Almost-Orthogonality of Restricted Haar Functions – Updated Version

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Abstract

We consider the set of Haar functions $\left\{ h_I \;\middle|\; I \text{ dyadic}, \; I \subset [0,1) \right\}$ restricted to sets $E \subset [0,1)$. We show that if $p > \frac{2}{3}$ and $E \subset [0,1)$ then the set of all normalized functions $h_I \mathbb{1}_E$ with $|I \cap E| \geq p|I|$ is a Riesz basic sequence. The proof can be seen as an instance of the Bellman function technique. For $p \leq \frac{2}{3}$ we provide a counterexample. We further extend this result to a slightly more general setting where for each p we additionally formulate a guess for the optimal constant that holds for all Riesz basic sequences. For certain sequences we prove this constant.

Contents

1 Introduction

1.1 Setting

Riesz basic sequences Let H be a Hilbert space and V a set of vectors. Then V is called a *Bessel sequence* if there is a C such that for all finite subsets $U \subset V$ and $u \in H$ we have

$$\sum_{v \in U} |\langle u, v\rangle|^2 \leq C\|u\|^2. \tag{1}$$

Note that this is equivalent to asking (1) to hold for at most countable subsets. This in turn is the definition of the analysis operator

$$u \mapsto \big(\langle u, v\rangle\big)_{v \in U}$$

being a map $H \to l^2$ with bound C. This is the case if and only if the synthesis operator

$$(a_v)_{v \in U} \mapsto \sum_{v \in U} a_v v,$$

the dual of the analysis operator, has bound C as an operator $l^2 \to H$. This means that the condition given by (1) is equivalent to

$$\|\sum_{v \in U} a_v v\|^2 \leq C \sum_{v \in U} |a_v|^2 \tag{2}$$

If (1) and for some c its reverse inequality

$$\|u\|^2 \leq c \sum_{v \in U} |\langle u, v\rangle|^2 \tag{3}$$

holds, the vectors are called *frame*. If instead the reverse inequality of (2)

$$\sum_{v \in U} |a_v|^2 \leq c\|\sum_{v \in U} a_v v\|^2 \tag{4}$$

holds in addition to (1), they are called *Riesz basic sequence*.

Example. Consider the case that V is finite. Then

- V is a Bessel sequence.
- V is a frame if and only if it is spanning.
- V is a Riesz basic sequence if and only if it is linearly independent.

Hence neither of the conditions given by (3) and (4) imply the respective other.

Restricted Haar functions For an interval $I := [a, b)$ we write $\mathfrak{l}(I) := [a, \frac{a+b}{2})$ and $\mathfrak{r}(I) := [\frac{a+b}{2}, b)$. We denote by h_I the *Haar function* of I,

$$h_I := -\mathbb{1}_{\mathfrak{l}(I)} + \mathbb{1}_{\mathfrak{r}(I)}$$

and further denote

$$h_{I,E} := h_I \mathbb{1}_E.$$

We call the latter a *restricted Haar function*.

We call two intervals I_1, I_2 *compatible* if they are disjoint or one is contained in one half of the other. We call a set of intervals $\mathbb{I}$ compatible if all $I, J \in \mathbb{I}$ with $I \neq J$ are compatible. For example the set of dyadic intervals contained in $[0, 1)$, which we denote by $\mathscr{D}$, is compatible. We will be mostly interested in the case of dyadic intervals, but the compatibility of intervals is actually the only property we need. Note that for a set of compatible intervals $\mathbb{I}$, $\left\{\frac{h_I}{\|h_I\|_2} \mid I \in \mathbb{I}\right\}$ is an orthogonal subset of $L^2\big([0, 1)\big)$. A few questions are even more easily answered in the setting of compatible intervals than in the dyadic setting.

Let $p \in [0, 1]$ and $E \subset \mathbb{R}$. An interval I is called (E, p)-*dominant* if $|I \cap E| \geq p|I|$.

1.2 Main Result

The main result of this work is the following Theorem 1.1.

Theorem 1.1. *$p > \frac{2}{3}$ if and only if for all $E \subset [0, 1)$ and compatible sets $\mathbb{I}$ of (E, p)-dominant intervals,*

$$\left\{ \frac{h_{I,E}}{\|h_{I,E}\|_2} \;\middle|\; I \in \mathbb{I} \right\} \tag{5}$$

is a Riesz basic sequence.

Proof. The theorem will be a consequence of Theorem 2.1 and Theorem 2.3. □

For the reverse direction of the theorem we show that for $\mathbb{I} = \mathscr{D}$, $p = \frac{2}{3}$, $E = [0, \frac{2}{3})$, (5) is not a Riesz basic sequence.

For every $p > \frac{2}{3}$ we also compute a constant $c(p)$ with respect to which all (5) are Riesz basic sequences; see Theorem 2.1. Then in Section 3 we compute the optimal constant for $E = [0, p)$ and a certain Riesz sequence; see Theorem 3.1 and Proposition 3.2. For p close to $\frac{2}{3}$ it is by a factor $\frac{8}{3}$ greater than the constant we find for the general case in Theorem 2.1. In Section 4 we show that the constant for that special case is actually valid for certain types of Riesz basic sequences; see Theorem 4.1 and Theorem 4.7. We currently believe that this constant is actually also the optimal constant for the general case, i.e. that Theorem 2.1 still holds with this constant.

Remark. It can also be proven directly from Theorem 1.1 that the constant of the Riesz basic sequence (5) can be chosen to only depend on p; see Lemma A.15.

Remark. The statements and constants do not change if we also allow the function $\frac{\mathbb{1}_E}{\|\mathbb{1}_E\|_2}$ to be in (5). We may also express this by allowing the interval $[-1, 1)$ to be in $\mathbb{I}$, even though $[-1, 1)$ is never E-dominant for $p > \frac{1}{2}$. For a proof; see Lemma A.14.

Note that any $I \subset [0, 1)$ is compatible to $[-1, 1)$. Allowing $[-1, 1) \in \mathbb{I}$ simplifies the arguments in Sections 3 and 4.

1.3 Related Topics

The initial question of the thesis was the following:

Question 1. *Let $\mathscr{D}$ be the set of dyadic intervals of $[0, 1)$. Let $[0, 1) = E_0 \cup E_1$ be a partition. Is there a partition $\mathscr{D} = \mathscr{D}_0 \cup \mathscr{D}_1$ such that for $i = 0, 1$*

$$\left\{ \frac{h_{I,E_i}}{\|h_{I,E_i}\|_2} \;\middle|\; I \in \mathscr{D}_i \right\} \tag{6}$$

is a Riesz basic sequence?

By this we mean that if $|I \cap E_i| = 0$ then we must put I into $\mathscr{D}_{1-i}$. We were not able to answer this question. However in Section 5 we prove a first result.

An initial approach to Question 1 could be to construct a partition by a majority decision: For $i = 0, 1$ take $\mathscr{D}_i$ s.t. for all $I \in \mathscr{D}_i$ we have

$$|I \cap E_i| \geq \frac{1}{2}|I|. \tag{7}$$

However by Theorem 1.1 with $p = \frac{1}{2}$ this strategy does not produce Riesz basic sequences for all E. It even fails if we take some $p \leq \frac{2}{3}$ and don't assign the intervals with $|I \cap E| < p|I|$ to either of $\mathscr{D}_1$ and $\mathscr{D}_2$.[1] What the majority decision (7) does achieve however, is that (6) is a Bessel sequence: For $i = 0, 1$ let $(a_I)_{I \in \mathscr{D}_i} \in l^2(\mathscr{D}_i)$. Define $\tilde{a}_I = (\frac{|I|}{|I \cap E_i|})^{\frac{1}{2}} a_I$. Then

$$\Big\| \sum_{I \in \mathscr{D}_i} a_I \frac{h_{I,E_i}}{\|h_{I,E_i}\|_2} \Big\|_2^2 = \Big\| \sum_{I \in \mathscr{D}_i} \tilde{a}_I \frac{h_{I,E_i}}{|I|^{\frac{1}{2}}} \Big\|_2^2 = \Big\| \sum_{I \in \mathscr{D}_i} \tilde{a}_I \frac{h_I}{|I|^{\frac{1}{2}}} \mathbb{1}_{E_i} \Big\|_2^2$$

[1] In the particular counterexample $E = [0, \frac{2}{3})$ in Theorem 2.3 we actually assign all intervals. However the proof idea still works for $E = [\frac{1}{4}, \frac{2}{3})$ where the dyadic interval $[0, \frac{1}{2})$ lies in E and $E^{\complement}$ with a portion of only $\frac{1}{2}$ each and hence is not assigned.

$$\leq \Big\| \sum_{I \in \mathscr{D}_i} \tilde{a}_I \frac{h_I}{|I|^{\frac{1}{2}}} \Big\|_2^2 = \sum_{I \in \mathscr{D}_i} |\tilde{a}_I|^2 \leq 2 \sum_{I \in \mathscr{D}_i} |a_I|^2$$

By the same argument, actually for any $p > 0$ and any set of compatible (E, p)-dominant intervals $\mathbb{I}$, (5) is a Bessel sequence with constant $\frac{1}{p}$.

In [1] Bownik, Casazza, Marcus and Speegle proved the following Theorem 1.2, based on the resolution of the Kadison-Singer problem by Marcus, Spielman and Srivastava in [3]. It directly implies Corollary 1.3, a weaker version of Theorem 1.1.

Theorem 1.2 (Corollary 6.5 in [1])**.** *Let* $0 < \delta_0 < \frac{1}{4}$, $\varepsilon_0 = \frac{1}{2} - \sqrt{2\delta_0(1 - 2\delta_0)}$. *Let* Φ *be a finite Bessel sequence with bound* 1 *where for all* $\varphi \in \Phi$ *we have* $\|\varphi\|_2^2 \geq 1 - \delta_0$. *Then there is a partition* $\Phi = \Phi_0 \cup \Phi_1$ *such that* Φ_0 *and* Φ_1 *are Riesz sequences with constant* ε_0.

As they note in [1], it can be extended to countable sequences.

Corollary 1.3. Let $p > \frac{3}{4}$ and $E \subset [0, 1)$. Then

$$\left\{ \frac{h_{I,E_0}}{\|h_{I,E_0}\|_2} \;\middle|\; I \in \mathscr{D},\ |I \cap E_0| \geq p|I| \right\} \tag{8}$$

$$\cup \left\{ \frac{h_{I,E_1}}{\|h_{I,E_1}\|_2} \;\middle|\; I \in \mathscr{D},\ |I \cap E_1| \geq p|I| \right\} \tag{9}$$

can be partitioned into two Riesz sequences with lower bound $\frac{1}{2} - \sqrt{2(1-p)(2p-1)}$.

Note that Theorem 1.1 says, that (8) and (9) are already Riesz basic sequences. And since they are orthogonal to one another, by Lemma A.2 also their union is already a Riesz basic sequence prior to partitioning. Also this already holds for $p > \frac{2}{3}$.

Proof of Corollary 1.3. Let $\delta_0 = 1 - p \in (0, \frac{1}{4})$. For $i = 0, 1$ consider

$$H_i := \left\{ \frac{h_{I,E_i}}{\|h_I\|_2} \;\middle|\; I \in \mathscr{D},\ |I \cap E_i| \geq p|I| \right\}$$

Similarly to what we already discussed above, H_0 and H_1 are Bessel sequences with bound 1. And because H_0 and H_1 are orthogonal to one another, by Lemma A.3 also $H_0 \cup H_1$ is a Bessel sequence. Furthermore for $I \in \mathscr{D}$ and $i = 0, 1$ with $|I \cap E_i| \geq p|I|$ we have

$$\|h_{I,E_i}\|_2^2 = |I \cap E_i| \geq p|I| = p\|h_I\|_2^2 = (1 - \delta_0)\|h_I\|_2^2.$$

Hence we can apply Theorem 1.2 and obtain that $H_0 \cup H_1$ can be partitioned into two Riesz basic sequences with constant $\frac{1}{2} - \sqrt{2(1-p)(2p-1)}$. Now for each I with the corresponding i replace $\frac{h_{I,E_i}}{\|h_I\|_2}$ by $\frac{h_{I,E_i}}{\|h_{I,E_i}\|_2}$. This only increases the norm of the vectors and hence conserves the constant of the Riesz sequence. □

The advantage of Corollary 1.3 over Theorem 1.1 is that the former still has the chance to be strengthened to $p \leq \frac{2}{3}$. Assume that we can apply Corollary 1.3 to (8)∪(9) with $p = \frac{1}{2}$ and obtain a partition $\Phi_0 \cup \Phi_1$. Now for all $I \in \mathscr{D}$ there is an $i \in \{0, 1\}$ such that $|I \cap E_i| \geq \frac{1}{2}|I|$, which means h_{I,E_0} or h_{I,E_1} appears in $\Phi_0 \cup \Phi_1$. Unfortunately this kind of partition might still not be the partition we are looking for in Question 1: Φ_0 and Φ_1 might each contain some restricted Haar functions of E_0-dominant intervals and some of E_1-dominant ones. In fact, it is even somewhat likely that they do, because we might also prove Corollary 1.3 as follows: Use Theorem 1.2 to show that (8) and (9) can separately be partitioned into two Riesz basic sequences each. The partition of (8) is orthogonal to the partition of (9). That means by Lemma A.2 we can combine these two partitions to one partition of (8)∪(9) into two Riesz basic sequences.

Theorem 1.1 is not a consequence of the fact that $\{h_{I,E} \mid I \in \mathbb{I}\}$ is only a small perturbation of the orthogonal set $\{h_I \mid I \in \mathbb{I}\}$, in the sense that $\|h_I - h_{I,E}\|_2^2 \leq (1-p)\|h_I\|_2^2 < \frac{1}{3}\|h_I\|_2^2$. Take the following example: Assume that $u_1, \ldots, u_n$ are orthonormal. Abbreviate $u \coloneqq u_1 + \ldots + u_n$ and for $i = 1, \ldots, n$ set

$$u_i' \coloneqq u_i - \frac{1}{n}u.$$

Then

$$\|u_i - u_i'\|^2 = \frac{1}{n^2}\|u\|^2 = \frac{1}{n^2}\sum_{i=1}^{n}\|u_i\|^2 = \frac{1}{n}.$$

but

$$u_1' + \ldots + u_n' = u - u = 0,$$

i.e. $\{u_1', \ldots, u_n'\}$ is an arbitrarily small perturbation of $\{u_1, \ldots, u_n\}$ but is no Riesz basic sequence.

2 Proof of Theorem 1.1

2.1 The Case $p > \frac{2}{3}$

First we introduce some notation: Let $\mathbb{I}$ be a set of intervals and $S = \{(a_I, I) \mid I \in \mathbb{I}\}$ be a set of pairs of real numbers and intervals. In order to reduce the number of symbols we will usually omit the (,) and write $S = \{a_I I \mid I \in \mathbb{I}\}$.

Let $E \subset [0, 1)$. Then define

$$F_E(S) \coloneqq \sum_{aI \in S} a h_{I,E},$$
$$A_E(S) \coloneqq \sum_{aI \in S} \|a h_{I,E}\|_2^2,$$

$$B_E(S) := \| \sum_{aI\in S} ah_{I,E}\|_2^2.$$

In order to reduce the number of symbols we will also write $F_E(a_1I_1, a_2I_2, \dots)$ for $F_E(\{a_1I_1, a_2I_2, \dots\})$.

For any set S of pairs of coefficients and intervals we will denote the set of intervals by $\mathbb{I}(S)$.

Now one direction of Theorem 1.1 reads as follows:

Theorem 2.1. *Let $\frac{2}{3} < p \le 1$. Then there is a $c(p) > 0$ s.t. for all $E \subset [0,1)$ and for all S where $\mathbb{I}(S)$ is compatible and consists of (E,p)-dominant intervals, we have*

$$\frac{B_E(S)}{A_E(S)} \ge c(p)$$

with

$$c(p) = \frac{81}{8}(p - \frac{2}{3})^2 + \mathcal{O}(p - \frac{2}{3})^3.$$

We first introduce a function g that we need for the proof of Theorem 2.1.

Definition. For $p > \frac{2}{3}$ define $\varepsilon := p - \frac{2}{3}$,

$$\begin{aligned} a &:= \frac{4}{27}\frac{1}{\varepsilon} + \frac{10}{9} + \frac{4}{3}\varepsilon, \\ b &:= 1 + \frac{3}{2}\varepsilon \end{aligned}$$

and $g : [0,1) \to \mathbb{R}$ by

$$g(q) := \begin{cases} \frac{a-q}{b-q} & q \ge p \\ g(p)\frac{q}{p} & q \le p \end{cases} \tag{10}$$

This definition makes sense because $b > 1$. Also g is continuous. Proposition 2.2 lists all the properties of g that we use in the proof of Theorem 2.1.

Proposition 2.2. g as defined above has the following properties:

1. For all $q \in (0,1]$ we have

$$1 \le \frac{1}{p} \le \frac{g(q)}{q} \le g(1) \tag{11}$$

and

$$g(1) = \frac{8}{81}\frac{1}{\varepsilon^2} + \mathcal{O}(\frac{1}{\varepsilon}).$$

2. For all $q_1, q_2 \in [0,1]$ with $\frac{q_1+q_2}{2} =: q \in [p,1]$ and $x \in \mathbb{R}$ we have

$$G(x, q_1, q_2) := \frac{(1-x)^2}{2} g(q_1) + \frac{(1+x)^2}{2} g(q_2) - g(q) \geq x^2 \geq qx^2. \qquad (12)$$

For $q \in [0,1]$, g is convex.

Remark. Note that (12) for $q \in [0,1]$ and $x = 0$ would be midpoint convexity of g. In order to prove Theorem 2.1 for the case $\mathbb{I}(S) \subset \mathscr{D}$ we will in fact only use midpoint convexity and not convexity. However we get convexity from midpoint convexity by Lemma A.20 anyways.

Remark. We do not actually need $\frac{g(q)}{q} \geq \frac{1}{p}$ and $G(x, q_1, q_2) \geq x^2$: For the proof of Theorem 2.1 $\frac{g(q)}{q} \geq 1$ and $G(x, q_1, q_2) \geq qx^2$ suffice. However the g we found also happens to satisfy the stronger bounds. $G(x, q_1, q_2) \geq x^2$ probably comes from the fact that we actually found g by solving the ODE that arises when sending $q_1 \to q_2$ in $G(x, q_1, q_2) \geq x^2$. We did this because we could not solve the corresponding ODE for $G(x, q_1, q_2) \geq qx^2$. Details on how we came up with the explicit function g can be found in subsubsection 2.1.4.

