Almost-Orthogonality of Restricted Haar-Functions

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Abstract

We consider the Haar functions h_I on dyadic intervals. We show that if $p > \frac{2}{3}$ and $E \subset [0,1]$ then the set of all functions $\|h_I 1_E\|_2^{-1} h_I 1_E$ with $|I \cap E| \geq p|I|$ is a Riesz sequence. For $p \leq \frac{2}{3}$ we provide a counterexample.

1 Introduction

In this paper we prove a stability result for perturbed Haar functions. It grew out of the author's Master's thesis [6], written in Bonn under the supervision of Professor Christoph Thiele. It was motivated by an idea on how to to extend the result in [3] to three general functions.

The **Haar function** of the interval $I = [a, b)$ is given by

$$h_I = -1_{I^{\mathrm{l}}} + 1_{I^{\mathrm{r}}},$$

where $I^{\mathrm{l}} = [a, \frac{a+b}{2})$ and $I^{\mathrm{r}} = [\frac{a+b}{2}, b)$ are the left and right halves of I. We consider the Haar functions of the **dyadic intervals**

$$\mathcal{D} = \big\{ \big[k2^n, (k+1)2^n\big) \;\big|\; n, k \in \mathbb{Z} \big\}.$$

The main result of this paper is the following theorem.

Theorem 1.1. *For each $p > \frac{2}{3}$ there is a constant $c > 0$ such that for all measurable sets $E \subset [0,1)$ and all sequences $(a_I)_{I \in \mathcal{D}}$ with $a_I = 0$ if $|I \cap E| < p|I|$, we have*

$$\Big\| \sum_{I \in \mathcal{D}} a_I h_I 1_E \Big\|_2^2 \geq c \sum_{I \in \mathcal{D}} \|a_I h_I 1_E\|_2^2, \tag{1}$$

whenever the right-hand side is finite. For $p \leq \frac{2}{3}$ there is no such constant $c > 0$.

Remark. The proof strategy of Theorem 1.1 resembles the well known Bellman function technique as for example in [5]. A rephrasing of the proof which resembles the Bellman function technique more closely can be found in Section 2.1.5 in [6].

The proof also yields an explicit value for c. We discuss its optimality in Section 4. Furthermore, if the right-hand side of (1) converges, then the sum on the left-hand-side converges in L^2 because for any finite subset $D_0 \subset \mathcal{D}$ we have

$$\Big\| \sum_{I \in D_0} a_I h_I 1_E \Big\|_2^2 \le \Big\| \sum_{I \in D_0} a_I h_I \Big\|_2^2 = \sum_{I \in D_0} \|a_I h_I\|_2^2 \le \frac{1}{p} \sum_{I \in D_0} \|a_I h_I 1_E\|_2^2. \tag{2}$$

This also implies (2) with $D_0 = \mathcal{D}$, which means that a reverse inequality of (1) holds as for all $p > 0$.

In a more general setting, a sequence V in a Hilbert space is called a **Bessel sequence** if

$$\Big\| \sum_{v \in V} a_v v \Big\|^2 \le C \sum_{v \in V} |a_v|^2$$

holds, and a **Riesz sequence** if in addition also

$$\Big\| \sum_{v \in V} a_v v \Big\|^2 \ge c \sum_{v \in V} |a_v|^2$$

holds, where the constants $c > 0$ and $C < \infty$ respectively are independent of $(a_v)_{v \in V} \subset \ell^2(V)$. Inserting $\frac{a_I}{\|h_I 1_E\|_2}$ for a_I in (2) shows that for all $p > 0$, the sequence

$$V = \left\{ \frac{h_I 1_E}{\|h_I 1_E\|_2} \;\middle|\; I \in \mathcal{D},\ |I \cap E| \ge p|I| \right\} \tag{3}$$

is a Bessel sequence with constant $\frac{1}{p}$. Theorem 1.1 states that if $p > \frac{2}{3}$ then (3) is also a Riesz sequence. A weaker result already follows from the well-known **Kadison-Singer Problem**, which was resolved recently by Marcus, Spielman and Srivastava in [4]. In doing so, they also solved the numerous equivalent problems, one of which is the **Feichtinger Conjecture**, which states that every Bessel sequence can be partitioned into finitely many Riesz sequences. This means it can already be concluded from (2) that there is a finite partition of (3) into Riesz sequences. Building upon [4], Bownik, Casazza, Marcus and Speegle also proved a quantitative version of the Feichtinger Conjecture in [1]. For the specific setting of restricted Haar functions, their Corollary 6.5 in [1] reads that if $p > \frac{3}{4}$ then (3) can be partitioned into two Riesz sequences. Theorem 1.1 improves on that because it already applies for $p > \frac{2}{3}$ and states that (3) is already a Riesz sequence prior to partitioning.

