\mathcal{L}(Q_i\cap\{f>\lambda\})}{\mathcal{L}(Q_i)}\le\frac{1}{2}.$$

By Lemma 2.2, we have

$$\mathcal{L}(Q_i)^{\frac{n-1}{n}}\le 2^{(n+1)\frac{n-1}{n}}\mathcal{L}(Q_i\cap\{f>\lambda\})^{\frac{n-1}{n}}\le C_2\mathcal{H}^{n-1}(Q_i\cap\partial_*\{f>\lambda\}),$$

where $C_2=2^nC_1$. Since

$$\mu(Q\cap\{f>\lambda\})\le\mu\Big(\bigcup_i Q_i\Big)$$

by Lemma 2.1 and

$$\frac{\mu(Q_i)}{\mathcal{L}(Q_i)^{\frac{n-1}{n}q}} \le \mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu(x)$$

for every $x \in Q_i$, it follows that

$$\begin{aligned}\mu(Q \cap \{f > \lambda\})^{\frac{1}{q}} &\le \sum_i \mu(Q_i)^{\frac{1}{q}} \\ &\le C_2 \sum_i \frac{\mu(Q_i)^{\frac{1}{q}}}{\mathcal{L}(Q_i)^{\frac{n-1}{n}}} \mathcal{H}^{n-1}(Q_i \cap \partial_*\{f > \lambda\}) \\ &\le C_2 \sum_i \int_{Q_i \cap \partial_*\{f>\lambda\}} (\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu)^{\frac{1}{q}} \,\mathrm{d}\mathcal{H}^{n-1} \\ &\le C_2 \int_{Q \cap \partial_*\{f>\lambda\}} (\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu)^{\frac{1}{q}} \,\mathrm{d}\mathcal{H}^{n-1}.\end{aligned}$$

Integrating both sides in λ and applying Lemma 6.4, we obtain

$$\begin{aligned}\int_{\max\{m_f, f_Q\}}^{\infty} \mu(Q \cap \{f > \lambda\})^{\frac{1}{q}} \,\mathrm{d}\lambda &\le C_2 \int_{\max\{m_f, f_Q\}}^{\infty} \int_{Q \cap \partial_*\{f>\lambda\}} (\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu)^{\frac{1}{q}} \,\mathrm{d}\mathcal{H}^{n-1} \mathrm{d}\lambda \\ &\le C_2 \int_Q |\nabla f| \, (\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu)^{\frac{1}{q}} \,\mathrm{d}x.\end{aligned}$$

We have bounded both summands in (18) which means we can conclude that

$$\left(\int_Q |f - f_Q|^q \,\mathrm{d}\mu\right)^{\frac{1}{q}} \le C \int_Q |\nabla f| \, (\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu)^{\frac{1}{q}} \,\mathrm{d}x,$$

where $C = 8(1 + 2^{n-1})C_1$. □

7 Examples against weighted (q, p)-Poincaré inequalities for $p > 1$ and against a larger fractional parameter

In this section, we prove that the corresponding L^p-versions of the weighted fractional and classical Poincaré inequalities Theorem 4.1 and Corollary 6.5 do not hold. This is motivated by [15, Theorems 2.4 & 2.9] where sub-optimal results were obtained.

More precisely we show that for every cube $Q \subset \mathbb{R}^n$, $1 < p < n$, $p \le q \le \frac{np}{n-p}$, $\alpha = n - \frac{q}{p}(n-p)$, and $C > 0$ there is a Radon measure $\mu \ll \mathcal{L}$ and a Lipschitz function f with

$$\left(\int_Q |f - f_Q|^q \,\mathrm{d}\mu\right)^{\frac{1}{q}} > C\left(\int_Q |\nabla f|^p \, (\mathrm{M}_\alpha\mu)^{\frac{p}{q}} \,\mathrm{d}x\right)^{\frac{1}{p}}, \tag{19}$$

and that for any $0 < \delta < 1$, $1 < p < \min\{\frac{n}{\delta}, \frac{1}{1-\delta}\}$, $p \le q \le \frac{np}{n-\delta p}$, $\alpha = n - \frac{q}{p}(n - \delta p)$, and $C > 0$ there is a Radon measure $\mu \ll \mathcal{L}$ and a Lipschitz function f with

$$\left(\int_Q |f - f_Q|^q \,\mathrm{d}\mu\right)^{\frac{1}{q}} > C(1-\delta)^{\frac{1}{p}}\left(\int_Q \int_Q \frac{|f(x) - f(y)|^p}{|x - y|^{n+\delta p}} \mathrm{d}y \, (\mathrm{M}_\alpha\mu(x))^{\frac{p}{q}} \,\mathrm{d}x\right)^{\frac{1}{p}}. \tag{20}$$

We do not know if the condition $p < \frac{1}{1-\delta}$ is necessary.