The constant in Theorem 2.1 which we find is

$$\Big[\sup_{q \in (0,1]} \frac{g(q)}{q}\Big]^{-1}, \qquad (13)$$

otherwise the value of $g(1)$ is not important. Hence if there is another g that also satisfies Proposition 2.2 but with an even smaller upper bound for $\frac{g(q)}{q}$, then Theorem 2.1 can be proven with a greater constant. However we will show later that if we minimize $g(1)$ over all a, b and g given by (10) under which the other statements of Proposition 2.2 hold, then we get $\frac{8}{81}\frac{1}{\varepsilon^2} + \mathcal{O}(\frac{1}{\varepsilon})$ as the minimal value. Since by (11) we have

$$\frac{g(1)}{1} \leq \sup_{q \in (0,1]} \frac{g(q)}{q} \leq g(1)$$

this means that $\frac{81}{8}\varepsilon^2 + \mathcal{O}(\varepsilon^3)$ is the maximal value for (13) among all such g. That means the choice of a, b in the definition of g is optimal for p close to $\frac{2}{3}$. It is not clear however if the function with the minimal value for $g(1)$ that satisfies the other properties of Proposition 2.2 has to be of the form (10). It actually seems rather unlikely, since as remarked above, the g we found satisfies stronger conditions than necessary for the proof of Theorem 2.1.

Before we prove Proposition 2.2 we use it to prove Theorem 2.1.

2.1.1 Proof of Theorem 2.1

We start with the proof for the special case $\mathbb{I}(S) \subset \mathscr{D} \cup \{[-1,1)\}$ which is a bit easier to understand but already showcases the central idea used in the proof for a general S.

For each n denote by $\mathscr{D}_n$ the set of dyadic intervals of length between 2^{-n} and 1 plus the interval $[-1,1)$.

Proof of Theorem 2.1 for $\mathbb{I}(S) \subset \mathscr{D} \cup \{[-1,1)\}$. We only need to prove the theorem for finite S. That means there is an n with $\mathbb{I}(S) \subset \mathscr{D}_n$. It suffices to consider the case $\mathbb{I}(S) = \mathscr{D}_n$ since adding intervals with coefficient zero does not change A or B, also if the intervals are not E-dominant.

Let g be given by Proposition 2.2. Then define $f : [0,1] \to \mathbb{R}$ on $(0,1]$ by

$$f(q) := \frac{g(q)}{q}$$

and $f(0) := f(p)$.[2] Then for each n, $I \in \mathscr{D}_{n+1}$ and $X \subset I \cap E$ set

$$\mu_n(X) := f\Big(\frac{|E \cap I|}{|I|}\Big)|X|$$

This defines a measure μ_n on E. [3]

Claim. For each n and S with $\mathbb{I}(S) = \mathscr{D}_n$ we have

$$\|F_E(S)\|^2_{L^2(\mu_n)} \geq A_E(S).$$

By Proposition 2.2 we have for all $q \in [0,1]$ that

$$f(q) \leq g(1)$$

and thus by the claim

$$B_E(S) = \|F_E(S)\|_2^2 \geq \frac{1}{g(1)}\|F_E(S)\|_{L^2(\mu_n)} \geq \frac{1}{g(1)}A_E(S)$$

which implies the theorem.

Proof of claim. We proceed by induction on n. If $n = 1$ then $\tilde{S} = \{a[-1,1)\}$. By Proposition 2.2 we have $f(|E|) \geq \frac{1}{p} \geq 1$ and thus $\mu_0(|E|) \geq |E|$ so that

$$\|F_E(\tilde{S})\|^2_{L^2(\mu_0)} = a^2\mu_0(|E|) \geq a^2|E| = A_E(\tilde{S}).$$

[2]Note that this makes f constant on $[0,p]$ by (10). We don't use this fact though. Also, the value of f at 0 does not actually matter.

[3]We don't actually need the σ-properties of the measure μ_n because we will only integrate simple functions w.r.t. $\{I \cap E \mid I \in \mathscr{D}_{n+1}\}$.

Now let $n \geq 0$, $\mathbb{I}(S) = \mathscr{D}_{n+1}$. Define $\tilde{S} = \{aI \mid aI \in S,\ I \in \mathscr{D}_n\}$. Then we may apply the inductive hypothesis to $\tilde{S}$. Since

$$A_E(S) = A_E(\tilde{S}) + \sum_{I \in \mathscr{D}_{n+1} \setminus \mathscr{D}_n,\ aI \in S} a^2 \|h_{I,E}\|_2^2$$

it suffices to show that

$$\|F_E(S)\|_{L^2(\mu_{n+1})} \geq \|F_E(\tilde{S})\|_{L^2(\mu_n)} + \sum_{I \in \mathscr{D}_{n+1} \setminus \mathscr{D}_n,\ aI \in S} a^2 \|h_{I,E}\|_2^2 \tag{14}$$

in order to prove the induction hypothesis for S. For that in turn it suffices to prove for all $I \in \mathscr{D}_{n+1} \setminus \mathscr{D}_n$ that

$$\|F_E(S)\|_{L^2(\mu_{n+1}, I)} \geq \|F_E(\tilde{S})\|_{L^2(\mu_n, I)} + a^2 \|h_{I,E}\|_2^2 \tag{15}$$

because then summing over $I \in \mathscr{D}_{n+1} \setminus \mathscr{D}_n$ will lead to (14). Now we write (15) out

$$\int_I [F_E(\tilde{S}) + a h_{I,E}]^2 \,\mathrm{d}\mu_{n+1} \geq \int_I F_E(\tilde{S})^2 \,\mathrm{d}\mu_n + a^2 \|h_{I,E}\|_2^2. \tag{16}$$

$F_E(\tilde{S})$ is constant on I. If $F_E(\tilde{S})$ is zero on I then (16) follows from $f \geq 1$, similarly to the case $n = 0$. Otherwise we divide everything by $F_I(\tilde{S})^2$. After renaming a we then only have to show

$$\int_I [1 + a h_{I,E}]^2 \,\mathrm{d}\mu_{n+1} \geq \mu_n(I) + a^2 \|h_{I,E}\|_2^2.$$

Written out, this is

$$\begin{aligned} &(1-a)^2 |\mathfrak{l}(I) \cap E| f\Big(\frac{|\mathfrak{l}(I) \cap E|}{|\mathfrak{l}(I)|}\Big) + (1+a)^2 |\mathfrak{r}(I) \cap E| f\Big(\frac{|\mathfrak{r}(I) \cap E|}{|\mathfrak{r}(I)|}\Big) \\ &\geq |I \cap E| f\Big(\frac{|I \cap E|}{|I|}\Big) + a^2 |I \cap E|. \end{aligned}$$

Now we divide by $|I|$ and call

$$\begin{aligned} q_1 &= 2\frac{|\mathfrak{l}(I) \cap E|}{|I|} = \frac{|\mathfrak{l}(I) \cap E|}{|\mathfrak{l}(I)|}, \\ q_2 &= 2\frac{|\mathfrak{r}(I) \cap E|}{|I|} = \frac{|\mathfrak{r}(I) \cap E|}{|\mathfrak{r}(I)|}, \\ x &= a, \end{aligned}$$

so that

$$\frac{q_1 + q_2}{2} = \frac{|I \cap E|}{|I|}.$$

Then we obtain exactly (12). Now there are two cases to consider. If I is E-dominant, then $\frac{q_1+q_2}{2} \geq p$ and $q_1, q_2 \in [2p-1, 1]$, where (12) is valid. If I is not E-dominant, then we assumed $x = a = 0$ where (12) is also valid since g is convex. □

Now the claim is proven and hence so is Theorem 2.1 for the case $\mathbb{I}(\tilde{S}) \subset \mathscr{D} \cup \{[-1,1)\}$. $\square$

Proof of Theorem 2.1. First we need to establish a few notions. For a finite set of intervals $\mathbb{I}$ that contains $[0,1)$ and subintervals of $[0,1)$ such that for all $I, J \in \mathbb{I}$ $I \cap J \in \{\emptyset, I, J\}$ define

$$\mathrm{leaves}(\mathbb{I}) \coloneqq \{I \in \mathbb{I} \mid \forall J \in \mathbb{I}, J \neq I : J \not\subset I\}$$

For $n = 0, 1, \dots$ define

$$\mathbb{I}_n \coloneqq \mathrm{leaves}\big(\mathbb{I} \setminus \bigcup_{i<n} \mathbb{I}_i\big).$$

Note, that $\mathbb{I}_0, \mathbb{I}_1, \dots$ is a partition of $\mathbb{I}$. Define inductively for $n = 0, 1, \dots$

$$\tilde{\mathbb{I}}_n \coloneqq \bigcup_{n=0,1,\dots} \left\{ I \setminus \bigcup_{i<n} \bigcup \tilde{\mathbb{I}}_i \,\middle|\, I \in \mathbb{I}_n \right\}$$

and

$$p(\mathbb{I}) \coloneqq \tilde{\mathbb{I}}_0 \cup \tilde{\mathbb{I}}_1 \cup \dots.$$

Note that $p(\mathbb{I})$ is a partition of $[0,1)$.

Let g be given by Proposition 2.2. Then define $f : [0,1] \to \mathbb{R}$ on $(0,1]$ by

$$f(q) \coloneqq \frac{g(q)}{q}$$

and $f(0) \coloneqq f(p)$.[4] Now assume that P is a partition of $[0,1)$ and $E \subset [0,1)$. Then for each $M \in P$ and $X \subset M \cap E$ set

$$\mu_P(X) \coloneqq f\Big(\frac{|M \cap E|}{|M|}\Big)|X|. \tag{17}$$

This defines a measure μ_P on E.

Let S be a compatible sequence of intervals. If $[-1,1)$ is not already in $\mathbb{I}(S)$ we may add it to S with coefficient 0. Then with

$$h(S) \coloneqq \left\{ \mathfrak{l}(I), \mathfrak{r}(I) \,\middle|\, I \in \mathbb{I}(S) \right\} \setminus \{[-1,0)\}$$

we may define $p(h(S))$ and $\mu_{p(h(S))}$. Note that for each $M \in p(h(S))$ we have that $F_E(S)$ is constant on $E \cap M$.

[4] Note that this make f constant on $[0,p]$ by (10). We don't use this fact though. Also, the value of f at 0 does not matter actually.

Claim. For each S we have

$$\|F_E(S)\|^2_{L^2(\mu_{p(h(S))})} \geq A_E(S).$$

By Proposition 2.2 we have for all $q \in [0,1]$ that

$$f(q) \leq g(1)$$

and thus by the claim

$$B_E(S) = \|F_E(S)\|_2^2 \geq \frac{1}{g(1)} \|F_E(S)\|_{L^2(\mu_{p(h(S))})} \geq \frac{1}{g(1)} A_E(S)$$

which implies the theorem.

Proof of claim. We proceed by induction on n. If $n = 1$ then $S = \{a[-1,1)\}$ and $p(h(S)) = \{[-1,0),[0,1)\}$. By Proposition 2.2 we have $f(|E|) \geq \frac{1}{p} \geq 1$ and thus $\mu_S(|E|) \geq |E|$ so that

$$\|F_E(S)\|^2_{L^2(\mu_{p(h(S))})} = a^2 \mu_S(|E|) \geq a^2|E| = A_E(S).$$

So assume it holds for $n \geq 1$ and let $|S| = n+1$. Then there is an $I \in \text{leaves}(\mathbb{I}(S))$ and $aI \in S$. Define $\tilde{S} = S \setminus \{aI\}$. Then there is a set $M \in p(h(\tilde{S}))$ which contains I. Then note that

$$\begin{aligned}
p\Big(h(\tilde{S}) \cup \{I\}\Big) &= p\Big(h(\tilde{S})\Big) \setminus \{M\} \cup \{M \setminus I, I\} \\
p\Big(h\big(\tilde{S} \cup \{aI\}\big)\Big) &= p\Big(h(\tilde{S}) \cup \{\mathfrak{l}(I), \mathfrak{r}(I)\}\Big) \\
&= p\Big(h(\tilde{S}) \cup \{I\}\Big) \setminus \{I\} \cup \{\mathfrak{l}(I), \mathfrak{r}(I)\} \\
&= p\Big(h(\tilde{S})\Big) \setminus \{M\} \cup \{M \setminus I, \mathfrak{l}(I), \mathfrak{r}(I)\}.
\end{aligned}$$

$F_E(\tilde{S})$ and $F_E(\tilde{S} \cup \{aI\})$, and $\mu_{p(h(\tilde{S}\cup\{aI\}))}$ and $\mu_{p(h(\tilde{S}))}$ are equal on the complement of $M \cap E$. Hence it suffices to show

$$\int_M (F_E(\tilde{S}) + ah_{I,E})^2 \,\mathrm{d}\mu_{p(h(\tilde{S}\cup\{aI\}))} - \int_M F_E(\tilde{S})^2 \,\mathrm{d}\mu_{p(h(\tilde{S}))} \geq a^2 \|h_{I,E}\|_2^2$$

in order to conclude (17) for S from the inductive hypothesis (17) for $\tilde{S}$. $F_E(\tilde{S})$ is constant on M. If $F_E(\tilde{S})$ is zero on I then (16) follows from $f \geq 1$, similarly to the case $n = 0$. Otherwise we divide everything by $F_I(\tilde{S})^2$. After renaming a we then only have to show

$$\int_M (1 + ah_{I,E})^2 \,\mathrm{d}\mu_{p(h(\tilde{S}\cup\{aI\}))} - \mu_{p(h(\tilde{S}))}(M) \geq a^2 \|h_{I,E}\|_2^2. \tag{18}$$

Firstly take $t := \frac{|I|}{|M|}$ so that $1 - t = \frac{|M \setminus I|}{|M|}$ and

$$t\frac{|I \cap E|}{|I|} + (1-t)\frac{|(M \setminus I) \cap E|}{|M \setminus I|} = \frac{|M \cap E|}{|M|}.$$

Then we get by the convexity of g that

$$\begin{aligned}\frac{1}{|M|}\mu_{p(h(\tilde{S}))}(M) &= \frac{|M \cap E|}{|M|} f\Big(\frac{|M \cap E|}{|M|}\Big) = g\Big(\frac{|M \cap E|}{|M|}\Big) \\ &\leq tg\Big(\frac{|I \cap E|}{|I|}\Big) + (1-t)g\Big(\frac{|(M \setminus I) \cap E|}{|M \setminus I|}\Big) \\ &\leq \frac{|I \cap E|}{|M|} f\Big(\frac{|I \cap E|}{|I|}\Big) + \frac{|(M \setminus I) \cap E|}{|M|} f\Big(\frac{|(M \setminus I) \cap E|}{|M \setminus I|}\Big) \\ &= \frac{1}{|M|}\mu_{p(h(\tilde{S}) \cup \{I\})}(M). \end{aligned} \tag{19}$$

$\mu_{p(h(\tilde{S} \cup \{aI\}))}$ equals $\mu_{p(h(\tilde{S}) \cup \{I\})}$ on $I^{\complement}$ and $ah_{I,E}$ is 0 on $I^{\complement}$ so that

$$\int_M (1 + ah_{I,E})^2 \, \mathrm{d}\mu_{p(h(\tilde{S} \cup \{aI\}))} = \int_I (1 + ah_{I,E})^2 \, \mathrm{d}\mu_{p(h(\tilde{S} \cup \{aI\}))} + \mu_{p(h(\tilde{S}) \cup \{I\})}(M \setminus I).$$

This means by (19) it suffices to prove

$$\int_I (1 + ah_{I,E})^2 \, \mathrm{d}\mu_{p(h(\tilde{S} \cup \{aI\}))} \geq \mu_{p(h(\tilde{S}) \cup \{I\})}(I) + a^2 \|h_{I,E}\|_2^2$$

to get (18). Calculating both sides this means

$$\begin{aligned}&(1-a)^2|\mathfrak{l}(I) \cap E| f\Big(\frac{|\mathfrak{l}(I) \cap E|}{|\mathfrak{l}(I)|}\Big) + (1+a)^2|\mathfrak{r}(I) \cap E| f\Big(\frac{|\mathfrak{r}(I) \cap E|}{|\mathfrak{r}(I)|}\Big) \\ &\geq I \cap E| f\Big(\frac{|I \cap E|}{|I|}\Big) + a^2|I \cap E|\end{aligned}$$

or equivalently

$$\begin{aligned}&\frac{(1-a)^2}{2}\frac{|\mathfrak{l}(I) \cap E|}{|\mathfrak{l}(I)|} f\Big(\frac{|\mathfrak{l}(I) \cap E|}{|\mathfrak{l}(I)|}\Big) + \frac{(1+a)^2}{2}\frac{|\mathfrak{r}(I) \cap E|}{|\mathfrak{r}(I)|} f\Big(\frac{|\mathfrak{r}(I) \cap E|}{|\mathfrak{r}(I)|}\Big) \\ &\geq \frac{|I \cap E|}{|I|} f\Big(\frac{|I \cap E|}{|I|}\Big) + a^2\frac{|I \cap E|}{|I|}.\end{aligned} \tag{20}$$

Now since $\frac{|I \cap E|}{|I|} \geq p$ and $\frac{|\mathfrak{l}(I) \cap E|}{|\mathfrak{l}(I)|}, \frac{|\mathfrak{r}(I) \cap E|}{|\mathfrak{r}(I)|} \geq 2p - 1$ and

$$\frac{1}{2}\frac{|\mathfrak{l}(I) \cap E|}{|\mathfrak{l}(I)|} + \frac{1}{2}\frac{|\mathfrak{r}(I) \cap E|}{|\mathfrak{r}(I)|} = \frac{|I \cap E|}{|I|}$$

we may invoke (12) in order to conclude (20). □

Now the claim is proven and hence so is Theorem 2.1. □

2.1.2 Idea of the Proof of Theorem 2.1

If one wants to prove that $\{\frac{h_I}{\|h_I\|_2} \mid I \in \mathscr{D}\}$ is a Riesz basic sequence, one could proceed as follows: First sort $\mathscr{D}$ decreasing in scale. Then assume we have already linearly combined the first $n-1$ Haar functions to the function F_{n-1}. Now check, that after adding ah_{I_n}, the increase of $\int_{I_n} F_{n-1}^2$ to $\int_{I_n} F_n^2$ is just $\|ah_{I_n}\|_2^2$.