For more details on the relation of this paper to other work; see Section 4.

2 Proof of the Case $p > \frac{2}{3}$

For $n \in \mathbb{N}_0$ denote $\mathcal{D}_n = \{I \in \mathcal{D} \mid |I| \ge 2^{-n}\}$. The idea is to first prove a weighted inequality,

$$\Big\| \sum_{I \in \mathcal{D}_n} a_I h_I 1_E \Big\|_{L^2(w_n)}^2 \ge \sum_{I \in \mathcal{D}_n} \|a_I h_I 1_E\|_2^2.$$

The weights are introduced in order to allow a proof by induction on n. They will be uniformly bounded from above, so that the case $p > \frac{2}{3}$ of Theorem 1.1 follows from the weighted inequality.

The weights w_n look as follows: Fix $\frac{2}{3} < p \leq 1$. Define $g : [0,1] \to \mathbb{R}$ by

$$g(q) = \begin{cases} 1 + \frac{p(2-p)}{(3p-2)(3p-2q)} & q \geq p, \\ g(p)\frac{q}{p} & q \leq p. \end{cases} \tag{4}$$

It is well defined because $2q \leq 2 < 3p$. Now on each interval $I \in \mathcal{D}_{n+1} \setminus \mathcal{D}_n$ abbreviate $q_I = \frac{|I \cap E|}{|I|}$ and assign w_n the constant value $\frac{g(q_I)}{q_I}$ if $q_I > 0$. If $q_I = 0$ then the value of w_n does not matter. The properties of g that we need are collected in the following proposition.

Proposition 2.1. Let $\frac{2}{3} < p \leq 1$. Then the function g has the following properties:

1) For all $q_1, q_2 \in [0,1]$ and $a \in \mathbb{R}$ with $\frac{q_1+q_2}{2} \geq p$ or $a = 0$ we have

$$\frac{(1-a)^2}{2} g(q_1) + \frac{(1+a)^2}{2} g(q_2) - g\Big(\frac{q_1+q_2}{2}\Big) \geq a^2. \tag{5}$$

2) There is a constant $C > 0$ s.t. for all $q \in [0,1]$ we have

$$q \leq g(q) \leq Cq. \tag{6}$$

Proposition 2.1 is the crucial step. After that it requires not much more than bookkeeping to conclude the case $p > \frac{2}{3}$ of Theorem 1.1. In order to prove Proposition 2.1, we first show the Lemmas 2.2 and 2.3.

Define $\tilde{g} : [0,1] \to \mathbb{R}$ by

$$\tilde{g}(q) = 1 + \frac{p(2-p)}{(3p-2)(3p-2q)},$$

so that $g = \tilde{g}$ on $q \geq p$.

Lemma 2.2. Let $q_1, q_2 \in [0,1]$, $a \in \mathbb{R}$. Then

$$\frac{(1-a)^2}{2} \tilde{g}(q_1) + \frac{(1+a)^2}{2} \tilde{g}(q_2) - \tilde{g}\Big(\frac{q_1+q_2}{2}\Big) \geq a^2.$$

Proof. For $i = 1,2$ take x_i s.t.

$$\tilde{g}(q_i) = 1 + \frac{1}{x_i}.$$

Then $x_i > 0$ and

$$\tilde{g}\Big(\frac{q_1+q_2}{2}\Big) = 1 + \frac{2}{x_1+x_2}.$$

Hence it suffices to confirm the positivity of

$$\begin{aligned}
&\frac{(1-a)^2}{2}\Big(1+\frac{1}{x_1}\Big) + \frac{(1+a)^2}{2}\Big(1+\frac{1}{x_2}\Big) - \Big(1+\frac{2}{x_1+x_2}\Big) - a^2 \\
&= \frac{1}{2}a^2\Big(\frac{1}{x_1}+\frac{1}{x_2}\Big) + a\Big(\frac{1}{x_2}-\frac{1}{x_1}\Big) + \frac{1}{2}\Big(\frac{1}{x_1}+\frac{1}{x_2}-4\frac{1}{x_1+x_2}\Big) \\
&= \frac{1}{x_1x_2}\Big[\frac{1}{2}a^2(x_1+x_2) + a(x_1-x_2) + \frac{1}{2}\Big(x_1+x_2-4\frac{x_1x_2}{x_1+x_2}\Big)\Big]
\end{aligned}$$