Moreover, we show that given $q \geq 1$ (and δ), the value $\alpha = n-q(n-1)$ (or $\alpha = n-q(n-\delta)$ respectively) for the fractional parameter is the best possible for which Theorem 4.1 and Corollary 6.5 hold in the following sense. For any $\varepsilon \geq 0$ we have the pointwise inequality $\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu(x) \leq \mathrm{l}(Q)^{\varepsilon}\mathrm{M}^{\mathrm{d}}_{\alpha-\varepsilon,Q}\mu(x)$ for $x \in Q$. Hence, Theorem 4.1 and Corollary 6.5 also hold with $\mathrm{l}(Q)^{\varepsilon}\mathrm{M}^{\mathrm{d}}_{\alpha-\varepsilon,Q}\mu(x)$ instead of $\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu(x)$. This argument clearly only works for $\varepsilon \geq 0$. And indeed, we show that Theorem 4.1 and Corollary 6.5 fail when we replace $\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu(x)$ by $\mathrm{l}(Q)^{-\varepsilon}\mathrm{M}^{\mathrm{d}}_{\alpha+\varepsilon,Q}\mu(x)$ with any $\varepsilon > 0$. We show this even for the fractional maximal function $\mathrm{M}_{\alpha}\mu$ which is larger than $\mathrm{M}^{\mathrm{d}}_{\alpha,Q}\mu$ up to a constant. More precisely, we show that for any $1 \leq q \leq \frac{n}{n-1}$, $\alpha = n - q(n-1)$, $\varepsilon > 0$ and $C > 0$ there is a Radon measure $\mu \ll \mathcal{L}$ and a Lipschitz function f

$$\left(\int_Q |f - f_Q|^q \mathrm{d}\mu\right)^{\frac{1}{q}} > C\frac{1}{\mathrm{l}(Q)^{\frac{\varepsilon}{q}}}\int_Q |\nabla f| \,(\mathrm{M}_{\alpha+\varepsilon}\mu)^{\frac{1}{q}}\mathrm{d}x, \tag{21}$$

and that for any $0 < \delta < 1$, $1 \leq q \leq \frac{n}{n-\delta}$, $\alpha = n - q(n-\delta)$, $\varepsilon > 0$ and $C > 0$ there is a Radon measure $\mu \ll \mathcal{L}$ and a Lipschitz function f with

$$\left(\int_Q |f - f_Q|^q \mathrm{d}\mu\right)^{\frac{1}{q}} > C\frac{1-\delta}{\mathrm{l}(Q)^{\frac{\varepsilon}{q}}}\int_Q\int_Q \frac{|f(x)-f(y)|}{|x-y|^{n+\delta}}\mathrm{d}y\,(\mathrm{M}_{\alpha+\varepsilon}\mu(x))^{\frac{1}{q}}\mathrm{d}x. \tag{22}$$

We proceed with the proof of (19) to (22). Let $Q_0 \subset B(0,1)$ be the cube with center 0 and sidelength $1/\sqrt{n}$. By translation and dilation it suffices to find a function f and a measure μ which satisfy (19) to (22) on Q_0. Consider the sequence of absolutely continuous measures and Lipschitz functions

$$\mu_k(A) = \frac{\mathcal{L}(B(0,e^{-k}) \cap A)}{\mathcal{L}(B(0,e^{-k}))} \qquad \text{and} \qquad f_k(x) = \min\{-\log|x|, k\}$$

with $k \in \mathbb{N}$. Denote

$$\int_{Q_0} |f_k|\mathrm{d}x \leq \int_{Q_0} \big|\log|x|\big|\mathrm{d}x = c < \infty.$$

It holds that $f_k(x) = k$ for $|x| \leq e^{-k}$. Thus, for $k \geq c$ we have

$$\left(\int_{Q_0} |f_k - (f_k)_{Q_0}|^q \mathrm{d}\mu_k\right)^{\frac{1}{q}} \geq k - c \tag{23}$$

for any $q \geq 1$. Furthermore for any $0 \leq \alpha \leq n$ we have

$$\mathrm{M}_{\alpha}\mu_k(x) \leq \frac{C_1}{(|x| + e^{-k})^{n-\alpha}} \leq \frac{C_1}{|x|^{n-\alpha}}$$