This strategy is not going to work for restricted Haar functions as F_n might even have a smaller L^2-norm on I_n than F_{n-1}. We can however make the strategy work again by integrating F_{n-1}^2 and F_n^2 with respect to weights. The corresponding measures will nevertheless be comparable to the Lebesgue measure. Consider for example the case $|I_n \cap E| = p|I|$, $|\mathfrak{l}(I_n) \cap E| = |\mathfrak{l}(I_n)|$. Then the interesting case is $a \geq 0$. After having added ah_{I_n} it does not make any sense to add any more Haar functions with support intersecting $\mathfrak{l}(I_n)$ because they are orthogonal to F_k for each $k \geq n$; $\mathfrak{l}(I_n)$ is "used up" now. That means that F_k and F_n agree on $\mathfrak{l}(I_n)$. On $\mathfrak{r}(I_n)$ it is different: Many more functions with support in $\mathfrak{r}(I_n)$ may be added to F_n that might reduce $\int_{\mathfrak{r}(I_n)} F_n^2$ and contribute to A_k. Hence it is reasonable to weigh the L^2-norm of F_n less on $\mathfrak{r}(I_n)$ than on $\mathfrak{l}(I_n)$. $\frac{g(q)}{q}$ plays the role of such a weight where here $q = \frac{|\mathfrak{r}(I_n)\cap E|}{|\mathfrak{r}(I_n)|}$. Note that $\frac{g(q)}{q}$ is constant on $[0,p]$ and increasing on $[0,1]$. In the case of $p = 1$ we could choose the weight to be constantly 1.

2.1.3 Proof of Proposition 2.2

Proof of Proposition 2.2. Lemma A.22 with $x = 0$ implies that $\frac{a-q}{b-q}$ is midpoint convex and thus convex by Lemma A.20. By definition, on $[2p-1, p]$ g is just the linear interpolation between $g(2p-1)$ and $g(p)$. Furthermore $g(p) = \frac{a-p}{b-p}$ and by Lemma A.23 we have $g(2p-1) \geq \frac{a-(2p-1)}{b-(2p-1)}$. Therefore for all $q \in [2p-1, p]$ we have

$$g(q) \geq \frac{a-q}{b-q}.$$

By Lemma A.22 this implies (12) for $q := \frac{q_1+q_2}{2} \geq p$ as

$$G(q_1, q_2, x) \geq \frac{(1-x)^2}{2}\frac{a-q_1}{b-q_1} + \frac{(1+x)^2}{2}\frac{a-q_2}{b-q_2} - \frac{a-q}{b-q} \geq x^2.$$

Now we prove that g is convex on $[0,1]$. Since g is continuous, by Lemma A.20 it suffices to show that g is midpoint convex, i.e. that for all $q_1, q_2 \in D$ we have

$$\frac{1}{2}g(q_1) + \frac{1}{2}g(q_2) \geq g(\frac{q_1+q_2}{2}). \tag{21}$$

For $q_1, q_2 \leq p$ this is true since g is linear there and for $\frac{q_1+q_2}{2} \geq p$ we just showed it. Hence it remains to consider the case $\frac{q_1+q_2}{2} \leq p,\ q_2 \geq p$ upon renaming. Then

$$\begin{aligned}\frac{1}{2}g(q_1) - g(\frac{q_1+q_2}{2}) &= \frac{g(p)}{p}(\frac{q_1}{2} - \frac{q_1+q_2}{2}) \\ &= \frac{g(p)}{p}(-\frac{q_2}{2})\end{aligned}$$

Now take $2p - 1 \leq \tilde{q}_1 \leq p$ s.t. $\frac{\tilde{q}_1+q_2}{2} = p$ and

$$\begin{aligned}&= \frac{g(p)}{p}(\frac{\tilde{q}_1}{2} - \frac{\tilde{q}_1+q_2}{2}) \\ &= \frac{1}{2}g(\tilde{q}_1) - g(\frac{\tilde{q}_1+q_2}{2})\end{aligned}$$

and since we already proved (12) for $\frac{\tilde{q}_1+q_2}{2} = p$ we have

$$\geq -\frac{1}{2}g(q_2)$$

Now it remains to prove that for all $q \in [0, 1]$

$$\frac{q}{p} \leq g(q) \leq g(1)q.$$

Since g is convex and $g(0) = 0$ we already get the upper bound. Since $1 \leq b \leq a$ we have $g(p) \geq 1 = \frac{p}{p}$. Because for $q \in [0, p]$ g is linear we also get $g(q) \geq \frac{q}{p}$ there. Now by convexity we may extend this to $[0, 1]$. □

Remark. Lemma A.23 says that our choice of a, b is even optimal for p close to $\frac{2}{3}$.

2.1.4 Motivation for the Choice of g

First we motivate why it is reasonable to choose g linear on $[0, p]$. Let S be compatible and E-dominant and $S_0 \subset S$ and $M \in p(h(S_0))$ with $|M \cap E| \leq p|M|$. Then the union of all intervals $\mathbb{I}(S) \setminus \mathbb{I}(S_0)$ can only cover a part of M of size $\frac{1}{p}|M \cap E| \leq |M|$. This means the uncovered empty space of size $\frac{1}{p}|M \cap E| - |M|$ in M does not have any effect. The measures given by (17) reflect this, because if we remove that uncovered part, i.e. take some $M' \in P'$ with $E \subset M' \subset M$ and $|M' \cap E'| = p|M'|$, then g being linear on $[0, p]$ means that with $q := \frac{|M \cap E|}{|M|} \leq p = \frac{|M' \cap E|}{|M'|}$ we have

$$\mu_{p(h(S_0))}(M \cap E) = \frac{g(q)}{q}|M \cap E| = \frac{g(p)}{p}|M \cap E| = f(p)|M' \cap E| = \mu_{P'}(M' \cap E).$$

It also seems reasonable to take g linear on $[0, p]$ from the point of view of Proposition 2.2, because we mainly want g to be convex on $[0, p]$, with $\frac{g(q)}{q}$ having the smallest possible maximum, while g should still be somewhat large on $[2p - 1, p]$. A linear increasing function likely could have these properties.

Now we motivate how we chose g on $[p, 1]$. By Lemma A.21 we have for any function $g : (0, 1) \to (1, \infty)$ and any $q_1, q_2 \in (0, 1)$ that

$$\begin{aligned} &\inf_x [g(q_1) + g(q_2) - 2][G(q_1, q_2, x) - x^2] \\ &\quad = -[g(q_1) + g(q_2)]g(\frac{q_1 + q_2}{2}) + 2g(q_1)g(q_2) - g(q_1) - g(q_2) + 2g(\frac{q_1 + q_2}{2}). \end{aligned} \tag{22}$$

Claim. Letting $q_1, q_2 \to q$ in (22) ≥ 0 we obtain the ODE

$$g''(g - 1) - 2(g')^2 \geq 0. \tag{23}$$

Proof. We do a Taylor expansion of (22) around $\frac{q_1+q_2}{2}$. Abbreviate $g(\frac{q_1+q_2}{2}) = y$ and $\Delta q = q_2 - \frac{q_1+q_2}{2} = -[q_1 - \frac{q_1+q_2}{2}]$. The terms constant in Δq vanish. The terms linear in Δq are

$$-[y'(-\Delta q) + y'\Delta q]y + 2yy'(-\Delta q) + 2yy'\Delta q - y'(-\Delta q) - y'\Delta q = 0$$

The quadratic terms are

$$\begin{aligned} &-[\frac{1}{2}y''\Delta q^2 + \frac{1}{2}y''\Delta q^2]y + 2\frac{1}{2}y''y\Delta q^2 - 4\frac{1}{2}y'y'\Delta q^2 + 2\frac{1}{2}yy''\Delta q^2 \\ &\quad - \frac{1}{2}y''\Delta q^2 - \frac{1}{2}y''\Delta q^2 \\ &= [y''(y - 1) - 2(y')^2]\Delta q^2 \end{aligned}$$

Hence dividing (22) ≥ 0 by Δq^2 and letting $\Delta q \to 0$, all terms vanish except (23). $\square$

All functions of the form

$$q \mapsto \frac{a - q}{b - q}$$

are solutions of (23) $= 0$. Interestingly we have shown in the proof of Proposition 2.2 that they also satisfy (22) $= 0$, even though the calculation above shows only the reverse direction (22) $= 0 \implies$ (23) $= 0$.

Now we want to find a, b such that g satisfies all the properties of Proposition 2.2 with $g(1)$ as small as possible. $q \leq g(q) \leq g(1)q$ implies g to be bounded and positive. This requires $a, b \geq 1$ or $a, b \leq 0$. Since in the proof of the claim we used $g(q) \geq 1$ the cases $a \geq b \geq 1$ and $a \leq b \leq 0$ remain. Now

$$\frac{a - q}{b - q} = 1 + \frac{a - b}{b - q}$$

$$\frac{\mathrm{d}}{\mathrm{d}q}\frac{a-q}{b-q} = \frac{a-b}{(b-q)^2}$$
$$\frac{\mathrm{d}^2}{\mathrm{d}q^2}\frac{a-q}{b-q} = 2\frac{a-b}{(b-q)^3}$$

Note, that the functions $q \mapsto \frac{a-q}{b-q}$ arising from $a, b \leq 0$ are just the ones arising from $a, b \geq 1$, mirrored at $q = \frac{1}{2}$. Now since we want $q \leq g(q) \leq g(1)q$ we are rather looking for functions with positive derivatives, i.e. those with $a \geq b$. Actually when looking at the idea of the proof of Theorem 2.1, subsubsection 2.1.2, we even expect $\frac{g(q)}{q}$ to have a positive derivative. Since we also want g to be convex, $a \geq b$ requires $b \geq q$. And since we want it to be well defined at $q = 1$ we actually need $b > 1$. That means it is reasonable to consider only

$$1 < b \leq a.$$

We calculated in Lemma A.23 how we should choose a, b in detail. The other properties of g then follow as presented in the proof of Proposition 2.2, subsubsection 2.1.3.

2.1.5 Bellman Function Interpretation

There are strong parallels between the proof of Theorem 2.1 for the case $\mathbb{I}(S) \subset \mathscr{D}$, and the strategy of the following instances of the Bellman function technique: Lemma 3.3, (5.1) and Theorem 9.1 in [4]. Note that by Lemma A.14 the case $\mathbb{I}(S) \subset \mathscr{D} \cup \{[-1, 1)\}$ is already a consequence of the case $\mathbb{I}(S) \subset \mathscr{D}$.

In what follows the two strategies are reformulated and written in one go. At the places where the strategies differ, this is how we mark the $\boxed{\text{Bellman function technique in [4]}}$ and $\boxed{\text{the proof of Theorem 2.1 for } \mathbb{I}(S) \subset \mathscr{D}}$.

The goal is for some $c > 0$, N, $(x_I)_{I \in \mathscr{D}_N}$ and

$$\boxed{|X_{[0,1)}| \in [0, \infty)} \qquad \boxed{(|X_I|)_{I \in \mathscr{D}_{N+1} \setminus \mathscr{D}_N} \subset [0, \infty)}$$

to establish a bound

$$\boxed{c \sum_{I \in \mathscr{D}_N} x_I |I| \leq |X_{[0,1)}|} \qquad \boxed{c \sum_{I \in \mathscr{D}_N} x_I |I| \leq \sum_{I \in \mathscr{D}_{N+1} \setminus \mathscr{D}_N} |I||X_I|}. \tag{24}$$

We will see later how $x_I, |X_I|$ can be chosen such that (24) becomes Theorem 2.1. Note that usually the sum on the left hand side of (24) is over all $I \in \mathscr{D}$ but it suffices to obtain a uniform bound for all N. As you can see on the right hand sides, the number N is a bit more important in our technique than in the Bellman function technique. In order to prove (24) they come up with a positive function B with arguments $I \in \mathscr{D}$ and a tuple of parameters X. Here $|\cdot|$ denotes just denotes a function from the parameter space to $[0,\infty)$. B satisfies a concavity convexity condition: For all $I \in \mathscr{D}$ and $X, X_1, X_2 \in \mathbb{R}$ such that in some sense $X = \frac{X_1+X_2}{2}$, we have

$$B_I(X) - \frac{B_{\mathfrak{l}(I)}(X_1) + B_{\mathfrak{r}(I)}(X_2)}{2} \geq x_I \qquad \frac{B_{\mathfrak{l}(I)}(X_1) + B_{\mathfrak{r}(I)}(X_2)}{2} - B_I(X) \geq x_I. \tag{25}$$

Furthermore

$$c B_I(X) \leq |X_I|. \tag{26}$$

That way for each n and $X = 2^{-(n+1)} \sum_{I \in \mathscr{D}_{n+1} \setminus \mathscr{D}_n} X_I$ by induction

$$\sum_{I \in \mathscr{D}_n} x_I |I| \leq B_{[0,1)}(X) - \sum_{I \in \mathscr{D}_{n+1} \setminus D_n} |I| B_I(X_I)$$

$$\sum_{I \in \mathscr{D}_n} x_I |I| \leq \sum_{I \in \mathscr{D}_{n+1} \setminus \mathscr{D}_n} |I| B_I(X_I) - B_{[0,1)}(X).$$

This implies

$$\sum_{I \in \mathscr{D}_N} x_I |I| \leq B_{[0,1)}(X) \qquad \sum_{I \in \mathscr{D}_N} x_I |I| \leq \sum_{I \in \mathscr{D}_{N+1} \setminus \mathscr{D}_N} |I| B_I(X_I)$$

and by (26) we are done.

Now we show how to choose $B_I, x_I, X_I, |\cdot|$ in order to get Theorem 2.1 using this strategy from above. Recall that Theorem 2.1 states

$$c \sum_{I \in \mathscr{D}_N} a_I^2 \|h_{I,E}\|_2^2 \leq \|\sum_{I \in \mathscr{D}_N} a_I h_{I,E}\|_2^2.$$

Here X is of the form

$$X = (q, s), \qquad q \in [0,1],\ s \in [0,\infty)$$

For the interval I we will have the interpretation

$$q = \frac{|I \cap E|}{|I|},$$

$$s = \sum_{I \subset \mathfrak{r}(J)} a_J - \sum_{I \subset \mathfrak{l}(J)} a_J$$

We further have

$$x_I = q_I(\frac{s_{\mathfrak{r}(I)} - s_{\mathfrak{l}(I)}}{2})^2 = q_I a_I^2$$

and

$$|X| = qs^2.$$

The Bellman function would be

$$B(X) = g(q)s^2.$$

This means

$$\sum_{I \in \mathscr{D}_N} x_I |I| = \sum_{I \in \mathscr{D}_N} q_I a_I^2 |I| = \sum_{I \in \mathscr{D}_N} |E \cap I| a_I^2 = \sum_{I \in \mathscr{D}_N} \|a_I h_{I,E}\|_2^2$$

and

$$\begin{aligned}
\sum_{I \in \mathscr{D}_{N+1} \setminus \mathscr{D}_N} |I||X_I| &= \sum_{I \in \mathscr{D}_{N+1} \setminus \mathscr{D}_N} |E \cap I| (\sum_{I \subset \mathfrak{r}(J)} a_J - \sum_{I \subset \mathfrak{l}(J)} a_J)^2 \\
&= \sum_{I \in \mathscr{D}_{N+1} \setminus \mathscr{D}_N} \int_E \mathbb{1}_I (\sum_{I \subset \mathfrak{r}(J)} a_J - \sum_{I \subset \mathfrak{l}(J)} a_J)^2 \\
&= \int_E \sum_{I \in \mathscr{D}_{N+1} \setminus \mathscr{D}_N} (\sum_{I \subset \mathfrak{r}(J)} \mathbb{1}_{I \cap \mathfrak{r}(J)} a_J - \sum_{I \subset \mathfrak{l}(J)} \mathbb{1}_{I \cap \mathfrak{l}(J)} a_J)^2 \\
&= \int_E \sum_{I \in \mathscr{D}_{N+1} \setminus \mathscr{D}_N} (\sum_{J \in \mathscr{D}_N} \mathbb{1}_{I \cap \mathfrak{r}(J)} a_J - \sum_{J \in \mathscr{D}_N} \mathbb{1}_{I \cap \mathfrak{l}(J)} a_J)^2 \\
&= \int_E \sum_{I \in \mathscr{D}_{N+1} \setminus \mathscr{D}_N} (\sum_{J \in \mathscr{D}_N} \mathbb{1}_I a_J h_J)^2 \\
&= \int_E \sum_{I \in \mathscr{D}_{N+1} \setminus \mathscr{D}_N} \mathbb{1}_I (\sum_{J \in \mathscr{D}_N} a_J h_J)^2 \\
&= \int_E (\sum_{J \in \mathscr{D}_N} a_J h_J)^2 \\
&= \| \sum_{J \in \mathscr{D}_N} a_J h_{J,E} \|_2^2
\end{aligned}$$

That means (24) reads

$$\| \sum_{I \in \mathscr{D}_N} a_I h_{I,E} \|_2^2 \geq c \sum_{I \in \mathscr{D}_N} \|a_I h_{I,E}\|_2^2.$$

which is what we want to show.

As for an idea why our technique uses equalities that are somehow converse to those of the Bellman function technique, note that Lemma 3.3, (5.1) and Theorem 9.1 in [4] resemble operator bounds from above, while Theorem 2.1 is an operator bound from below.

2.2 The Case $p \leq \frac{2}{3}$

The remaining direction of Theorem 1.1 follows from the following Theorem 2.3.

Theorem 2.3. *For* $E = [0, \frac{2}{3})$,

$$\left\{ \frac{h_{I,E}}{\|h_{I,E}\|_2} \;\middle|\; I \in \mathscr{D},\ |I \cap E| = \frac{2}{3}|I| \right\} \tag{27}$$

is no Riesz basic sequence.

Proof. In this proof we denote numbers by their binary representation instead of their decimal representation.