which is a quadratic polynomial in a. Since $x_1 + x_2 \geq 0$ and $\frac{1}{x_1 x_2} \geq 0$ it has a positive leading coefficient. The discriminant is

$$\begin{aligned}&(x_1+x_2)\Big(x_1+x_2-4\frac{x_1x_2}{x_1+x_2}\Big)-(x_1-x_2)^2\\&=(x_1+x_2)^2-4x_1x_2-(x_1-x_2)^2=0,\end{aligned}$$

and so the minimum of the polynomial is zero. □

Lemma 2.3.

$$g(2p-1)=\tilde{g}(2p-1).$$

Proof.

$$\begin{aligned}g(2p-1)=g(p)\frac{2p-1}{p}&=\Big[1+\frac{p(2-p)}{p(3p-2)}\Big]\frac{2p-1}{p}\\&=\frac{2p}{3p-2}\frac{2p-1}{p}=1+\frac{p}{3p-2}=\tilde{g}(2p-1).\end{aligned}$$

□

Proof of Proposition 2.1. Lemma 2.2 with $a = 0$ implies that $\tilde{g}$ is midpoint convex and thus convex. By Lemma 2.3 and by the definition of $g(p)$ we have that $g(q) = \tilde{g}(q)$ at the two values $q = 2p-1, p$. This means that on $[0, p]$ the function g describes the line that passes through these two distinct points. On $[p, 1]$ recall that $g = \tilde{g}$. It follows from this that also g is convex. This means that (5) holds for $a = 0$ and so it suffices to consider the case $\frac{q_1+q_2}{2} \in [p, 1]$. There we have $q_1 \geq 2p - q_2 \geq 2p-1$ and similarly $q_2 \geq 2p-1$. From the considerations above we then get $g(q_1) \geq \tilde{g}(q_1)$, $g(q_2) \geq \tilde{g}(q_2)$, $g\big(\frac{q_1+q_2}{2}\big) = \tilde{g}\big(\frac{q_1+q_2}{2}\big)$ which implies that for all a we have

$$\begin{aligned}&\frac{(1-a)^2}{2}g(q_1)+\frac{(1+a)^2}{2}g(q_2)-g\Big(\frac{q_1+q_2}{2}\Big)\\&\geq\frac{(1-a)^2}{2}\tilde{g}(q_1)+\frac{(1+a)^2}{2}\tilde{g}(q_2)-\tilde{g}\Big(\frac{q_1+q_2}{2}\Big)\geq a^2,\end{aligned}$$

where the last inequality follows from Lemma 2.2. This finishes the proof of (5).

The upper bound in (6) holds for $C = g(1)$ because g is convex and non-negative and $g(0) = 0$. For the lower bound, recall that for $q \in [0, p]$ we have $g(q) = \frac{q}{p}g(p)$, so that convexity implies $g(q) \geq \frac{q}{p}g(p)$ for all $q \in [0, 1]$. It follows from the definition of g that $g(p) \geq 1$ and therefore $g(q) \geq \frac{q}{p} \geq q$. □

The following lemma translates Proposition 2.1 into our setting of Haar functions on weighted L^2 spaces.

Lemma 2.4. For every $n \in \mathbb{N}$ we have

$$\Big\|\sum_{I\in\mathcal{D}_{n+1}}a_Ih_I1_E\Big\|^2_{L^2(w_{n+1})}-\Big\|\sum_{I\in\mathcal{D}_n}a_Ih_I1_E\Big\|^2_{L^2(w_n)}\geq\sum_{I\in\mathcal{D}_{n+1}\setminus\mathcal{D}_n}\|a_Ih_I1_E\|_2^2 \quad (7)$$

and

$$1\leq w_n\leq C. \quad (8)$$

The constant C is the same as in Proposition 2.1.