for some dimensional constant C_1 and

$$|\nabla f_k(x)| = \frac{1}{|x|}1_{\{|\cdot| \geq e^{-k}\}}(x).$$

Let $p > 1$, q and α be as specified for (19). Then

$$\begin{aligned}\int_{Q_0} |\nabla f_k|^p \, (\mathrm{M}_\alpha \mu_k)^{\frac{p}{q}} \,\mathrm{d}x &\le \int_{B(0,1)} |\nabla f_k|^p \, (\mathrm{M}_\alpha \mu_k)^{\frac{p}{q}} \,\mathrm{d}x \\ &\le n\sigma_n C_1^{\frac{p}{q}} \int_0^1 \frac{1}{r^p} 1_{\{r \ge e^{-k}\}} r^{(\alpha-n)\frac{p}{q}} r^{n-1} \mathrm{d}r \\ &= C_2 \int_{e^{-k}}^1 r^{-p+(\alpha-n)\frac{p}{q}+n-1} \mathrm{d}r \\ &= C_2 \int_{e^{-k}}^1 r^{-p-(n-p)+n-1} \mathrm{d}r \\ &= C_2 \int_{e^{-k}}^1 \frac{1}{r} \mathrm{d}r \\ &= C_2(\log(1) - \log(e^{-k})) \\ &= C_2 k,\end{aligned} \tag{24}$$

where $C_2 = n\sigma_n C_1^{\frac{p}{q}}$. Choosing $k \in \mathbb{N}$ large enough, for example

$$k > (C C_2^{\frac{1}{p}} + c)^{\frac{p}{p-1}},$$

we use (23) and (24) to conclude (19) for $p > 1$.

We use the same sequence of functions and measures to satisfy the remaining inequalities (20) to (22). In order to find k such that μ_k, f_k satisfies (21), let p, α, q and $\varepsilon > 0$ be as specified there. Then we have

$$\begin{aligned}\frac{1}{1(Q_0)^{\frac{\varepsilon}{q}}} \int_{Q_0} |\nabla f_k| \, (\mathrm{M}_{\alpha+\varepsilon} \mu_k)^{\frac{1}{q}} \,\mathrm{d}x &\le n^{\frac{\varepsilon}{2q}} \int_{B(0,1)} |\nabla f_k| \, (\mathrm{M}_{\alpha+\varepsilon} \mu_k)^{\frac{1}{q}} \,\mathrm{d}x \\ &\le n^{1+\frac{\varepsilon}{2q}} \sigma_n C_1^{\frac{1}{q}} \int_0^1 \frac{1}{r} 1_{\{r \ge e^{-k}\}} r^{(\alpha+\varepsilon-n)\frac{1}{q}} r^{n-1} \mathrm{d}r \\ &= C_3 \int_{e^{-k}}^1 r^{(\alpha+\varepsilon-n)\frac{1}{q}+n-2} \mathrm{d}r \\ &= C_3 \int_{e^{-k}}^1 r^{\frac{\varepsilon}{q}-1} \mathrm{d}r \\ &= \frac{C_3 q}{\varepsilon}(1 - e^{-\frac{\varepsilon}{q}k}),\end{aligned} \tag{25}$$

where $C_3 = n^{1+\frac{\varepsilon}{2q}} \sigma_n C_1^{\frac{1}{q}}$. Choosing $k \in \mathbb{N}$ large enough, for example

$$k > \frac{2^{\frac{\varepsilon}{q}} C C_3 q}{\varepsilon} + c,$$

we use (23) and (25) to conclude (21) for $\varepsilon > 0$.

In order to find k such that μ_k, f_k satisfies (20) and (22), we first note that by for example [15, Lemma 3.6], both $(\mathrm{M}_\alpha \mu)^{\frac{p}{q}}$ and $(\mathrm{M}_{\alpha+\varepsilon} \mu)^{\frac{1}{q}}$ are A_1-weights with

$$\left[(\mathrm{M}_\alpha \mu)^{\frac{p}{q}}\right]_{A_1} \le \frac{15^n 4n}{p\delta}$$

$$\left[(\mathrm{M}_{\alpha+\varepsilon}\mu)^{\frac{1}{q}}\right]_{A_1} \leq \frac{15^n 4n}{\delta + \frac{(n-\delta)}{n-\alpha}\varepsilon}.$$

Since by assumption we have $\delta > 0$ or $\delta > \frac{p-1}{p}$ respectively, we can apply Corollary 6.3 with $w = (\mathrm{M}_\alpha \mu)^{\frac{p}{q}}$ and $w = (\mathrm{M}_{\alpha+\varepsilon}\mu)^{\frac{1}{q}}$, and we conclude (20) and (22) from (19) and (21) respectively.

Funding Open Access funding provided by Aalto University.