Let I_i be the dyadic interval with $|I_i| = 2^{-i}$ which contains $\frac{2}{3} = .1010\cdots$. For any dyadic I with $|I| = 2^{-i}$ there is a sequence of binary digits s of length $i-1$ such that

$$h_I : x \mapsto \begin{cases} -1 & x \in [.s00, .s01), \\ 1 & x \in [.s01, .s10), \\ 0 & \text{else} \end{cases}$$

For $i = 0$ this is ment in such a way that s is empty and everything that comes after s is moved one place further to the left. For $h_{I_{2i}}$, s consists of the sequence of digits $101010\cdots$ of length $2i-1$. Note, that the sequence starts and ends with the digit 1. Since $.1010\cdots$ is between $.\underbrace{1010\cdots}_{2i-1}01$ and $.\underbrace{1010\cdots}_{2i-1}10$ this means

$$\begin{aligned} h_{I_{2i},E} : x \mapsto &\begin{cases} -1 & x \in [.\underbrace{1010\cdots}_{2i-1}00, .\underbrace{1010\cdots}_{2i-1}01), \\ 1 & x \in [.\underbrace{1010\cdots}_{2i-1}01, .1010\cdots), \\ 0 & \text{else} \end{cases} \\ = &\begin{cases} -1 & x \in [.\underbrace{1010\cdots}_{2i}0, .\underbrace{1010\cdots}_{2i}1), \\ 1 & x \in [.\underbrace{1010\cdots}_{2i}1, .1010\cdots), \quad . \\ 0 & \text{else} \end{cases} \end{aligned}$$

Thus

$$|I_i \cap E| = .1010\cdots - .\underbrace{1010\cdots}_{2i} = .\underbrace{00\cdots}_{2i}101\cdots = 2^{-2i}\frac{2}{3} = \frac{2}{3}|I_i|$$

which means that I_i is selected in (27).

For each I denote by a_I the coefficient in front $h_{I,E}$. Set

$$a_{I_0} := 1,$$
$$i \geq 1 : \quad a_{I_{2i}} := 2^{i-1}$$

and abbreviate

$$F_n = \sum_{i=0}^{n} a_{I_{2i}} h_{I_{2i},E}.$$

Claim. For all n

$$F_n(x) = \begin{cases} -1 & x \in [0, .1), \\ 0 & x \in [.1, .\underbrace{10\cdots}_{2n+1}), \\ 2^n & x \in [.\underbrace{10\cdots}_{2n+1}, .1010\cdots). \end{cases}$$

Proof. We proceed by induction on n. The claim is clear for $n = 0$. Assume the claim holds for n. When adding $a_{I_{2(n+1)}} h_{I_{2(n+1)}}$ the domain $[0, .\underbrace{1010\cdots}_{2(n+1)})$ remains unchanged. On the domain $[0, .\underbrace{1010\cdots}_{2(n+1)}0, .\underbrace{1010\cdots}_{2(n+1)}1)$, F_{n+1} is zeroed. On the domain $[.\underbrace{1010\cdots}_{2(n+1)}1, .1010\cdots)$, F_{n+1} is increased by 2^n to 2^{n+1}. This proves the claim for $n+1$. □

The next claim clearly implies the statement of the theorem:

Claim. For all n we have

$$\sum_{i=0}^{n} \|a_{I_{2i}} h_{I_{2i},E}\|_2^2 = \frac{2}{3} + \frac{n}{6},$$
$$\|F_n\|_2^2 = \frac{2}{3}$$

Proof. For each i we have

$$|\mathfrak{l}(I_{2(i+1)}) \cap E| = \frac{1}{4}|\mathfrak{l}(I_{2i}) \cap E|,$$
$$|\mathfrak{r}(I_{2(i+1)}) \cap E| = \frac{1}{4}|\mathfrak{r}(I_{2i}) \cap E|.$$

So by the choice of the coefficients we get

$$\|a_{I_0} h_{0,E}\|_2^2 = \frac{2}{3},$$
$$i \geq 1: \qquad \|a_{I_{2i}} h_{I_{2i},E}\|_2^2 = \frac{1}{6}.$$

The size of the domain where $F_n(x) = 2^n$ is $.\underbrace{00\cdots}_{2n+1}0101\cdots = \frac{1}{3}(\frac{1}{2})^{2n+1}$. So

$$\|F_n\|_2^2 = \frac{1}{2} \cdot 1 + \frac{1}{3}(\frac{1}{2})^{2n+1} 2^{2n} = \frac{2}{3}.$$

□

The claim is proven and hence so is the theorem. □

3 Explicit Optimum for a Special Case

3.1 A Special Sequence

We first introduce a few notions: We extend the definition of A, B, F, because in the following sections we will not exclusively be dealing with restricted Haar functions anymore but with slightly more general functions. So if S is a finite set of functions in $L^2(\mathbb{R})$ we write

$$F_E(S) := \sum_{f \in S} f \mathbb{1}_E,$$
$$A_E(S) := \sum_{f \in S} \|f \mathbb{1}_E\|_2^2,$$
$$B_E(S) := \|\sum_{f \in S} f \mathbb{1}_E\|_2^2.$$

For a set of pairs of numbers and intervals S we identify (a, I) with ah_I so that this definition agrees with the original one.

Definition. Let $p \in [\frac{1}{2}, 1)$[5]. Let $E \subset I$ be two intervals with the same left boundary. An interval $J \subset I$ with

$$|E \cap J| = p|J|,$$

[5] If $p = 1$ we may call every interval J with $|E \cap J| = |J|$ most p-antiparallel. The case $p = 1$ is never interesting in this work though. Also we could obviously extend this definition to $p \in (0, \frac{1}{2})$. However there the notion does not reflect the intended meaning anymore, as Lemma A.13 does not hold for $p \in (0, \frac{1}{2})$. And we will never consider the case $p \in (0, \frac{1}{2})$ anyways.

$$|J| = \begin{cases} \frac{|E|}{p} & |E| \le p|I| \\ \frac{|I|-|E|}{1-p} & |E| \ge p|I| \end{cases}$$

is called a *most p-antiparallel interval to I w.r.t. E.*

If E is not contained in I we say that J most antiparallel to I w.r.t. E if it is most antiparallel to I w.r.t. $E \cap I$. We sometimes leave p and E away if it is clear what they are.

Lemma A.13 provides a characterisation of this notion.

Fix $\frac{2}{3} < p \le 1$. From now on throughout this entire section we will have $E = [0, p)$. We define $(I_n^p)_n$ inductively as follows: Set $I_0^p := [-1, 1)$ and for I_n^p given let I_{n+1}^p be the most antiparallel interval to $\mathfrak{r}(I_n^p)$. Now define[6]

$$\begin{aligned} \mathbb{I}_n^p &:= \{I_0^p, \dots, I_n^p\}, \\ \mathbb{I}^p &:= \{I_0^p, I_1^p, \dots\}, \\ S_n(r) &:= \{I_0^p\} \cup \left\{ r(1+r)^{k-1} I_k^p \;\middle|\; 1 \le k \le n \right\}, \\ S_\infty(r) &:= S_0(r) \cup S_1(r) \cup \dots. \end{aligned}$$

Further define

$$\begin{aligned} A_n(r) &:= A(S_n(r)), \\ B_n(r) &:= B(S_n(r)), \\ A_\infty &:= \lim_{n\to\infty} A_n(r), \\ B_\infty &:= \lim_{n\to\infty} B_n(r). \end{aligned}$$

The main result of this section is the following Theorem 3.1:

Theorem 3.1. *Let $p \ge \frac{2}{3}$. Then*

$$\inf\left\{ \frac{B(S)}{A(S)} \;\middle|\; \mathbb{I}(S) = \mathbb{I}^p \right\} = \inf_{r\in\mathbb{R}} \frac{B_\infty(r)}{A_\infty(r)}. \tag{28}$$

We believe that the left hand side in (28) can actually be replaced by

$$\inf\left\{ \frac{B_E(S)}{A_E(S)} \;\middle|\; E \subset [0,1),\ S \text{ compatible and } (E,p)\text{-dominant} \right\}.$$

We couldn't prove that though. But in the next section 4 we will show some reductions that allow to enlarge the domain of the infimum in (28) a bit, see Theorem 4.1 and

[6]We don't mention the parameter p in $S_n(r)$ because p is fixed in this section anyways. $\mathbb{I}_n^p$, $\mathbb{I}^p$ will also be used outside this section though.

Theorem 4.7. We will soon also compute the right hand side in (28); see Proposition 3.2. But first, we will show that for all p of the form

$$p = \frac{2^e}{d} > \frac{2}{3} \tag{29}$$

for any $e, d \in \mathbb{N}$, we also get a dyadic version of Theorem 3.1, i.e. there is an E_p and a set of dyadic E_p, p-dominant intervals $\mathbb{I}^{\text{dyadic}}$ with

$$\inf\left\{ \frac{B_{E_p}(S)}{A_{E_p}(S)} \;\middle|\; \mathbb{I}(S) = \mathbb{I}^{\text{dyadic}} \right\} = \inf_{r\in\mathbb{R}} \frac{B_\infty(r)}{A_\infty(r)}. \tag{30}$$

Note that the set of such p satisfying (29) is dense in $(\frac{2}{3}, 1]$ so that we can approach the value $\inf_r \frac{B_\infty(r)}{A_\infty(r)}$ by dyadic S for any $p > \frac{2}{3}$.

The intervals in $\mathbb{I}_n^p$ are indeed not necessarily dyadic. All that we do in the construction of I^{dyadic} is replace each I_n^p by a bunch of small translates and dilates of I_n^p that are dyadic and have in total the same length as I_n^p. The parameter p is just chosen in such a way that this is possible. We redistribute E_p accordingly.

Lets do this in detail. For $p = \frac{2^e}{d}$ set $N_p := 2^{e+1}\frac{p-\frac{1}{2}}{p} = 2^{e+1}(1-\frac{1}{2p}) = 2^{e+1} - d$. Denote by

$$\mathscr{T} := \{s \mid s = s_1 \ldots s_n,\ i = 1, \ldots, n \,:\, s_i = 1, \ldots, N_p,\ n = 0, 1, \ldots\}$$

the set of all finite sequences s over $\{1, 2, \ldots, N_p\}$. This set $\mathscr{T}$ will be the index set of our tree of intervals

$$\mathbb{I}^{\text{dyadic}} := \{[-1,1)\} \cup \left\{ I_s^p \;\middle|\; s \in \mathscr{T} \right\},$$
$$\mathbb{I}_n^{\text{dyadic}} := \{[-1,1)\} \cup \left\{ I_s^p \;\middle|\; s \in \mathscr{T},\ |s| \le n-1 \right\}.$$

We define the intervals inductively. Set

$$I_\varnothing^p := [0,1).$$

Given I_s^p and $k \in \{1, \ldots, N_p\}$ we take I_{sk}^p to be the kth dyadic interval of size $|I_s^p|2^{-(e+1)}$ from the left boundary of in $\mathfrak{r}(I_s^p)$. Then $I_{sk}^p \subset \mathfrak{r}(I_s^p)$ because

$$k|I_s^p|2^{-(e+1)} \le N_p|I_s^p|2^{-(e+1)} = |I_s^p|(1 - \frac{1}{2p}) \le \frac{1}{2}|I_s^p|.$$

This means that for each $s \in \mathscr{T}$ we have

$$\sum_{k=1}^{N_p} |I_{sk}^p| = N_p|I_s^p|2^{-(e+1)} = (1 - \frac{1}{2p})|I_s^p|, \tag{31}$$

$$\sum_{t\in\mathscr{T},s\subset t}|I_t^p| = |I_s^p|\frac{1}{1-(1-\frac{1}{2p})} = 2p|I_s^p|. \tag{32}$$

Now set $E_p := \bigcup_{s\in\mathscr{T}} \mathfrak{l}(I_s^p)$. Note that this union is disjoint. Then by (32) for each $s \in \mathscr{T}$

$$|I_s^p \cap E_p| = p|I_s^p|.$$

For any $r \in \mathbb{R}$ and $n = 0, 1, \dots$ set

$$S_n^{\text{dyadic}}(r) := \{[-1,1)\} \cup \left\{ r(1+r)^{|s|} I_s^p \;\middle|\; s \in \mathscr{T},\ |s| \le n-1 \right\}$$

where $|s|$ is the length of the sequence s.

This way for each n we have that $\{I_s^p \mid |s| = n\}$ consists of disjoint intervals and

$$\sum_{|s|=n} |I_s^p| = |I_{n+1}^p|.$$

So

$$A_{E_p}(S_n^{\text{dyadic}}(r)) = A_{[0,p)}(S_n(r)),$$
$$B_{E_p}(S_n^{\text{dyadic}}(r)) = B_{[0,p)}(S_n(r)).$$

hold. This already implies '$\le$' in (30). Thus it remains to show that for each n and S with $\mathbb{I}(S) \subset \mathbb{I}_n^{\text{dyadic}}$ there is a finite T with $\mathbb{I}(T) \subset \mathbb{I}_n^p$ and

$$\frac{B_{[0,p)}(T)}{A_{[0,p)}(T)} \le \frac{B_{E_p}(S)}{A_{E_p}(S)}. \tag{33}$$

Let $\mathbb{I}(S) \subset \mathbb{I}_n^{\text{dyadic}}$. Then S is of the form

$$S = \{a\mathbb{1}_{E_p}, bh_{[0,1),E_p}\} \cup T_1 \cup \ldots \cup T_{N_p}.$$

We claim that there is a partition $[0,1) = I_1 \cup I_2 \cup \ldots$, an E and disjointly supported $S_1, S_2, \ldots$ where for each $i = 1, 2, \ldots$ $a\mathbb{1}_{I_i\cap E} \in S_i$ and $\mathbb{I}(S_i)$, $E \cap I_i$ is a dilate and translate of $\mathbb{I}_n^p$, $[0,p)$ and

$$A_{E_p}(S) = \sum_i A_E(S_i),$$
$$B_{E_p}(S) = \sum_i B_E(S_i).$$

Then (33) will follow from Lemma A.9 and by rescaling.

We prove the claim by by induction on n. For $n = 0$ there is nothing to be done. So assume the claim holds for $n - 1$. Then we may apply the inductive hypothesis for each $k = 1, \ldots, N_p$ to

$$\left\{(a+b)\mathbb{1}_{I_k^{\text{dyadic}} \cap E_p}\right\} \cup T'_k$$

and obtain a partition $\{I_{ki} \mid k = 1, \ldots, N_p,\ i = 1, 2, \ldots\}$ of I_2^p, an $E' \subset I_2^p \subset [\frac{1}{2}, 1)$ and T'_{ki} for $k = 1, \ldots, N_p,\ i = 1, 2, \ldots$ with $(a+b)\mathbb{1}_{I'_{ki} \cap E'} \in T'_{ki}$. Add $[0, \frac{1}{2})$ to E' and set

$$S' := \left\{a\mathbb{1}_{E'}, bh_{[0,1),E'}\right\} \cup \bigcup_{k,i} T_{ki} \setminus \left\{(a+b)\mathbb{1}_{I'_{ki} \cap E'}\right\}.$$

Then

$$\begin{aligned} A_{E_p}(S) &= A_{E'}(S'), \\ B_{E_p}(S) &= B_{E'}(S'). \end{aligned}$$

Applying Lemma A.16 to S', E' proves the claim. This completes the proof of (30).

3.2 Computation of the Constant

Proposition 3.2. Abbreviate $\frac{2p}{2p-1} = (2p)'$. Then

$$\inf_{r \in \mathbb{R}} \frac{B_\infty(r)}{A_\infty(r)} = \frac{1}{2p} \frac{(2 - \sqrt{(2p)'})^2}{(1 - \sqrt{(2p)'})^2} = 27(p - \frac{2}{3})^2 + \mathcal{O}(p - \frac{2}{3})^3$$

Because this term will appear quite often, abbreviate

$$f(r) = \frac{1}{1 - (1 - \frac{1}{2p})(1+r)^2}.$$

Note that $I_1^p, I_2^p, \ldots$ all satisfy the first case $|I_{n+1}^p| = \frac{|\mathfrak{r}(I_n^p) \cap E|}{p}$ in the definition of 'most antiparallel'. Thus for $n \geq 1$ we have

$$|I_n^p| = (1 - \frac{1}{2p})^{n-1}.$$

Therefore

$$A_n(r) = |E \cap I_0^p| + \sum_{k=0}^{n-1} (1+r)^{2k} r^2 |E \cap I_{k+1}^p|$$

$$= p + pr^2 \sum_{k=0}^{n-1} [(1+r)^2(1-\frac{1}{2p})]^k$$
$$= p + pr^2 \begin{cases} \left\{1 - \left[(1-\frac{1}{2p})(1+r)^2\right]^n\right\} f(r) & (1-\frac{1}{2p})(1+r)^2 \neq 1 \\ n & (1-\frac{1}{2p})(1+r)^2 = 1 \end{cases}$$

This converges for $n \to \infty$ if and only if

$$(1-\frac{1}{2p})(1+r)^2 < 1$$
$$|1+r| < \frac{\sqrt{2p}}{\sqrt{2p-1}} = \sqrt{(2p)'}.$$

Hence

$$A_\infty(r) = \begin{cases} p + pr^2 f(r) & r \in \left(-\sqrt{(2p)'} - 1, \sqrt{(2p)'} - 1)\right) \\ \infty & \text{else} \end{cases}$$

Now to $B_n(r)$. First, by the definition of $S_k(r)$ it can be checked inductively that for $0 \leq k \leq n$ on $\mathfrak{r}(I_k^p \cap E)$ the function $F_k(r)$ attains the value

$$(1+r)^{k-1} + r(1+r)^{k-1} = (1+r)^k$$

and for $k \geq 1$ on $\mathfrak{l}(I_k^p \cap E)$ it attains the value

$$(1+r)^{k-1} - r(1+r)^{k-1} = (1-r)(1+r)^{k-1}.$$

The latter is also the value that $F_n(r)$ attains on $\mathfrak{l}(I_k^p \cap E)$. Furthermore

$$|\mathfrak{r}(I_k^p \cap E)| = (p - \frac{1}{2})|I_k^p| = p(1-\frac{1}{2p})^k,$$
$$|\mathfrak{l}(I_k^p \cap E)| = |\mathfrak{l}(I_k^p)| = \frac{1}{2}(1-\frac{1}{2p})^{k-1}.$$

Now as $\mathfrak{l}(I_1^p) \cup \ldots \cup \mathfrak{l}(I_n^p) \cup \mathfrak{r}(I_n^p \cap E)$ is a partition of E this means

$$B_n(r) = \sum_{k=0}^{n-1} \frac{1}{2}(1-\frac{1}{2p})^k (1-r)^2 (1+r)^{2k} + p(1-\frac{1}{2p})^n (1+r)^{2n} \tag{34}$$
$$= \frac{1}{2}(1-r)^2 \begin{cases} \left\{1 - \left[(1-\frac{1}{2p})(1+r)^2\right]^n\right\} f(r) & (1-\frac{1}{2p})(1+r)^2 \neq 1 \\ n & (1-\frac{1}{2p})(1+r)^2 = 1 \end{cases}$$
$$+ p\left[(1-\frac{1}{2p})(1+r)^2\right]^n$$

$$= \begin{cases} \frac{1}{2}(1-r)^2\big\{1-\big[(1-\frac{1}{2p})(1+r)^2\big]^n\big\}f(r) & \\ \qquad +p\big[(1-\frac{1}{2p})(1+r)^2\big]^n & (1-\frac{1}{2p})(1+r)^2 \neq 1 \\ \frac{1}{2}(1-r)^2 n + p & (1-\frac{1}{2p})(1+r)^2 = 1 \end{cases}$$

Hence just like for $A_\infty(r)$ we get

$$B_\infty(r) = \begin{cases} \frac{1}{2}(1-r)^2 f(r) & r \in \big(-\sqrt{(2p)'}-1, \sqrt{(2p)'}-1\big), \\ \infty & \text{else} \end{cases}. \tag{35}$$

Lemma 3.3. The allowed range of r, $\big(-\sqrt{(2p)'}-1, \sqrt{(2p)'}-1\big)$, lies left of the point 1.