Proof. Recall that on $I \in \mathcal{D}_{n+1} \setminus \mathcal{D}_n$ we assign w_n the constant value $\frac{g(q_I)}{q_I}$ with $q_I = \frac{|I \cap E|}{|I|}$ if $q_I > 0$. Where $|I \cap E|$ vanishes, the value of w_n does not matter because the integrated function in (7) is zero a.e. on such I anyways.

First note that (8) is equivalent to (6). We prove (7) using mostly (5). Partition the domain of integration on the left-hand-side of (7) into $\mathcal{D}_{n+1} \setminus \mathcal{D}_n$ so that the inequality becomes

$$\sum_{J \in \mathcal{D}_{n+1} \setminus \mathcal{D}_n} \int_{E \cap J} \Big[\Big(\sum_{I \in \mathcal{D}_{n+1}} a_I h_I \Big)^2 w_{n+1} - \Big(\sum_{I \in \mathcal{D}_n} a_I h_I \Big)^2 w_n \Big]$$
$$\geq \sum_{J \in \mathcal{D}_{n+1} \setminus \mathcal{D}_n} \int_E (a_J h_J)^2.$$

We prove this inequality for each summand $J \in \mathcal{D}_{n+1} \setminus \mathcal{D}_n$ individually. So fix $J \in \mathcal{D}_{n+1} \setminus \mathcal{D}_n$. Then for each $I \in \mathcal{D}_n$ the function h_I is constant on J. That means we may write

$$1_J \sum_{I \in \mathcal{D}_n} a_I h_I = b_J 1_J$$

for some $b_J \in \mathbb{R}$. For $I \in \mathcal{D}_{n+1} \setminus \mathcal{D}_n$ the function h_I is nonzero if and only if $I = J$, so that

$$1_J \sum_{I \in \mathcal{D}_{n+1}} a_I h_I = b_J 1_J + a_J h_J.$$

So it suffices to show

$$\int_{E \cap J} \Big[(b_J 1_J + a_J h_J)^2 w_{n+1} - (b_J 1_J)^2 w_n \Big] \geq \int_E (a_J h_J)^2 \tag{9}$$

in order to prove (7). Write

$$q_1 = \frac{|J^{\mathrm{l}} \cap E|}{|J^{\mathrm{l}}|}, \qquad q_2 = \frac{|J^{\mathrm{r}} \cap E|}{|J^{\mathrm{r}}|}, \qquad \frac{q_1 + q_2}{2} = \frac{|J \cap E|}{|J|},$$

so that we have

$$1_{J^{\mathrm{l}}} w_{n+1} = \frac{g(q_1)}{q_1} 1_{J^{\mathrm{l}}}, \qquad 1_{J^{\mathrm{r}}} w_{n+1} = \frac{g(q_2)}{q_2} 1_{J^{\mathrm{r}}}, \qquad 1_J w_n = \frac{g\big(\frac{q_1+q_2}{2}\big)}{\frac{q_1+q_2}{2}} 1_J.$$

if the respective denominators are positive. Evaluating the integrals, (9) then reads

$$(b_J - a_J)^2 |J^{\mathrm{l}}| g(q_1) + (b_J + a_J)^2 |J^{\mathrm{r}}| g(q_2) - b_J^2 |J| g\Big(\frac{q_1 + q_2}{2}\Big) \geq |J| \frac{q_1 + q_2}{2} a_J^2,$$

also if q_1 or q_2 are zero because $g(0) = 0$. Then divide both sides by $|J|$. For $b_J = 0$ we obtain

$$a_J^2 \frac{g(q_1) + g(q_2)}{2} \geq \frac{q_1 + q_2}{2} a_J^2.$$

This inequality holds due to the lower bound in (6). In case $b_J \neq 0$ we additionally divide by b_J^2 and obtain

$$\Big(1 - \frac{a_J}{b_J}\Big)^2 \frac{1}{2} g(q_1) + \Big(1 + \frac{a_J}{b_J}\Big)^2 \frac{1}{2} g(q_2) - g\Big(\frac{q_1 + q_2}{2}\Big) \geq \frac{q_1 + q_2}{2} \Big(\frac{a_J}{b_J}\Big)^2.$$

This inequality is a consequence of (5) because $\frac{q_1+q_2}{2} \leq 1$. Note that we can also obtain the case $b_J = 0$ from (5) by sending $a_J \to \infty$, instead of from (6). That way we would even get the stronger inequality without the factor $\frac{q_1+q_2}{2}$. $\square$