Data Availibility Statement Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

1. Alberico, A., Cianchi, A., Pick, L., Slavíková, L.: On the limit as $s \to 0^+$ of fractional Orlicz–Sobolev spaces. J. Fourier Anal. Appl. **26**(6), Paper No. 80, 19 (2020)
2. Bourgain, J., Brezis, H., Mironescu, P.: Another look at Sobolev spaces. In: Optimal Control and Partial Differential Equations, pp. 439–455. IOS, Amsterdam (2001)
3. Bourgain, J., Brezis, H., Mironescu, P.: Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications, vol. 87, Dedicated to the memory of Thomas H. Wolff, pp. 77–101 (2002)
4. Brezis, H.: How to recognize constant functions. A connection with Sobolev spaces. Uspekhi Mat. Nauk **57**(4(346)), 59–74 (2002)
5. Brezis, H., Van Schaftingen, J., Yung, P.-L.: A surprising formula for Sobolev norms. Proc. Natl. Acad. Sci. USA **118**, no. 8, Paper No. e2025254118, 6 (2021)
6. Canto, J., Pérez, C.: Extensions of the John-Nirenberg theorem and applications. Proc. Am. Math. Soc. **149**(4), 1507–1525 (2021)
7. Cejas, M.E., Mosquera, C., Pérez, C., Rela, E.: Self-improving Poincaré–Sobolev type functionals in product spaces. Journal d'Analyse Mathématique. **149**, 1–48 (2022)
8. Chua, S.-K.: Embedding and compact embedding for weighted and abstract Sobolev spaces. Pac. J. Math. **303**(2), 519–568 (2019)
9. Drelichman, I., Durán, R.G.: The Bourgain–Brézis–Mironescu formula in arbitrary bounded domains. Proc. Am. Math. Soc. **150**(2), 701–708 (2022)
10. Dyda, B., Ihnatsyeva, L., Vähäkangas, A.V.: On improved fractional Sobolev–Poincaré inequalities. Ark. Mat. **54**(2), 437–454 (2016)
11. Dyda, B., Lehrbäck, J., Vähäkangas, A.V.: Fractional Poincaré and localized Hardy inequalities on metric spaces. Advances in Calculus of Variations (2022)
12. Evans, L.C., Gariepy, R.F.: Measure Theory and Fine Properties of Functions, revised CRC Press, Boca Raton (2015)
13. Franchi, B., Pérez, C., Wheeden, R.L.: Self-improving properties of John–Nirenberg and Poincaré inequalities on spaces of homogeneous type. J. Funct. Anal. **153**(1), 108–146 (1998)
14. Franchi, B., Pérez, C., Wheeden, R.L.: Sharp geometric Poincaré inequalities for vector fields and non-doubling measures. Proc. Lond. Math. Soc. (3) **80**(3), 665–689 (2000)
15. Hurri-Syrjänen, R., Martínez-Perales, J.C., Pérez, C., Vähäkangas, A.V.: On the BBM-phenomenon in fractional Poincaré–Sobolev inequalities with weights. Int. Math. Res. Not. (2022). https://doi.org/10.1093/imrn/rnac246
16. Hurri-Syrjänen, R., Martínez-Perales, J.C., Pérez, C., Vähäkangas, A.V.: On the weighted inequality between the Gagliardo and Sobolev seminorms, Israel J. Math. (2023)

17. Karadzhov, G.E., Milman, M., Xiao, J.: Limits of higher-order Besov spaces and sharp reiteration theorems. J. Funct. Anal. **221**(2), 323–339 (2005)
18. Lerner, A.K., Lorist, E., Ombrosi, S.: Operator-free sparse domination, Forum Math. Sigma **10**, Paper No. e15, 28 (2022)
19. Maz'ya, V., Shaposhnikova, T.: On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces. J. Funct. Anal. **195**(2), 230–238 (2002)
20. Maz'ya, V.: Sobolev Spaces with Applications to Elliptic Partial Differential Equations, 2nd, augmented Edition, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 342, Springer, Heidelberg (2011)
21. Meyers, N.G., Ziemer, W.P.: Integral inequalities of Poincaré and Wirtinger type for BV functions. Am. J. Math. **99**(6), 1345–1360 (1977)
22. Pérez, C., Rela, E.: Degenerate Poincaré–Sobolev inequalities. Trans. Am. Math. Soc. **372**(9), 6087–6133 (2019)
23. Ponce, A.C.: An estimate in the spirit of Poincaré's inequality. J. Eur. Math. Soc. (JEMS) **6**(1), 1–15 (2004)
24. Swanson, D.: Area, coarea, and approximation in $W^{1,1}$. Ark. Mat. **45**(2), 381–399 (2007)
25. Wendel, J.G.: Note on the gamma function. Am. Math. Mon. **55**, 563–564 (1948)