Proof. Since $2p > \frac{4}{3}$ we have

$$(2p)' < (\frac{4}{3})' = \frac{\frac{4}{3}}{\frac{4}{3}-1} = \frac{\frac{4}{3}}{\frac{1}{3}} = 4 \tag{36}$$

so that $\sqrt{(2p)'} - 1 < 2 - 1 = 1$. □

Proof of Proposition 3.2.

$$\frac{B_\infty(r)}{A_\infty(r)} = \frac{\frac{1}{2}(1-r)^2 f(r)}{p + pr^2 f(r)} = \frac{1}{2p}\frac{(1-r)^2}{1-(1-\frac{1}{2p})(1+r)^2+r^2}.$$

This means

$$\begin{aligned} &p\big[1-(1-\frac{1}{2p})(1+r)^2+r^2\big]^2 \frac{\mathrm{d}}{\mathrm{d}r}\frac{B_\infty(r)}{A_\infty(r)} \\ &= -(1-r)[1-(1-\frac{1}{2p})(1+r)^2+r^2] - (1-r)^2[-(1-\frac{1}{2p})(1+r)+r] \\ &= -1+r-r^2+r^3-r+2r^2-r^3+(1-\frac{1}{2p})(1-r^2)[1+r+1-r] \\ &= -1+r^2+(1-\frac{1}{2p})(1-r^2)2 \\ &= (1-r^2)[1-\frac{1}{p}]. \end{aligned} \tag{37}$$

By Lemma 3.3 in the allowed range of r, (37) is positive for $r \leq -1$ and negative for $r \geq -1$. Thus the infimum of $\frac{B_\infty(r)}{A_\infty(r)}$ will be approached at the boundaries, where $(1-\frac{1}{2p})(1+r)^2 = 1$ so that

$$\frac{B_\infty(r)}{A_\infty(r)} \to \frac{1}{2p}\frac{(1-r)^2}{r^2}$$

and thus

$$r \to -\sqrt{(2p)'} - 1 : \qquad \frac{B_\infty(r)}{A_\infty(r)} \to \frac{1}{2p}\frac{(2+\sqrt{(2p)'})^2}{(1+\sqrt{(2p)'})^2},$$
$$r \to \sqrt{(2p)'} - 1 : \qquad \frac{B_\infty(r)}{A_\infty(r)} \to \frac{1}{2p}\frac{(2-\sqrt{(2p)'})^2}{(1-\sqrt{(2p)'})^2}.$$

Now note that $2p \leq 2$ so that $(2p)' \geq 2' = 2$ so that together with (36), Lemma A.24 says that the smaller of the two limits is

$$\frac{1}{2p}\frac{(2-\sqrt{(2p)'})^2}{(1-\sqrt{(2p)'})^2} \tag{38}$$

which thus is the global infimum of $\frac{B_\infty(r)}{A_\infty(r)}$.

It remains to compute (38) for p close to $\frac{2}{3}$. First note that if $a > 0$ and $f \geq 0$, $f(x) = \mathcal{O}(x)$ then

$$\frac{1}{a + f(x)} = \frac{1}{a} + \mathcal{O}(x). \tag{39}$$

Thus since $2p - 1 = \frac{1}{3} + 2(p - \frac{2}{3})$ we have

$$\frac{1}{2p-1} = 3 + \mathcal{O}(p - \frac{2}{3})$$

and

$$(2p)' - 4 = \frac{2p - 8p + 4}{2p - 1} = -\frac{6}{2p-1}(p - \frac{2}{3}) = -18(p - \frac{2}{3}) + \mathcal{O}(p - \frac{2}{3})^2.$$

Therefore by $\sqrt{4+x} = 2 + \frac{1}{4}x + \mathcal{O}(x^2)$ and with $x = (2p)' - 4$ we get

$$\sqrt{(2p)'} - 2 = \frac{1}{4}[(2p)' - 4] + \mathcal{O}[(2p) - 4]^2 = -\frac{9}{2}(p - \frac{2}{3}) + \mathcal{O}(p - \frac{2}{3})^2. \tag{40}$$

Invoking (39) once more and $2p = \frac{4}{3} + 2(p - \frac{2}{3})$ we get

$$\frac{1}{2p} = \frac{4}{3} + \mathcal{O}(p - \frac{2}{3}),$$

and by (39) and (40)

$$\frac{1}{\sqrt{(2p)'} - 1} = 1 + \mathcal{O}(p - \frac{2}{3}).$$

Altogether this implies

$$\frac{1}{2p}\frac{(2-\sqrt{(2p)'})^2}{(1-\sqrt{(2p)'})^2} = \frac{4}{3}\frac{9^2}{2^2}(p - \frac{2}{3})^2 + \mathcal{O}(p - \frac{2}{3})^3 = 27(p - \frac{2}{3})^2 + \mathcal{O}(p - \frac{2}{3})^3.$$

□

3.3 Proof of Theorem 3.1

Lemma 3.4. There is a $r_{\min} \geq 0$ such that

$$B_\infty(r_{\min}) = \inf\{B_E(S) \mid 1\mathbb{1}_E \in S,\ \mathbb{I}(S) = \mathbb{I}^p\}. \tag{41}$$

Proof. Let m be the infimum on the right hand side of (41). By Theorem 2.1 we have $m > 0$. Applying the minimization to the interval $[\frac{1}{2}, p)$ instead of $[0, p)$ we obtain

$$m = \inf_{r\in\mathbb{R}} \frac{1}{2}(1-r)^2 + \frac{(p-\frac{1}{2})}{p}(1+a)^2 m.$$

Since the right hand side is a quadratic polynomial, the infimum is attained for some $r_{\min}$, i.e.

$$m = \frac{1}{2}(1-r_{\min})^2 + (1-\frac{1}{2p})(1+r_{\min})^2 m. \tag{42}$$

First note that we can't have $r_{\min} = 1$ because then by Lemma 3.3 we'd have $(1-\frac{1}{2p})(1+r_{\min})^2 > 1$, and together with $m > 0$ this would contradict (42). Thus $\frac{1}{2}(1-r_{\min})^2 > 0$ so that (42) and $m > 0$ give

$$(1-\frac{1}{2p})(1+r_{\min})^2 < 1. \tag{43}$$

Now we insert (42) for m in the right hand side of (42) n times and get

$$m = \sum_{k=0}^{n} \frac{1}{2}(1-\frac{1}{2p})^k (1+r_{\min})^{2k}(1-r_{\min})^2 + \left[(1-\frac{1}{2p})(1+r_{\min})^2\right]^{n+1} m. \tag{44}$$

Comparing this with (34) we see that the right hand side of (44) tends to $B_\infty(r_{\min})$ for $n \to \infty$ because the second summand in (44) tends to 0 by (43). □

Remark. Another way to prove Lemma 3.4 is to compute the left hand side of (41) using (35), and to compute the right hand side of (41) by induction on n where we only take the infimum over S with $|S| = n$. This way we could also avoid the use of Theorem 2.1.

Lemma 3.5. For $r > r_{\min}$ in the allowed range of r we have

$$\frac{\partial}{\partial r} B_\infty(r) > 0.$$

Proof.

$$\frac{\partial}{\partial r} f(r) = 2(1-\frac{1}{2p})(1+r)f(r)^2$$

$$\begin{aligned}\frac{\partial}{\partial r}B_\infty(r) &= -(1-r)f(r) + \frac{1}{2}(1-r)^2\frac{\partial}{\partial r}f(r)\\ &= f(r)^2(1-r)\Big[-\big[1-(1-\frac{1}{2p})(1+r)^2\big] + (1-r)(1-\frac{1}{2p})(1+r)\Big]\\ &= f(r)^2(1-r)\Big[-1+2(1-\frac{1}{2p})(1+r)\Big]\end{aligned}$$

The first two factors are strictly greater than zero by Lemma 3.3. Since we know by Lemma 3.4 that $B_\infty(r)$ has a minimum at $r_{\min}$, the third factor must be zero at $r_{\min}$. Since the third factor is a first order polynomial in r with strictly positive derivative, it will be positive for all $r > r_{\min}$. Hence such will be the entire product. □

Proof of Theorem 3.1. We only have to consider finite S on the left hand side of (28). By Lemma A.1 it suffices to consider the case that all coefficients in S are nonnegative. Now let I_n^p be the largest interval in S with nonzero coefficient. By scaling we may assume that the coefficient is 1. Orthogonally split

$$h_{I_n^p,E} = -\mathbb{1}_{\mathfrak{l}(I_n^p)\cap E} + \mathbb{1}_{\mathfrak{r}(I_n^p)\cap E}.$$

We apply Lemma A.8 with $u = -\mathbb{1}_{\mathfrak{l}(I_n^p)\cap E}$, $v = \mathbb{1}_{\mathfrak{r}(I_n^p)\cap E}$ and then apply Lemma A.10. The validity of the hypothesis of Lemma A.10 can be seen when looking at the calculations of the proof of Lemma A.1. So this means it suffices to consider $S \setminus \{h_{I_n^p,E}\} \cup \{\mathbb{1}_{\mathfrak{r}(I_n^p)\cap E}\}$. Then by dilating and translating from $\mathfrak{r}(I_n^p)$ to $[0,1)$, we may pass to an S with $\mathbb{I}(S) \subset \mathbb{I}^p$, $1\cdot[-1,1) \in S$. Since we may furthermore always add intervals with coefficients 0, it suffices to consider the case that for some n, $\mathbb{I}(S) = \mathbb{I}_n^p$.

So we have shown that it remains to prove that for all n and S with $\mathbb{I}(S) = \mathbb{I}_n^p$, $1\cdot[-1,1) \in S$ there is an $r \geq 0$ with $(1-\frac{1}{2p})(1+r)^2 < 1$ such that

$$\begin{aligned}A_E(S_\infty(r)) &\geq A_E(S),\\ B_E(S_\infty(r)) &\leq B_E(S).\end{aligned}$$

We proceed by induction on n.

The case $n = 0$ works with $r = 0$. Now assume it holds for n and let S be a sequence with $\mathbb{I}(S) = \mathbb{I}_{n+1}^p$, $1\cdot[-1,1) \in S$. Again by Lemma A.1 it suffices to consider the case that all coefficients are positive. Denote the coefficient in front of I_1^p by r_1. Now set

$$T \coloneqq S \setminus \left\{I_0^p, r_1 I_1^p\right\} \cup \left\{(1+r_1)\mathbb{1}_{I_2^p\cap E}\right\}.$$

Then by the inductive hypothesis translated and dilated to I_2^p instead of $[0,1)$ and with the coefficients scaled by $(1+r_1)$, there is an $r_2 \geq 0$ with $(1-\frac{1}{2p})(1+r_2)^2 < 1$ such that

$$T' \coloneqq \{(1+r_1)\mathbb{1}_{I_2^p\cap E}, (1+r_1)r_2 I_2^p, (1+r_1)(1+r_2)r_2 I_3^p, (1+r_1)(1+r_2)^2 r_2 I_4^p, \dots\}$$

satisfies

$$A_E(T') \geq A_E(T),$$
$$B_E(T') \leq B_E(T).$$

Now with

$$S' \coloneqq T' \setminus \{(1+r_1)\mathbb{1}_{I_2^p \cap E}\} \cup \{I_0^p, r_1 I_1^p\}$$

we have

$$\begin{aligned}
A_E(S') &= A_E(T') - \|(1+r_1)\mathbb{1}_{I_2^p \cap E}\|_2^2 + \|h_{I_0^p,E}\|_2^2 + \|r_1 h_{I_1^p,E}\|_2^2 \\
&\geq A_E(T) - \|(1+r_1)\mathbb{1}_{I_2^p \cap E}\|_2^2 + \|h_{I_0^p,E}\|_2^2 + \|r_1 h_{I_1^p,E}\|_2^2 \\
&= A_E(S), \\
B_E(S') &= \|F_E(S')\|^2_{L^2([0,\frac{1}{2}))} + \|F_E(S')\|_{L^2([\frac{1}{2},1))} \\
&= \|h_{I_0^p,E} + r_1 h_{I_1^p,E}\|^2_{L^2([0,\frac{1}{2}))} + \|F_E(T')\|^2_{L^2([\frac{1}{2},1))} \\
&\leq \|h_{I_0^p,E} + r_1 h_{I_1^p,E}\|^2_{L^2([0,\frac{1}{2}))} + \|F_E(T)\|^2_{L^2([\frac{1}{2},1))} \\
&= B_E(S).
\end{aligned}$$

So if we can show that we may now pass to $r = r_1 = r_2$ while further increasing A and decreasing B, then note that we have transformed S' into the form $S_\infty(r)$ needed for the induction step. Denote

$$A(r_1, r_2) \coloneqq A_E(S'),$$
$$B(r_1, r_2) \coloneqq B_E(S').$$

Then

$$\begin{aligned}
A(r_1, r_2) &= p + pr_1^2 + (1 - \frac{1}{2p})(1+r_1)^2(A_\infty(r_2) - p) \\
&= p + pr_1^2 + (1 - \frac{1}{2p})(1+r_1)^2 pr_2^2 f(r_2), \\
B(r_1, r_2) &= \frac{1}{2}(1-r_1)^2 + (1 - \frac{1}{2p})(1+r_1)^2 B_\infty(r_2) \\
&= \frac{1}{2}(1-r_1)^2 + (1 - \frac{1}{2p})(1+r_1)^2 \frac{1}{2}(1-r_2)^2 f(r_2).
\end{aligned}$$

It remains to show that there is an $r \geq 0$ with $(1 - \frac{1}{2p})(1+r)^2 < 1$ such that $A_\infty(r) \geq A(r_1, r_2)$ and $B_\infty(r) \leq B(r_1, r_2)$. If with $r_{\min}$ from Lemma 3.4 we have $A(r_1, r_2) \leq$

$A(r_{\min})$, then by Lemma 3.4 $r := r_{\min}$ already does the job. So it suffices to show that for each $c > A_\infty(r_{\min})$

$$\inf\Big\{ B(r_1,r_2) \;\Big|\; r_1 \geq 0, r_2 \geq 0, (1-\frac{1}{2p})(1+r_2)^2 < 1,\ A(r_1,r_2)=c \Big\}$$

is attained on the diagonal $r_1 = r_2$. Denote the strip

$$D := \Big\{ (r_1,r_2) \;\Big|\; (1-\frac{1}{2p})(1+r_2)^2 < 1 \Big\}$$

and for $c > A_\infty(r_{\min})$ denote

$$D_c := \Big\{ (r_1,r_2) \in D \;\Big|\; A(r_1,r_2) = c \Big\}.$$

Then on $D \cap \{r_1 \geq 0\}$, A is smooth and increases with r_1 and r_2, i.e. ∇A points into the upper right quadrant. Furthermore on $D \cap \{r_1 \geq 0\}$, A tends to ∞ whenever (r_1,r_2) approaches the boundary of D, while $A(0,0) \leq A_\infty(r_{\min}) < c$. That means $D_c \cap \{r_1, r_2 \geq 0\}$ is a smooth curve of finite length that starts and ends somewhere on $r_1 = 0$. In order to get the infimum of B on $D_c \cap \{r_1, r_2 \geq 0\}$, we look for the points where ∇A and ∇B are parallel.

$$\begin{aligned}
\partial_1 A(r_1,r_2) &= 2pr_1 + 2(1-\frac{1}{2p})(1+r_1)pr_2^2 f(r_2) \\
&= f(r_2)\Big\{ 2pr_1\big[1-(1-\frac{1}{2p})(1+r_2)^2\big] + 2(1-\frac{1}{2p})(1+r_1)pr_2^2 \Big\} \\
&= f(r_2)\Big\{ 2pr_1 + (2p-1)\big[-r_1(1+r_2)^2 + (1+r_1)r_2^2\big] \Big\} \\
&= f(r_2)\Big\{ 2pr_1 + (2p-1)\big[-r_1 - 2r_1r_2 + r_2^2\big] \Big\} \\
&= f(r_2)\Big\{ r_1 + (2p-1)r_2(-2r_1 + r_2) \Big\} \\
\partial_2 A(r_1,r_2) &= (1-\frac{1}{2p})(1+r_1)^2 p \Big\{ 2r_2 f(r_2) + r_2^2(1-\frac{1}{2p})2(1+r_2)f(r_2)^2 \Big\} \\
&= (1-\frac{1}{2p})(1+r_1)^2 p f(r_2)^2 \Big\{ 2r_2\big[1-(1-\frac{1}{2p})(1+r_2)^2\big] \\
&\qquad + r_2^2(1-\frac{1}{2p})2(1+r_2) \Big\} \\
&= (1-\frac{1}{2p})(1+r_1)^2 p f(r_2)^2 \Big\{ r_2(1-\frac{1}{2p})(1+r_2)\big[-2(1+r_2)+2r_2\big] \\
&\qquad + 2r_2 \Big\}
\end{aligned}$$

$$
\begin{aligned}
&= (1-\frac{1}{2p})(1+r_1)^2 p f(r_2)^2 \left\{ -2r_2(1-\frac{1}{2p})(1+r_2) + 2r_2 \right\} \\
&= (1-\frac{1}{2p})(1+r_1)^2 2pr_2 \big[1-(1-\frac{1}{2p})(1+r_2)\big] f(r_2)^2 \\
&= (1-\frac{1}{2p})(1+r_1)^2 r_2 \big[1-(2p-1)r_2\big] f(r_2)^2
\end{aligned}
$$

$$
\begin{aligned}
\partial_1 B(r_1,r_2) &= -(1-r_1) + (1-\frac{1}{2p})(1+r_1)(1-r_2)^2 f(r_2) \\
&= f(r_2)\left\{ -(1-r_1)\big[1-(1-\frac{1}{2p})(1+r_2)^2\big] + (1-\frac{1}{2p})(1+r_1)(1-r_2)^2 \right\} \\
&= f(r_2)\left\{ -(1-r_1) + (1-\frac{1}{2p})\big[(1-r_1)(1+r_2)^2 + (1+r_1)(1-r_2)^2\big] \right\} \\
&= f(r_2)\left\{ -(1-r_1) + (1-\frac{1}{2p})2\big[1+r_2^2-2r_1r_2\big] \right\}
\end{aligned}
$$

$$
\begin{aligned}
\partial_2 B(r_1,r_2) &= (1-\frac{1}{2p})(1+r_1)^2 \left\{ -(1-r_2)f(r_2) + (1-r_2)^2(1-\frac{1}{2p})(1+r_2)f(r_2)^2 \right\} \\
&= (1-\frac{1}{2p})(1+r_1)^2 f(r_2)^2 \\
&\qquad \cdot \left\{ -(1-r_2)\big[1-(1-\frac{1}{2p})(1+r_2)^2\big] + (1-r_2)^2(1-\frac{1}{2p})(1+r_2) \right\} \\
&= (1-\frac{1}{2p})(1+r_1)^2 f(r_2)^2 \\
&\qquad \cdot \left\{ (1-r_2)(1-\frac{1}{2p})(1+r_2)(1+r_2+1-r_2) - (1-r_2) \right\} \\
&= (1-\frac{1}{2p})(1+r_1)^2 2(1-r_2)\big[(1-\frac{1}{2p})(1+r_2) - \frac{1}{2}\big] f(r_2)^2
\end{aligned}
$$