Proof of Theorem 1.1 in case $p > \frac{2}{3}$. We use Lemma 2.4. Because $\mathcal{D}_0 = \{[0,1)\}$ consists of only one interval we get

$$\Big\| \sum_{I\in\mathcal{D}_0} a_I h_I 1_E \Big\|^2_{L^2(w_0)} \geq \sum_{I\in\mathcal{D}_0} \|a_I h_I 1_E\|_2^2 \tag{10}$$

from the lower bound in (8). Summing up (10) and (7) for $n = 1, \ldots, k-1$ we get

$$\Big\| \sum_{I\in\mathcal{D}_k} a_I h_I 1_E \Big\|^2_{L^2(w_k)} \geq \sum_{I\in\mathcal{D}_k} \|a_I h_I 1_E\|_2^2,$$

and the upper bound in (8) allows us to get rid of the weight

$$C\Big\| \sum_{I\in\mathcal{D}_k} a_I h_I 1_E \Big\|^2_2 \geq \Big\| \sum_{I\in\mathcal{D}_k} a_I h_I 1_E \Big\|^2_{L^2(w_k)}.$$

This proves the part $p > \frac{2}{3}$ of Theorem 1.1 with $c = \frac{1}{C}$ if we only consider finite sums. For infinite sums where the right hand side of (1) converges, we may also pass to the limit $n \to \infty$ with the help of (2). $\square$

3 Proof of the Case $p \leq \frac{2}{3}$

Fix $E = [0, \frac{2}{3}]$. We build the counterexample from the sequence of dyadic intervals $(I_{2n})_{n=0,1,\ldots}$, defined inductively by

$$\begin{aligned} I_0 &= [0,1], \\ I_{2n+1} &= {I_{2n}}^{\mathrm{r}}, \\ I_{2n+2} &= {I_{2n+1}}^{\mathrm{l}}. \end{aligned}$$

Lemma 3.1. For all $n = 0, 1, \ldots$ we have

$$|I_n| = 2^{-n}, \qquad |I_{2n} \cap E| = \frac{2}{3}|I_{2n}|, \qquad |I_{2n+1} \cap E| = \frac{1}{3}|I_{2n+1}|.$$

Proof. It is clear that we have $|I_n| = 2^{-n}$. For the other statements we proceed by induction on n. For $n = 0$ we have

$$\begin{aligned} \left|[0,1] \cap \left[0, \frac{2}{3}\right]\right| &= \frac{2}{3} = \frac{2}{3}|[0,1]|, \\ \left|\left[\frac{1}{2}, 1\right] \cap \left[0, \frac{2}{3}\right]\right| &= \frac{2}{3} - \frac{1}{2} = \frac{1}{6} = \frac{1}{3}\left|\left[\frac{1}{2}, 1\right]\right|. \end{aligned}$$

Now assume the lemma holds for n. That means that the point $\frac{2}{3}$ lies $\frac{1}{3}|I_{2n+1}|$ to the right from the left boundary of I_{2n+1}, i.e. $\frac{2}{3}|I_{2(n+1)}|$ to the right from the left boundary of $I_{2(n+1)}$. Therefore we have $|I_{2(n+1)} \cap E| = \frac{2}{3}|I_{2(n+1)}|$. That in turn implies that the point $\frac{2}{3}$ lies $\frac{1}{6}|I_{2(n+1)}|$ to the right from the midpoint of $I_{2(n+1)}$, i.e.$\frac{1}{3}|I_{2(n+1)+1}|$ to the right from the left boundary of $I_{2(n+1)+1}$. Therefore we have $|I_{2(n+1)+1} \cap E| = \frac{1}{3}|I_{2(n+1)+1}|$, finishing the proof of the lemma for $n+1$. $\square$

Further set

$$a_0 = 1,$$
$$a_{2n} = 2^{n-1}, \qquad n \geq 1.$$

The following proposition proves the case $p \leq \frac{2}{3}$ of Theorem 1.1.

Proposition 3.2. For each n we have

$$\sum_{k=0}^{n} \|a_{2k} h_{I_{2k}} 1_E\|_2^2 = \frac{2}{3} + \frac{n}{6}, \tag{11}$$

$$\Big\|\sum_{k=0}^{n} a_{2k} h_{I_{2k}} 1_E\Big\|_2^2 = \frac{2}{3}. \tag{12}$$

Proof. By Lemma 3.1 we have $|I_{2n} \cap E| = \frac{2}{3} 2^{-2n}$. Thus

$$\|a_0 h_{I_0} 1_E\|_2^2 = \frac{2}{3},$$
$$\|a_{2n} h_{I_{2n}} 1_E\|_2^2 = \frac{2}{3} 2^{-2n} 2^{2(n-1)} = \frac{1}{6}, \qquad n \geq 1,$$

which implies (11).

In order to prove (12), first note that the support of $h_{I_{2(n+1)}} 1_E$ is ${I_{2n}}^{\mathrm{r}} \cap E$. Therefore it follows by induction on n that

$$\sum_{k=0}^{n} a_{2k} h_{I_{2k}} 1_E = -1_{[0,\frac{1}{2})} + 2^n 1_{{I_{2n}}^{\mathrm{r}} \cap E},$$

since

$$2^n 1_{{I_{2n}}^{\mathrm{r}} \cap E} + 2^n h_{I_{2(n+1)}} 1_E = 2^{n+1} 1_{{I_{2(n+1)}}^{\mathrm{r}} \cap E}.$$

By Lemma 3.1 we have $|{I_{2n}}^{\mathrm{r}} \cap E| = \frac{1}{3} |{I_{2n}}^{\mathrm{r}}| = \frac{1}{3} 2^{-2n-1}$ so that we obtain

$$\Big\|\sum_{k=0}^{n} a_{2k} h_{I_{2k}} 1_E\Big\|_2^2 = \frac{1}{2} \cdot (-1)^2 + \frac{1}{3} 2^{-2n-1} 2^{2n} = \frac{2}{3}.$$

$\square$

4 Remarks

4.1 Optimality of the Constant c

From the proof we get an explicit expression for the constant in (1)

$$c = \frac{1}{C} = \frac{1}{g(1)} = \frac{(3p-2)^2}{(3p-2)^2 + p(2-p)} = \frac{\left(p - \frac{2}{3}\right)^2}{\left(p - \frac{2}{3}\right)^2 + \frac{1}{9} p(2-p)}$$

and since

$$p(2-p) = (p - \frac{2}{3})\left[\frac{4}{3} - (p - \frac{2}{3})\right] + \frac{2}{3}\left[\frac{4}{3} - (p - \frac{2}{3})\right] = \frac{8}{9} + \mathcal{O}\Big(p - \frac{2}{3}\Big)$$

we have

$$c = \frac{\left(p - \frac{2}{3}\right)^2}{\frac{8}{81} + \mathcal{O}(p - \frac{2}{3})} = \frac{81}{8}\Big(p - \frac{2}{3}\Big)^2 \frac{1}{1 + \mathcal{O}(p - \frac{2}{3})} = \frac{81}{8}\Big(p - \frac{2}{3}\Big)^2 + \mathcal{O}\Big(p - \frac{2}{3}\Big)^3.$$

However this constant c is likely not maximal because g satisfies a stronger bound than the required $g(q) \geq q$, and because we dropped a factor $\frac{q_1+q_2}{2}$ in the deduction of (5). We only did this because sending $q_1 \to q_2$ in (5) leads to an ODE with solution $\tilde{g}$, while with the factor $\frac{q_1+q_2}{2}$ in place we could not solve the ODE. We did however minimize C in some respect: There are multiple solutions to the ODE from (5) such that the corresponding g satisfies (6) and (5) with some C. Among all those, $\tilde{g}$ has the smallest C for $p \to \frac{2}{3}$. For a proof of this and for more details; see Section 2.1 in [6]. In Section 3 in [6] we also provide a set E for which we prove an explicit sharp constant c_{s} which satisfies $c_{\mathrm{s}} = 27\big(p - \frac{2}{3}\big)^2 + \mathcal{O}(p - \frac{2}{3})^3$. We conjecture that (1) holds already with this particular constant c_{s}. In Section 4 of [6] we prove that this is indeed the case at least for certain E.

4.2 Further remarks on Theorem 1.1

For $p > \frac{2}{3}$ inequality (1) still holds if we add the constant function to the sums, i.e. allow $a_{[0,2)} \neq 0$, even though usually $|[0,2) \cap E| < p|[0,2)|$.