The gradients are parallel if and only if the following expression equals zero:

$$
\begin{aligned}
& f(r_2)^{-3} \det\big(\nabla A(r_1,r_2) \;\big|\; \nabla B(r_1,r_2)\big) \\
&= \big[r_1 + (2p-1)r_2(-2r_1+r_2)\big](1-\frac{1}{2p})(1+r_1)^2 2(1-r_2)\big[(1-\frac{1}{2p})(1+r_2)-\frac{1}{2}\big] \\
&\qquad - (1-\frac{1}{2p})(1+r_1)^2 r_2\big[1-(2p-1)r_2\big]\left\{ -(1-r_1) + (1-\frac{1}{2p})2\big[1+r_2^2-2r_1r_2\big] \right\} \\
&= (1-\frac{1}{2p})(1+r_1)^2 \left\{ \big[r_1 + (2p-1)r_2(-2r_1+r_2)\big]2(1-r_2)\big[(1-\frac{1}{2p})(1+r_2)-\frac{1}{2}\big] \right. \\
&\qquad \left. - r_2\big[1-(2p-1)r_2\big]\Big(-(1-r_1) + (1-\frac{1}{2p})2\big[1+r_2^2-2r_1r_2\big]\Big) \right\}
\end{aligned}
$$

$$= (1 - \frac{1}{2p})(1 + r_1)^2$$

$$\cdot \Big\{ \Big[(r_1 - r_2) + r_2 + (2p - 1)r_2(-2(r_1 - r_2) - r_2)\Big]2(1 - r_2)\big[(1 - \frac{1}{2p})(1 + r_2) - \frac{1}{2}\big]$$

$$- r_2\big[1 - (2p - 1)r_2\big]$$

$$\cdot \Big(-1 + (r_1 - r_2) + r_2 + (1 - \frac{1}{2p})2\big[1 - r_2^2 - 2(r_1 - r_2)r_2\big]\Big)\Big\}$$

$$= (1 - \frac{1}{2p})(1 + r_1)^2$$

$$\cdot \Big\{ (r_1 - r_2)\Big(\big[1 - 2(2p - 1)r_2\big]2(1 - r_2)\big[(1 - \frac{1}{2p})(1 + r_2) - \frac{1}{2}\big]$$

$$- r_2\big[1 - (2p - 1)r_2\big]\big[1 - 4r_2(1 - \frac{1}{2p})\big]\Big)$$

$$+ \Big(r_2\big[1 - (2p - 1)r_2\big]2(1 - r_2)\big[(1 - \frac{1}{2p})(1 + r_2) - \frac{1}{2}\big]$$

$$- r_2\big[1 - (2p - 1)r_2\big]\big[-1 + r_2 + (1 - \frac{1}{2p})2(1 - r_2^2)\big]\Big)\Big\}$$

$$= (1 - \frac{1}{2p})(1 + r_1)^2(r_1 - r_2)$$

$$\cdot \Big\{ \big[1 - 2(2p - 1)((1 + r_2) - 1)\big]2[2 - (1 + r_2)]\big[(1 - \frac{1}{2p})(1 + r_2) - \frac{1}{2}\big]$$

$$[-(1 + r_2) + 1]\big[2p - (2p - 1)(1 + r_2)\big]\big[5 - \frac{2}{p} - 4(1 + r_2)(1 - \frac{1}{2p})\big]\Big\}$$

$$= (1 - \frac{1}{2p})(1 + r_1)^2(r_1 - r_2)$$

$$\Big\{ (1 + r_2)^2\Big[-2(2p - 1) - 8(2p - 1)(1 - \frac{1}{2p}) - [1 + 2(2p - 1)]2(1 - \frac{1}{2p})$$

$$+ (2p - 1)(5 - \frac{2}{p}) + 4(2p - 1) + 4(2p - 1)(1 - \frac{1}{2p})\Big]$$

$$+ (1 + r_2)\Big[4(2p - 1) + [1 + 2(2p - 1)] + 4[1 + 2(2p - 1)](1 - \frac{1}{2p})$$

$$- 2p(5 - \frac{2}{p}) - (2p - 1)(5 - \frac{2}{p}) - 4(2p - 1)\Big]$$

$$+ \Big[-2[1 + 2(2p - 1)] + 2p(5 - \frac{2}{p})\Big]\Big\}$$

$$= (1 - \frac{1}{2p})(1 + r_1)^2(r_1 - r_2)$$

$$
\begin{aligned}
&\cdot \Big\{ (1+r_2)^2(1-\frac{1}{2p})\Big[-4p - 8(2p-1) - 2 - 4(2p-1) \\
&\qquad\qquad\qquad + 2p(5-\frac{2}{p}) + 8p + 4(2p-1)\Big] \\
&\quad + (1+r_2)\Big[p(8+4+16-10-10-8) \\
&\qquad\qquad + (-4-1+4-8-8+4+5+4+4) + \frac{1}{p}(2-2)\Big] \\
&\quad + [2p-2]\Big\} \\
&= (1-\frac{1}{2p})(1+r_1)^2(r_1-r_2)\Big\{(1+r_2)^2(1-\frac{1}{2p})[-2p+2] + [2p-2]\Big\} \\
&= -2(1-p)(1-\frac{1}{2p})(1+r_1)^2(r_1-r_2)\frac{1}{f(r_2)}
\end{aligned}
$$

This equals zero if and only if $r_1 = r_2$. Hence $B(r_1, r_2)$ has exactly one local extremum on $D_c \cap \{r_1, r_2 \geq 0\}$ that is not an endpoint, because $A(r,r) = A_\infty(r)$ is strictly increasing for $r \geq 0$. So let $(\tilde{r}, \tilde{r}) \in D_c \cap \{r_1, r_2 \geq 0\}$ be that unique extremum. If it is a local mininmum, it is also the global minimum, because if there was another point $(r_1, r_2) \in D_c \cap \{r_1, r_2 \geq 0\}$ with $B(r_1, r_2) < B(\tilde{r}, \tilde{r})$, there would have to be a second local extremum in between $(\tilde{r}, \tilde{r})$ and (r_1, r_2).

Since for $r \geq r_{\min}$ we have $\frac{\mathrm{d}}{\mathrm{d}r}A(r,r) > 0$ and by Lemma 3.5 also $\frac{\mathrm{d}}{\mathrm{d}r}B(r,r) > 0$, ∇A and ∇B must be parallel and not antiparallel at $(\tilde{r}, \tilde{r})$. Further by our computation

$$
\operatorname{sign}\det\big(\nabla A(r_1, r_2) \bigm| \nabla B(r_1, r_2)\big) = \operatorname{sign}(r_2 - r_1).
$$

That means that if we move from $(\tilde{r}, \tilde{r})$ along $D_c \cap \{r_1, r_2 \geq 0\}$ in the direction where r_1 increases and r_2 decreases, $\det(\nabla A \mid \nabla B)$ becomes negative which means that ∇B bends into the direction of motion, implying that B increases. Since it gets the other sign if we move the other way, B also increases in that direction. Thus our local extremum is a minimum and we are done. □

4 Extensions of Theorem 3.1

4.1 The Case $E = [0, b)$

The main result of this subsection is the following Theorem 4.1.

Theorem 4.1. *Let $p > \frac{2}{3}$, $E \subset [0,1)$ and $S = \{a_1 I_1, a_2 I_2, \ldots\}$ be a set of compatible E-dominant intervals with some coefficients, where for $i = 1, 2\ldots$ we have $I_{i+1} \subset I_i$.*

Then there is a T with $\mathbb{I}(T) = \mathbb{I}_p$ and such that

$$\frac{B_{[0,p)}(T)}{A_{[0,p)}(T)} \le \frac{B_E(S)}{A_E(S)}.$$

Corollary 4.2. Let $p > \frac{2}{3}$, $b \in [0,1]$ and $E = [0,b)$ and S be a set of compatible E-dominant intervals with some coefficients. Then there is a T with $\mathbb{I}(T) = \mathbb{I}_p$ such that

$$\frac{B_{[0,p)}(T)}{A_{[0,p)}(T)} \le \frac{B_E(S)}{A_E(S)}.$$

Proof of Corollary 4.2. By Lemma A.10 we may remove those intervals from S that do not contain b, since all their respective restricted Haar functions are orthogonal to all other restricted Haar functions. Then call the remaining intervals $I_1, I_2, \ldots$. Because they are compatible and all contain b, they can be ordered in such a way such that for $i = 1, 2, \ldots$ we have $I_{i+1} \subset I_i$. □

We first show a few lemmas and propositions that provide the steps in the proof of Theorem 4.1.

Definition. For vectors $u, v \neq 0$ we define

$$\sphericalangle\big[u, v\big] := \frac{\langle u, v\rangle}{\|u\|\,\|v\|}.$$

If $0 \in \{u, v\}$ we set

$$\sphericalangle\big[u, v\big] := 0.$$

Definition. For $E \subset [0,1)$, an ordered set of compatible intervals

$$\mathbb{I} = \{I_1, I_2, \ldots\}$$

is called *nested* if for all $i = 1, 2, \ldots$ we have

$$I_{i+1} \subset \mathfrak{r}(I_i).$$

Remark. Let $\mathbb{I}$ be compatible and nested and for all $I \in \mathbb{I},\ I \neq [-1,1)$ assume $|\mathfrak{l}(I) \cap E| \ge |\mathfrak{r}(I) \cap E|$. Then for any $I, J \in \mathbb{I}$ we have $\sphericalangle\big[h_{I,E}, h_{J,E}\big] \le 0$.

Definition. For a nested sequence $S = \{a_0 I_0, a_1 I_1, \ldots\}$ define

$$U(S) := \Big\{ k \ge 1 \;\Big|\; |a_k| > |\sum_{i=0}^{k-1} a_i| \Big\}.$$

Proposition 4.3. Let $p > \frac{2}{3}$, $E \subset [0,1)$ and $S = \{a_0 I_0, \dots, a_n I_n\}$ be a nested sequence of compatible E-dominant intervals. Then there is an E' and a nested sequence T of compatible E'-dominant intervals with

$$\frac{B_{E'}(T)}{A_{E'}(T)} \leq \frac{B_E(S)}{A_E(S)} \tag{45}$$

and $U(T) = \emptyset$.

Proof. Assume $U(S) \neq \emptyset$ and take $k = \min U(S)$. Abbreviate

$$\begin{aligned} S_0 &:= \{a_0 I_0, \dots, a_{k-1} I_{k-1}\}, \\ S_1 &:= \{a_k I_k, \dots, a_n I_n\}. \end{aligned}$$

We will prove the proposition by induction on $|S_1| = |S| - \min U(S)$. In case $U(S) = \emptyset$ we set $\min U(S) = |S|$. This is the base case where there is nothing to be done. So assume $k = \min U(S) < |S|$.

If $F_E(S_0)\mathbb{1}_{I_k} = 0$ then $F_E(S_0)$ and $F_E(S_1)$ are orthogonal so that by Lemma A.9, (45) holds for $E' := E$ and $T := S_0$ or $T := S_1$. For the case $T = S_0$ we are done because $U(S_0) = \emptyset$. Since $S_1 \neq \emptyset$ which by definition implies $\min U(S_1) \geq 1$ we have for the case $T = S_1$ that $|S_1| - \min U(S_1) < |S_1| = |S| - \min U(S)$ so that we may apply the inductive hypothesis to S_1.

Thus it remains to consider $F_E(a_0 I_0, \dots, a_{k-1} I_{k-1})\mathbb{1}_{I_k} \neq 0$. Now for $|\mathfrak{l}(I_k) \cap E| \gtrless |\mathfrak{r}(I_k) \cap E|$ set

$$\begin{aligned} u &:= F_E(a_0 I_0, \dots, a_{k-1} I_k)\mathbb{1}_{I_k} = (\sum_{i=0}^{k-1} a_i)\mathbb{1}_{I_k \cap E}, \\ v &:= \pm(\sum_{i=0}^{k-1} a_i) h_{I_k, E}, \\ a &:= \pm \frac{a_k}{\sum_{i=0}^{k-1} a_i}, \end{aligned}$$

where we take (e.g.) the positive sign for the case of equality. Since $p > \frac{1}{2}$ we have $\langle u, v\rangle > -\|u\|\|v\|$ and by the choice of the signs also $\langle u, v\rangle \leq 0$. Furthermore since $I_k \in U(S)$ we have $|a| > 1$. Hence we may apply Lemma A.11. Replacing b by $b(\sum_{i=0}^{k-1} a_i)^{-1}$, (115) becomes

$$\begin{aligned} \|F_E(a_0 I_0, \dots, a_k I_k)\mathbb{1}_{I_k}\|_2^2 = \Big\| F_E\Big(a_0 I_0, \dots, a_{k-1} I_{k-1}, \sum_{i=0}^{k-1} a_i I_k\Big)\mathbb{1}_{I_k}\Big\|_2^2 \\ + \|b\mathbb{1}_{I_k \cap E} \pm b h_{I_k, E}\|_2^2 \end{aligned} \tag{46}$$

holds.

Now define $E^\perp$ as follows: If we are in the case $|\mathfrak{l}(I_k)\cap E| \geq |\mathfrak{r}(I_k)\cap E|$ simply set $E^\perp := E$. For later use define $J := \mathfrak{r}(I_k)$.

In the other case $|\mathfrak{l}(I_k) \cap E| < |\mathfrak{r}(I_k) \cap E|$ set $E^\perp \setminus I_k := E \setminus I_k$. Set $E^\perp \cap \mathfrak{l}(I_k) := E \cap \mathfrak{r}(I_k) - \frac{1}{2}|I_k|$. Set J to be the interval with the same left boundary as $\mathfrak{r}(I_k)$ but length $\frac{|E\cap\mathfrak{l}(I_k)|}{|E\cap\mathfrak{r}(I_k)|}|\mathfrak{r}(I_k)| < |\mathfrak{r}(I_k)|$. Then set $E^\perp \cap \mathfrak{r}(I_k)$ to be a translated and dilated version of $E \cap \mathfrak{r}(I_k)$ according to the transformation $\mathfrak{l}(I_k) \to J$. Then we have $|E^\perp \cap \mathfrak{l}(I_k)| = |E \cap \mathfrak{r}(I_k)|$, $|E^\perp \cap \mathfrak{r}(I_k)| = |E \cap \mathfrak{l}(I_k)|$.

This means in both cases we have $\|b\mathbb{1}_{I_k\cap E} \pm bh_{I_k,E}\|_2^2 = \|b\mathbb{1}_{I_k\cap E^\perp} + bh_{I_k,E^\perp}\|_2^2$ so that we may rewrite (46) as

$$\|F_E(a_0I_0,\dots,a_kI_k)\mathbb{1}_{I_k}\|_2^2 = \Big\|F_E\Big(a_0I_0,\dots,a_{k-1}I_{k-1},\sum_{i=0}^{k-1}a_iI_k\Big)\mathbb{1}_{I_k}\Big\|_2^2 + \|b\mathbb{1}_{I_k\cap E^\perp} + bh_{I_k,E^\perp}\|_2^2. \tag{47}$$

Now while the function on the left hand side of (47) also has mass on $\mathfrak{l}(I_k)$, both functions on the right are only supported on $\mathfrak{r}(I_k)$. In particular $F_E\big(a_0I_0,\dots,a_{k-1}I_{k-1},\sum_{i=0}^{k-1}a_iI_k\big)\mathbb{1}_{I_k}$ is just a scaled version of $F_E(a_0I_0,\dots,a_kI_k)\mathbb{1}_{I_k}$ and $b\mathbb{1}_{I_k\cap E^\perp}+bh_{I_k,E^\perp}$ is additionally dilated according to the transformation $\mathfrak{r}(I_k) \to J$. This means that for

$$F_{\text{tail}} := F_E(a_{k+1}I_{k+1},\dots,a_nI_n)$$

and $F_{\text{tail}}^\perp$, which shall be F_{tail} dilated according to the transformation $\mathfrak{r}(I_k) \to J$, and $c^\parallel, c^\perp$ with those signs such that the right hand sides of (48) are negative, we have[7]

$$\sphericalangle\big[F_E(a_0I_0,\dots,a_kI_k)\mathbb{1}_{I_k}, F_{\text{tail}}\big] \geq \sphericalangle\Big[F_E\Big(a_0I_0,\dots,a_{k-1}I_{k-1},\sum_{i=0}^{k-1}a_iI_k\Big)\mathbb{1}_{I_k}, c^\parallel F_{\text{tail}}\Big] = \sphericalangle\big[b\mathbb{1}_{I_k\cap E^\perp} + bh_{I_k,E^\perp}, c^\perp F_{\text{tail}}^\perp\big]. \tag{48}$$

Also fix the absolute values of $c^\parallel, c^\perp$ such that

$$(c^\parallel)^2|\mathfrak{r}(I_k)| + (c^\perp)^2|J| = \mathfrak{r}(I_k), \tag{49}$$

$$\frac{(c^\perp)^2|J|}{(c^\parallel)^2|\mathfrak{r}(I_k)|} = \frac{\|b\mathbb{1}_{I_k\cap E^\perp} + bh_{I_k,E^\perp}\|_2^2}{\big\|F_E\big(a_0I_0,\dots,a_{k-1}I_{k-1},\sum_{i=0}^{k-1}a_iI_k\big)\mathbb{1}_{I_k}\big\|_2^2}. \tag{50}$$

Now we add dilated and scaled versions of F_{tail} to both sides of (47) and get by (48), (49), (50) that

$$\|F_E(S)\mathbb{1}_{I_k}\|_2^2 \geq \Big\|F_E\Big(a_0I_0,\dots,a_{k-1}I_{k-1},\sum_{i=0}^{k-1}a_iI_k\Big)\mathbb{1}_{I_k} + c^\parallel F_{\text{tail}}\Big\|_2^2$$

[7](48) also holds for any other function supported on $\mathfrak{r}(I_k)$ instead of F_{tail}.

$$+ \|b\mathbb{1}_{I_k \cap E^\perp} + bh_{I_k,E^\perp} + c^\perp F_{\text{tail}}^\perp\|_2^2.$$

Adding $\|F_E(S)\mathbb{1}_{I_k^\complement}\|_2^2$ to both sides we get

$$\|F_E(S)\|_2^2 \geq \left\|F_E\Big(a_0 I_0, \ldots, a_{k-1} I_{k-1}, \sum_{i=0}^{k-1} a_i I_k\Big) + c^\parallel F_{\text{tail}}\right\|_2^2 + \|b\mathbb{1}_{I_k \cap E^\perp} + bh_{I_k,E^\perp} + c^\perp F_{\text{tail}}^\perp\|_2^2. \tag{51}$$

Furthermore by (116) we have

$$\begin{aligned}\|a_k h_{I_k,E}\|_2^2 &\leq \|\sum_{i=0}^{k-1} a_i h_{I_k,E}\|_2^2 + \|b\mathbb{1}_{I_k \cap E}\|_2^2 + \|bh_{I_k,E}\|_2^2 \\ &= \|\sum_{i=0}^{k-1} a_i h_{I_k,E}\|_2^2 + \|b\mathbb{1}_{I_k \cap E^\perp}\|_2^2 + \|bh_{I_k,E^\perp}\|_2^2. \end{aligned} \tag{52}$$

For $i = k+1, \ldots, n$ set $I_i^\perp$ to be the translate and dilate of I_i according to the transformation $\mathfrak{r}(I_k) \to J$. Then adding $\sum_{i \neq k} \|a_i h_{I_i,E}\|_2^2$ to both sides of (52) and applying (49) we get

$$\sum_{i=0}^{\infty} \|a_i h_{I_i,E}\|_2^2 \leq \sum_{i=0}^{k-1} \|a_i h_{I_i,E}\|_2^2 + \|\sum_{i=0}^{k-1} a_i h_{I_k,E}\|_2^2 + \sum_{i=k+1}^{\infty} \|c^\parallel h_{I_i,E}\|_2^2 + \|b\mathbb{1}_{I_k \cap E^\perp}\|_2^2 + \|bh_{I_k,E^\perp}\|_2^2 + \sum_{i=k+1}^{\infty} \|c^\perp a_i h_{I_i^\perp,E}\|_2^2. \tag{53}$$

Now set

$$T_1 := \Big\{a_0 I_0, \ldots, a_{k-1} I_{k-1}, (\sum_{i=0}^{k-1} a_i) I_k, c^\parallel a_{k+1} I_{k+1}, \ldots, c^\parallel a_n I_n\Big\}, \qquad E_1 := E,$$

$$T_2 := \Big\{b\mathbb{1}_{I_k}, bI_k, c^\perp a_{k+1} I_{k+1}^\perp, \ldots, c^\perp a_n I_n^\perp\Big\}, \qquad E_2 := E^\perp.$$

Then combining (51) and (53) we see that by Lemma A.25 there is an $i \in \{1, 2\}$ with

$$\frac{B_{E_i}(T_i)}{A_{E_i}(T_i)} \leq \frac{B_E(S)}{A_E(S)}.$$

Also we have

$$|T_i| - \min U(T_i) \leq (n+1) - (k+1) < (n+1) - k = |S| - \min U(S)$$

so that we may apply the inductive hypothesis to T_i, E_i. □

Proposition 4.4. Let $p > \frac{2}{3}$. Let $E \subset [0,1)$ and $S = \{a_1 I_1, a_2 I_2, \dots\}$ be compatible, E-dominant and nested with $U(S) = \emptyset$. Then there is an $E' \subset [0,1)$ and a T which is compatible, E'-dominant and nested with $U(T) = \emptyset$ and

$$\frac{B_{E'}(T)}{A_{E'}(T)} \leq \frac{B_E(S)}{A_E(S)}$$

and where for each $I \in \mathbb{I}(T),\ I \neq [-1,1)$ we have

$$|\mathfrak{l}(I) \cap E'| \geq |\mathfrak{r}(I) \cap E'|. \tag{54}$$

Proof. We prove the proposition by induction on the number of indices i for which (54) fails for I_i, E. In the base case there are no such indices so there is nothing to be done. So assume such indices exist and let i be the largest index for which (54) fails. Since $U(S) = \emptyset$ we have $|a_i| \leq |\sum_{j=1}^{i-1} a_j|$. Since we may simply remove $a_i I_i$ from S if $a_i = 0$ it suffices to consider $\sum_{j=1}^{i-1} a_j \neq 0$. Then by rescaling we may pass to $\sum_{j=1}^{i-1} a_j = 1$. Abbreviate

$$\begin{aligned} S_0 &\coloneqq \{a_1 I_1, \dots, a_{i-1} I_{i-1}\}, \\ S_1 &\coloneqq \{a_{i+1} I_{i+1}, a_{i+2} I_{i+2}, \dots\}. \end{aligned}$$

Then $F_E(S_0)\mathbb{1}_{I_i} = \mathbb{1}_{I_i \cap E}$. Now there are two cases to consider:

Case $a_i \geq 0$ Since (54) fails we have that $\sphericalangle\left[\mathbb{1}_{I_i \cap E}, h_{I_i,E}\right] \geq 0$. Therefore there is an $a \geq a_i$ s.t.

$$\|\mathbb{1}_{I_i \cap E} + a_i h_{I_i,E}\|_2^2 = \|\mathbb{1}_{I_i \cap E}\|_2^2 + \|a h_{I_i,E}\|_2^2 = \|\mathbb{1}_{I_i \cap E}\|_2^2 + \|a \mathbb{1}_{I_i \cap E}\|_2^2. \tag{55}$$

Now decompose I_i into two intervals $I_{\parallel} \cup I_{\perp} = I_i$ and take $\tilde{E}$ such that

- $\tilde{E} \setminus I_i = E \setminus I_i$,
- $|\tilde{E} \cap I_i| = |E \cap I_i|$,
- $I_{\parallel} \cap \tilde{E}$ is a dilate and translate of $\mathfrak{r}(I_i) \cap E$,
- $\|\mathbb{1}_{I_i \cap E} + a_i h_{I_i,E}\|_{L^2(\mathfrak{r}(I_i))}^2 = \|\mathbb{1}_{I_i \cap \tilde{E}}\|_{L^2(I_{\parallel})}^2 + \|a \mathbb{1}_{I_i \cap \tilde{E}}\|_{L^2(I_{\parallel})}^2$.

This is possible due to (55) and $\frac{|\mathfrak{r}(I_i) \cap E|}{|\mathfrak{r}(I_i)|} \geq \frac{|I_i \cap E|}{|I_i|}$. Now $\mathbb{1}_{I_{\parallel} \cap \tilde{E}}$ and $a\mathbb{1}_{I_{\parallel} \cap \tilde{E}}$ are translated, dilated and scaled versions of $(\mathbb{1}_{I_i \cap E} + a_i h_{I_i,E})\mathbb{1}_{\mathfrak{r}(I_i)}$. Take S_1^1 and S_1^2 to be their respective corresponding translated, dilated and scaled versions of S_1. Then

$$A_E(S_1) = A_{\tilde{E}}(S_1^1) + A_{\tilde{E}}(S_1^2),$$

$$\|\mathbb{1}_{\mathfrak{r}(I_i)\cap E} + a_i h_{I_i,E\cap\mathfrak{r}(I_i)} + F_E(S_1)\|_2^2 = \|\mathbb{1}_{I_\|\cap\tilde{E}} + F_{\tilde{E}}(S_1^1)\|_2^2 + \|a\mathbb{1}_{I_\|\cap\tilde{E}} + F_{\tilde{E}}(S_1^2)\|_2^2$$

and thus, recalling that $F_E(S_0)\mathbb{1}_{I_i} = \mathbb{1}_{I_i\cap E}$ and hence also $F_{\tilde{E}}(S_0)\mathbb{1}_{I_i} = \mathbb{1}_{I_i\cap\tilde{E}}$, we get

$$\begin{aligned}
A_E(S) &= A_E(S_0) + \|a_i h_{I_i,E}\|_2^2 + A_E(S_1)\\
&\le A_{\tilde{E}}(S_0) + \|a\mathbb{1}_{I_i\cap\tilde{E}}\|_2^2 + A_{\tilde{E}}(S_1^1) + A_{\tilde{E}}(S_1^2)\\
&= A_{\tilde{E}}(S_0\cup S_1^1) + A_{\tilde{E}}\big(\{a\mathbb{1}_{I_i\cap\tilde{E}}\}\cup S_1^2\big)\\
B_E(S) &= \|F_E(S_0)\|^2_{L^2(I_i^\complement)}\\
&\qquad + \|F_E(S_0) + a_i h_{I_i,E}\|^2_{L^2(I_i)}\\
&\qquad - \|F_E(S_0) + a_i h_{I_i,E}\|^2_{L^2(\mathfrak{r}(I_i))}\\
&\qquad + \|F_E(S)\|^2_{L^2(\mathfrak{r}(I_i))}\\
&= \|F_{\tilde{E}}(S_0)\|^2_{L^2(I_i^\complement)}\\
&\qquad + \|F_E(S_0)\|^2_{L^2(I_i)} + \|a\mathbb{1}_{I_i\cap E}\|_2^2\\
&\qquad - \|F_{\tilde{E}}(S_0)\|^2_{L^2(I_\|)} - \|a\mathbb{1}_{I_\|\cap\tilde{E}}\|_2^2\\
&\qquad + \|F_{\tilde{E}}(S_0\cup S_1^1)\|^2_{L^2(I_\|)} + \|a\mathbb{1}_{I_\|\cap\tilde{E}} + F_{\tilde{E}}(S_1^2)\|_2^2\\
&= \|F_{\tilde{E}}(S_0)\|^2_{L^2(I_i^\complement)} + \|F_{\tilde{E}}(S_0)\|^2_{L^2(I_i)} - \|F_{\tilde{E}}(S_0)\|^2_{L^2(I_\|)} + \|F_{\tilde{E}}(S_0\cup S_1^2)\|^2_{L^2(I_\|)}\\
&\qquad + \|a\mathbb{1}_{I_i\cap\tilde{E}}\|_2^2 - \|a\mathbb{1}_{I_\|\cap\tilde{E}}\|_2^2 + \|a\mathbb{1}_{I_\|\cap\tilde{E}} + F_{\tilde{E}}(S_1^2)\|_2^2\\
&= \|F_{\tilde{E}}(S_0\cup S_1^1)\|_2^2 + \|F_{\tilde{E}}\big(\{a\mathbb{1}_{I_i\cap\tilde{E}}\}\cup S_1^2\big)\|_2^2
\end{aligned}$$

The way they were scaled, both $S_0\cup S_1^1$ and $\{a\mathbb{1}_{I_i\cap\tilde{E}}\}\cup S_1^2$ still satisfy $U=\varnothing$. They also have at least one interval less that violates (54) w.r.t. $\tilde{E}$ than S does w.r.t. E, and by Lemma A.25 one of them has a smaller $\frac{B}{A}$ than S. We apply the induction hypothesis to this sequence and are done with the first case.

Case $a_i \le 0$ Since $U(S)=\varnothing$ we have $|a_i|\le 1$. Also, if $a_i=-1$ then all coefficients in S_1 must be zero so that we may remove S_1 completely. Then we can simply turn $|\mathfrak{l}(I_i)\cap E| < |\mathfrak{r}(I_i)\cap E|$ around while preserving A and B, by swapping $\mathfrak{l}(I_i)\cap E$ and $\mathfrak{r}(I_i)\cap E$ and flipping the sign of a_i. That way all properties of the proposition are preserved and also (54) holds for all intervals because we assumed that it already holds for $I_1,\ldots,I_{i-1}$ anyways.

So it suffices to consider $-1 < a_i \le 0$. Then by Lemma A.17 there is an a with $|a_i|\le|a|\le 1$ and

$$\|\mathbb{1}_{I_i\cap E} + a_i h_{I_i,E}\|^2_{L^2(\mathfrak{r}(I_i))} \le \|\mathbb{1}_{I_i\cap E} - a h_{I_i,E}\|^2_{L^2(\mathfrak{l}(I_i))}, \tag{56}$$

$$\|\mathbb{1}_{I_i\cap E} + a_i h_{I_i,E}\|_2^2 = \|\mathbb{1}_{I_i\cap E} - a h_{I_i,E}\|_2^2. \tag{57}$$

Now decompose $\mathfrak{r}(I_i)$ into two intervals $I_\| \cup I_\perp = \mathfrak{r}(I_i)$ and take $\tilde{E}$ such that

- $\tilde{E} \setminus I_i = E \setminus I_i$,
- $|\tilde{E} \cap \mathfrak{l}(I_i)| = |E \cap \mathfrak{r}(I_i)|$, $|\tilde{E} \cap \mathfrak{r}(I_i)| = |E \cap \mathfrak{l}(I_i)|$ which means

$$\|\mathbb{1}_{I_i\cap E} - a h_{I_i,E}\|^2_{L^2(\mathfrak{l}(I_i))} = \|\mathbb{1}_{I_i\cap \tilde{E}} + a h_{I_i,\tilde{E}}\|^2_{L^2(\mathfrak{r}(I_i))},$$
$$\|\mathbb{1}_{I_i\cap E} - a h_{I_i,E}\|^2_{L^2(\mathfrak{r}(I_i))} = \|\mathbb{1}_{I_i\cap \tilde{E}} + a h_{I_i,\tilde{E}}\|^2_{L^2(\mathfrak{l}(I_i))}.$$

- $I_\| \cap \tilde{E}$ is a dilate and translate of $\mathfrak{r}(I_i) \cap E$,
- $\|\mathbb{1}_{I_i\cap E} + a_i h_{I_i,E}\|^2_{L^2(\mathfrak{r}(I_i))} = \|\mathbb{1}_{I_i\cap \tilde{E}} + a h_{I_i,\tilde{E}}\|^2_{L^2(I_\|)}$.

This is possible due to (56) and $\frac{|\mathfrak{r}(I_i)\cap E|}{|\mathfrak{r}(I_i)|} \geq \frac{|\mathfrak{l}(I_i)\cap E|}{|\mathfrak{l}(I_i)|} = \frac{|\mathfrak{r}(I_i)\cap \tilde{E}|}{|\mathfrak{r}(I_i)|}$. Now let $\tilde{S}_1$ be S_1 translated and dilated according to $\mathfrak{r}(I_i) \to I_\|$, and with coefficients scaled according to $1 + a_i \mapsto 1 + a$. Then $A_{\tilde{E}}(\tilde{S}_1) = A_E(S_1)$ and

$$\|\mathbb{1}_{I_i\cap E} + a_i h_{I_i,E} + F_E(S_1)\|^2_{L^2(\mathfrak{r}(I_i))} = \|\mathbb{1}_{I_i\cap \tilde{E}} + a h_{I_i,E} + F_{\tilde{E}}(\tilde{S}_1)\|^2_{L^2(I_\|)}.$$

Therefore recalling that $F_E(S_0)\mathbb{1}_{I_i} = \mathbb{1}_{I_i\cap E}$ and $F_{\tilde{E}}(S_0)\mathbb{1}_{I_i} = \mathbb{1}_{I_i\cap \tilde{E}}$ we get

$$\begin{aligned}
A_E(S) &= A_E(S_0) + \|a_i h_{I_i,E}\|_2^2 + A_E(S_1)\\
&\leq A_{\tilde{E}}(S_0) + \|a h_{I_i,\tilde{E}}\|_2^2 + A_{\tilde{E}}(\tilde{S}_1)\\
&= A_{\tilde{E}}\big(S_0 \cup \{aI_i\} \cup \tilde{S}_1\big),\\
B_E(S) &= \|F_E(S_0)\|^2_{L^2(I_i^\complement)}\\
&\qquad + \|F_E(S_0) + a_i h_{I_i,E}\|^2_{L^2(I_i)}\\
&\qquad - \|F_E(S_0) + a_i h_{I_i,E}\|^2_{L^2(\mathfrak{r}(I_i))}\\
&\qquad + \|F_E(S_0) + a_i h_{I_i,E} + F_E(S_1)\|^2_{L^2(\mathfrak{r}(I_i))}\\
&= \|F_{\tilde{E}}(S_0)\|^2_{L^2(I_i^\complement)}\\
&\qquad + \|F_E(S_0) + a h_{I_i,\tilde{E}}\|^2_{L^2(I_i)}\\
&\qquad - \|F_{\tilde{E}}(S_0) + a h_{I_i,\tilde{E}}\|^2_{L^2(I_\|)}\\
&\qquad + \|F_{\tilde{E}}(S_0) + a h_{I_i,\tilde{E}} + F_{\tilde{E}}(\tilde{S}_1)\|^2_{L^2(I_\|)}
\end{aligned}$$

$$= \|F_{\tilde{E}}\big(S_0) \cup \{aI_i\} \cup \tilde{S}_1\big)\|_2^2$$

The way it is scaled, $S_0 \cup \{aI_i\} \cup \tilde{S}_1$ still satisfies $U = \emptyset$. It also has one interval less that violates (54) w.r.t. $\tilde{E}$ and a smaller $\frac{B}{A}$ than S w.r.t. E. So we may apply the induction hypothesis to $S_0 \cup \{aI_i\} \cup \tilde{S}_1$ and are also done with the second case. □

Lemma 4.5. Let $p \geq \frac{2}{3}$ and $E \subset [0, 1)$. Assume $n \geq 0$ and $a_0, \ldots, a_{n-1}, a \geq 0$ and let

$$S = \{a_0 I_0^p, \ldots, a_{n-1} I_{n-1}^p, aI\}$$

be compatible, E-dominant, with $I \subset \mathfrak{r}(I_{n-1}^p)$ and $U(S) = \emptyset$ and where for all $J \in \mathbb{I}(S)$, $J \neq [-1, 1)$ we have $|\mathfrak{l}(J) \cap E| \geq |\mathfrak{r}(J) \cap E|$. Furthermore assume $n = 0$ or $E = [0, p)$. Then there are $a_0', \ldots, a_n' \geq 0$ such that

$$S' = \{a_0' I_0^p, \ldots, a_n' I_n^p\}$$

satisfies

$$\begin{gathered} U(S') = \emptyset, \\ A_{[0,p)}(S') \geq A_E(S), \\ B_{[0,p)}(S') \leq B_E(S), \\ \|F_{[0,p)}(S')\|^2_{L^2(\mathfrak{r}(I_n^p))} \geq \|F_E(S)\|^2_{L^2(\mathfrak{r}(I))}. \end{gathered}$$

Proof. First consider $n = 0$. Take a_0 s.t.

$$\|a_0' \mathbb{1}_{[0,p)}\|_2^2 = \|ah_{I,E}\|_2^2.$$

This is already the inequality for both A and B. For the last one observe

$$\|a_0' \mathbb{1}_{[0,p)}\|^2_{L^2(\mathfrak{r}(I_0^p))} = \|a_0' \mathbb{1}_{[0,p)}\|^2_{L^2([0,1))} = \|ah_{I,E}\|_2^2 \geq \|ah_{I,E}\|^2_{L^2(\mathfrak{r}(I))}.$$

Also $U(\{a_0' I_0^p\}) = \emptyset$.