Furthermore, Theorem 1.1 is not a consequence of the fact that $\big\{h_I 1_E \;\big|\; |I \cap E| \geq p|I|\big\}$ is only a small perturbation of the orthogonal set $\big\{h_I \;\big|\; |I \cap E| \geq p|I|\big\}$, in the sense that

$$\|h_I - h_I 1_E\|_2^2 \leq (1-p)\|h_I\|_2^2 < \frac{1}{3}\|h_I\|_2^2.$$

In order to see this, consider the following example. Assume that $u_1, \ldots, u_n$ are orthonormal. Abbreviate $u = u_1 + \ldots + u_n$ and for $i = 1, \ldots, n$ set

$$u_i' = u_i - \frac{1}{n} u.$$

Then

$$\|u_i - u_i'\|^2 = \frac{1}{n^2}\|u\|^2 = \frac{1}{n^2}\sum_{i=1}^{n}\|u_i\|^2 = \frac{1}{n}.$$

but

$$\|u_1' + \ldots + u_n'\|^2 = \|u - u\|^2 = 0 \not\gtrsim \|u_1'\|^2 + \ldots + \|u_n'\|^2.$$

4.3 Related Topics

The starting point of this work was the following question, because its answer could provide ideas on how to to extend the result in [3] to three general functions.

Question 1. *Let $\mathcal{D}$ be the set of dyadic intervals of* $[0,1)$*. Let* $[0,1) = E_0 \cup E_1$ *be a partition. Is there a partition $\mathcal{D} = D_0 \cup D_1$ such that for $i = 0, 1$ the equivalence*

$$\Big\|\sum_{I \in D_i} a_I h_I 1_{E_i}\Big\|_2^2 \sim \sum_{I \in D_i} \|a_I h_I 1_{E_i}\|_2^2 \tag{13}$$

is true?

We started investigating this question in Section 5 in [6]. An initial approach to Question 1 could be to construct a partition by a majority decision: For $i = 0, 1$ take D_i s.t. for all $I \in D_i$ we have

$$|I \cap E_i| \geq \frac{1}{2}|I|. \tag{14}$$

However by the counterexample of Theorem 1.1 for $p = \frac{2}{3} \geq \frac{1}{2}$, this strategy does not result in the lower bound in (13). Although by (2) the majority decision (14) at least leads to the upper bound in (13). Another idea was to use the Feichtinger-Conjecture which was recently resolved by Marcus, Spielman and Srivastava in [4]. Based on [4], Bownik, Casazza, Marcus and Speegle proved a quantified version of the Feichtinger Conjecture in [1]. The following theorem is a reformulation of Corollary 6.5 in [1].

Theorem 4.1. *Let $C < \frac{4}{3}$ and $c = \frac{C}{2} - \sqrt{2(C-1)(2-C)}$. Let V be a sequence in a Hilbert space such that for all $(a_v)_{v \in V} \subset \mathbb{R}$*

$$\Big\| \sum_{v \in V} a_v v \Big\|^2 \leq C \sum_{v \in V} \|a_v v\|^2.$$

Then there is a partition $V = V_0 \cup V_1$ such that for $i = 0, 1$ we have

$$\Big\| \sum_{v \in V_i} a_v v \Big\|^2 \geq c \sum_{v \in V_i} \|a_v v\|^2.$$

Unfortunately it is not clear if Theorem 4.1 can be used to answer Question 1. The closest consequence of Theorem 4.1 in that direction that we found is the following corollary.

Corollary 4.2. Let $p > \frac{3}{4}$ and $c = \frac{1}{2p} - \sqrt{2(\frac{1}{p} - 1)(2 - \frac{1}{p})}$. Let $E \subset [0, 1)$ and $E = E_0 \cup E_1$ be a partition and for $i = 0, 1$ set

$$H_i = \big\{ h_I \mathbb{1}_{E_i} \;\big|\; I \in \mathcal{D},\ |I \cap E_i| \geq p|I| \big\}.$$

Then $H_0 \cup H_1$ can be partitioned into $G_0 \cup G_1$ where for $i = 0, 1$ we have

$$\Big\| \sum_{v \in G_i} a_v v \Big\|^2 \geq c \sum_{v \in G_i} \|a_v v\|^2. \tag{15}$$

Theorem 1.1 can be seen as an improvement of Corollary 4.2. That is because by Theorem 1.1 the two sequences H_0 and H_1 already satisfy (15) with some other constant $c > 0$, and since H_0 and H_1 are orthogonal to one another, also their union satisfies (15), even without partitioning. And this already holds for $p > \frac{2}{3}$.

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