It remains to take care of the other case $n \geq 1$ and $E = [0, p)$. This goes in a couple of steps.

1. First enlarge I while keeping $\frac{|I \cap E|}{|I|}$ constant until $I \cap E = \mathfrak{r}(I_{n-1}^p) \cap E$. This is possible because $|\mathfrak{r}(I_{n-1}^p) \cap E| \leq p|\mathfrak{r}(I_{n-1}^p)|$ and $|I \cap E| \geq p|I|$. At the same time decrease a so that $\|ah_{I,E}\|_2^2$ stays constant. Then $A_E(S)$ stays constant. $B_E(S)$ decreases because $h_{I,E}$ becomes more parallel to $\mathbb{1}_{I \cap E}$, which can be seen as a consequence of Lemma A.7. Furthermore by Lemma A.18, $\|F_E(S)\|^2_{L^2(\mathfrak{r}(I))}$ increases. Since a decreases also $U(S)$ stays empty.

2. Now we want to extend I further to the right. We do this according to Lemma A.19 applied to the intervals $E \cap I_n^p \subset I \subset I_n^p$ and coefficients $0 \leq a \leq \sum_{i=0}^{n-1} a_i$. If I ends up being I_n^p then we are done.

3. Otherwise we obtain $a = \sum_{i=0}^{n-1} a_i$. That means that on $\mathfrak{l}(I) \cap E = \mathfrak{l}(I)$ the function $F_E(a_0 I_0^p, \dots, a_{n-1} I_{n-1}^p, aI)$ vanishes. Now we keep extending I to the right. However this alone would decrease B and also

$$\|F_{[0,p)}\big(a_0 I_0^p, \dots, a_{n-1} I_{n-1}^p, aI\big)\|^2_{L^2(\mathfrak{r}(I))} \tag{58}$$

because $\mathfrak{l}(I) \cap E$ grows and $\mathfrak{r}(I) \cap E$ shrinks. In order to compensate the loss in (58) we additionally increase a_{n-1} and a by the same amount such that (58) stays unchanged. That way $F_E(a_0 I_0^p, \dots, a_{n-1} I_{n-1}^p, aI)$ stays zero on $\mathfrak{l}(I)$, decreases on $\mathfrak{l}(I_{n-1}^p)$ and remains unchanged on $(I_{n-1}^p)^\complement$, which means that B decreases. A increases due to the growth of I, a_{n-1} and a. We want to do this until I becomes I_n^p.

However if $n \geq 2$ we need to stop if a_{n-1} reaches $\sum_{i=0}^{n-2} a_i$ before that, so that we don't violate $U(S) = \emptyset$. In that case $F_E(a_0 I_0^p, \dots, a_{n-1} I_{n-1}^p, aI)$ becomes zero on $\mathfrak{l}(I_{n-1}^p)$. Then we repeat this step by increasing a_{n-2} and a_{n-1} by the same amount, and a as much as $a_{n-2} + a_{n-1}$. We further repeat this step for $n-3, n-4, \dots$ until either I becomes I_n^p and we are done, or we arrive at the point where we want to increase a_0. In the latter case however, we may increase as long as we like without ever violating $U(S) = \emptyset$ so that eventually I will become I_n^p and we are done.

□

Proposition 4.6. Let $p \geq \frac{2}{3}$, $0 \leq n \leq N$, $E \subset [0,1)$ and

$$S = \{a_0 I_0^p, \dots, a_{n-1} I_{n-1}^p, a_n I_n, \dots, a_N I_N\}$$

be a compatible E-dominant nested sequence with positive coefficients and $U(S) = \emptyset$ and such that for all $J \in \mathbb{I}(S)$, $J \neq [-1,1)$ we have $|\mathfrak{l}(J) \cap E| \geq |\mathfrak{r}(J) \cap E|$. Furthermore assume that for $i = 0, \dots, n-1$ we have $\mathfrak{l}(I_i^p) \subset E$. Let $n = 0$ or $|E| = p$. Then there is an $E' \subset [0,1)$ with $|E'| = p$ and where for $i = 0, \dots, n$ we have $\mathfrak{l}(I_i^p) \subset E'$, and there is a compatible nested E'-dominant sequence

$$S' = \{a_0' I_0^p, \dots, a_n' I_n^p, a_{n+1}' I_{n+1}', \dots, a_N' I_N'\}$$

with positive coefficients, $U(S') = \emptyset$ and s.t. for all $J \in \mathbb{I}(S')$, $J \neq [-1,1)$ we have $|\mathfrak{l}(J) \cap E'| \geq |\mathfrak{r}(J) \cap E'|$ and

$$\frac{B_{E'}(S')}{A_{E'}(S')} \leq \frac{B_E(S)}{A_E(S)}.$$

Proof. Set

$$S_0 := \{a_0 I_0^p, \dots, a_{n-1} I_{n-1}^p, a_n I_n\},$$
$$S_1 := \{a_{n+1} I_{n+1}, \dots, a_N I_N\}.$$

First we would like to apply Lemma 4.5 to E, S_0. We show that this is also possible for $n \geq 1$ even though there Lemma 4.5 requires $E = [0, p)$: Recall that for any k we have $\bigcup_{i=1}^k \mathfrak{l}(I_i^p) \subset [0, p)$ and $[0, p) \setminus \bigcup_{i=1}^k \mathfrak{l}(I_i^p) \subset \mathfrak{r}(I_k^p)$ and $|I_k^p \cap [0, p)| = p|I_k^p|$. Thus for any E with $|E| = p$ and $\bigcup_{i=1}^{n-1} \mathfrak{l}(I_i^p) \subset E$ and $|I_{n-1}^p \cap E| \geq p|I_{n-1}^p|$ must have $E \setminus \bigcup_{i=1}^{n-1} \mathfrak{l}(I_i^p) \subset \mathfrak{r}(I_{n-1}^p)$. This means $|\mathfrak{r}(I_{n-1}^p) \cap E| = |\mathfrak{r}(I_{n-1}^p) \cap [0, p)|$. So instead of E and I_n we may take $[0, p)$ and the interval $\tilde{I}_n$ which satisfies

$$|\mathfrak{l}(I_n) \cap E| = |\mathfrak{l}(\tilde{I}_n) \cap [0, p)|,$$
$$|\mathfrak{r}(I_n) \cap E| = |\mathfrak{r}(\tilde{I}_n) \cap [0, p)|.$$

Such $\tilde{I}_n$ exists by $|\mathfrak{l}(I_n) \cap E| \geq |\mathfrak{r}(I_n) \cap E|$. Apply Lemma 4.5 to $[0, p), S_0$ with $\tilde{I}_n$ instead of I_n. By Lemma A.12 the result of Lemma 4.5 also holds for E, S_0.

So let S_0' be given by Lemma 4.5. If $S_1 = \varnothing$ we set $E' := [0, p)$ and $S' := S_0'$ and are done with the proof. Otherwise note that

$$F_E(S_0)\mathbb{1}_{\mathfrak{r}(I_n)} = \sum_{i=0}^n a_i \mathbb{1}_{\mathfrak{r}(I_n) \cap E} \tag{59}$$
$$F_{[0,p)}(S_0')\mathbb{1}_{\mathfrak{r}(I_n^p)} = \sum_{i=0}^n a_i' \mathbb{1}_{\mathfrak{r}(I_n^p) \cap [0,p)} \tag{60}$$

Since $U(S) \neq \varnothing$ and not all coefficients are zero we have $a_0 > 0$, and since all coefficients are positive this means (60) is not zero. By Lemma 4.5 (60) has a greater L^2-norm than (59) and by the E-dominantness of I_n we have

$$\frac{|\mathfrak{r}(I_n^p) \cap [0, p)|}{|\mathfrak{r}(I_n^p)|} = 2p - 1 \leq \frac{|\mathfrak{r}(I_n) \cap E|}{|\mathfrak{r}(I_n)|}.$$

That means there is an interval $I \subset \mathfrak{r}(I_n^p)$ with

$$\|F_{[0,p)}(S_0')\mathbb{1}_I\|_2^2 = \|F_E(S_0)\mathbb{1}_{I_{n+1}}\|_2^2, \tag{61}$$
$$\frac{|I \cap [0, p)|}{|I|} = \frac{|I_{n+1} \cap E|}{|I_{n+1}|}.$$

Take E' with $E' \setminus I = [0, p) \setminus I$ and s.t. $E' \cap I$ is the image of $E \cap I_{n+1}$ under the linear transformation $I_{n+1} \to I$. Now denote

$$S_1' := \{a_{n+1}' I_{n+1}', \dots, a_N' I_N'\}$$

where $\mathbb{I}(S_1')$ is the image of $\mathbb{I}(S_1)$ under $I_{n+1} \to I$ and $a_{n+1}', a_{n+2}', \dots$ are $a_{n+1}, a_{n+2}, \dots$ multiplied by $\frac{\sqrt{|I_{n+1}|}}{\sqrt{|I|}}$ so that $A_{E'}(S_1') = A_E(S_1)$ and with (61) we have

$$\|F_E(S)\mathbb{1}_{I_{n+1}}\|_2^2 = \|F_{E'}(S')\mathbb{1}_I\|_2^2.$$

Define $S' = S_0' \cup S_1'$. By the way $a_{n+1}', a_{n+2}', \dots$ are scaled, $U(S')$ stays just the same, i.e. empty. Also $|E'| = p$ and for $i = 0, \dots, n$ we have $\mathfrak{l}(I_i) \subset E'$. Then according to Lemma 4.5 we have

$$A_E(S) = A_E(S_0) + A_E(S_1) \le A_{E'}(S_0') + A_{E'}(S_1').$$

Also by Lemma 4.5 we have

$$\|F_E(S_0)\|_2^2 \ge \|F_{E'}(S_0')\|_2^2$$

and by (61) and the choice of E' we have

$$\|F_E(S_0)\mathbb{1}_{I_{n+1}}\|_2^2 = \|F_{E'}(S_0')\mathbb{1}_I\|_2^2.$$

Therefore eventually we get

$$\begin{aligned}\|F_E(S)\|_2^2 &= \|F_E(S_0)\|_2^2 - \|F_E(S_0)\mathbb{1}_{I_{n+1}}\|_2^2 + \|F_E(S)\mathbb{1}_{I_{n+1}}\|_2^2\\ &\ge \|F_{E'}(S_0')\|_2^2 - \|F_{E'}(S_0')\mathbb{1}_I\|_2^2 + \|F_{E'}(S')\mathbb{1}_I\|_2^2\\ &= \|F_{E'}(S')\|_2^2.\end{aligned}$$

□

Proof of Theorem 4.1. First we want to pass to a nested S. We achieve this by going inductively through all $i = 1, 2, \dots$ with $\mathbb{I}_{i+1} \subset \mathfrak{l}(I_i)$ and reflecting the intervals $I_{i+1}, I_{i+2}, \dots$ and $E \cap I_i$ around the midpoint of I_i. We also have to flip the signs of $a_i, a_{i+1}, \dots$ because a reflected Haar function is minus a Haar function. After that, A and B have not changed.

Now by Proposition 4.3 we may pass to a nested S with $U(S) = \emptyset$. Then by Proposition 4.4 we may additionally demand that for all $I \in \mathbb{I}(S)$, $I \neq [-1, 1)$ we have $|\mathfrak{l}(I) \cap E| \ge |\mathfrak{r}(I) \cap E|$. Then for any two $I, J \in \mathbb{I}(S)$ we have $\sphericalangle\left[h_{I,E}, h_{J,E}\right] \le 0$, so that by Lemma A.1 it suffices to consider the case that all coefficients are positive. Further let I be the largest interval in $\mathbb{I}(S)$. If $I \neq [-1, 1)$ then $I \subset [0, 1)$. Then translate and dilate S such that $I = [0, 1)$. Now we may replace I by $[-1, 1)$ because all other intervals in $\mathbb{I}(S)$ are contained in $\mathfrak{r}(I)$ by nestedness, and $h_{I\cap E}$ equals $\mathbb{1}_E$ on $\mathfrak{r}(I)$, and it equals $-\mathbb{1}_E$ on $\mathfrak{l}(I)$. Thus $A_E(S)$ and $B_E(S)$ don't change after the replacement.

Now we inductively apply Proposition 4.6 and eventually for some N pass to an E and an S with $\mathbb{I}(S) = \mathbb{I}_N^p$ which is E-dominant and where for all $n = 1, \dots, N$ we have $\mathfrak{l}(I_n^p) \subset E$ and $|E| = p$. As we already argued in in beginning of the proof of Lemma 4.5, this already implies $A_E(S) = A_{[0,p)}(S)$ and $B_E(S) = B_{[0,p)}(S)$ by Lemma A.12. Now we add the remaining intervals of $\mathbb{I}^p$ with zero coefficients. □

4.2 The Case of 4.5 Scales

The main statement in this section is the following Theorem 4.7.

Theorem 4.7. *Let $p \geq \frac{2}{3}$, $E \subset [0,1)$ and $|E \cap [0, \frac{1}{2})| \geq |E \cap [\frac{1}{2}, 1)|$. Let $I_1, I_2, \dots$ be disjoint subintervals of $[0, \frac{1}{2})$. Let $J_1, J_2, \dots$ be disjoint subintervals of $[\frac{1}{2}, 1), I_1, I_2, \dots$. Let $\{K_{ij} \mid i, j = 1, 2, \dots\}$ be disjoint intervals so that for all i, j we have $K_{ij} \subset J_i$. Furthermore all intervals are assumed compatible and E, p-dominant. Let*

$$S = \{a_0[-1,1), a[0,1), a_1 I_1, a_2 I_2, \dots, b_1 J_1, b_2 J_2, \dots\} \cup \{c_{ij} K_{ij} \mid i, j \in \mathbb{N}\}.$$

Then there is a T with $\mathbb{I}(T) = \mathbb{I}_p$ such that

$$\frac{B_{[0,p)}(T)}{A_{[0,p)}(T)} \leq \frac{B_E(S)}{A_E(S)}. \tag{62}$$

Remark. If we only consider $S \subset \mathscr{D}$, then Theorem 4.7 states that (62) holds whenever S contains only intervals of the scales $2^1, 2^0, 2^{-1}, 2^{-2}$, and those intervals of scale 2^{-3} that are contained in $[0, \frac{1}{2})$.

The propositions that follow provide the main steps of the proof of Theorem 4.7. Most of them are also somehow special cases of Theorem 4.7 in that they reduce certain compatible sets to a nested sequences. Theorem 4.7 can also be seen as the combination of a few individual reductions to nested sequences which happen to work together.

Proposition 4.8. Let $p \geq \frac{1}{2}$, $E \subset I$ and $J_1, J_2, \dots \subset I$ be disjoint and E-dominant and $a, a_1, a_2, \dots \in \mathbb{R}$. Then there is an $\tilde{E}$ with $|\tilde{E}| = |E|$ and an $\tilde{E}$-dominant interval $\tilde{J}$ with

$$|\tilde{J} \cap \tilde{E}| \leq \sum_{i=1,2,\dots} |J_i \cap E|,$$
$$|\tilde{J} \cap \tilde{E}^{\complement}| \leq \sum_{i=1,2,\dots} |J_i \cap E^{\complement}|$$

and $\tilde{a}$ with

$$\|a\mathbb{1}_{\tilde{E}}\|_2^2 + \|\tilde{a}h_{\tilde{J},\tilde{E}}\|_2^2 = \|a\mathbb{1}_E\|_2^2 + \sum_{i=1,2,\dots} \|a_i h_{J_i,E}\|_2^2,$$
$$\|a\mathbb{1}_{\tilde{E}} + \tilde{a}h_{\tilde{J},\tilde{E}}\|_2^2 \leq \|a\mathbb{1}_E + \sum_{i=1,2,\dots} a_i h_{J_i,E}\|_2^2.$$

Proof. By symmetry it suffices to consider the case that $a \geq 0$ and that for $i = 1, 2, \dots$ we have $|\mathfrak{l}(J_i) \cap E| \geq |\mathfrak{r}(J_i) \cap E|$. Then it suffices to consider $a_1, a_2, \dots \geq 0$ which follows from $i \neq j : J_i \cap J_j = \emptyset$ for $a = 0$ and from Lemma A.1 for $a > 0$.

Now for each $i = 1, 2, \dots$ take $E_i' \subset J_i$ and J_i' so that $|E_i'| = |E \cap J_i|$ and J_i' is most antiparallel to I_i w.r.t. E_i'. Further set $a_i' = \sqrt{\frac{|J_i \cap E|}{|J_i' \cap E'|}} a_i$ and

$$E' \coloneqq \bigcup_{i=1,2,\dots} E_i' \cup \Big(E \setminus \bigcup_{i=1,2,\dots} J_i \Big).$$

Then

$$|J_i' \cap E'| \leq |J_i \cap E|, \qquad |J_i' \cap E'^{\complement}| \leq |J_i \cap E^{\complement}|, \qquad \|a_i' h_{J_i',E'}\|_2^2 = \|a_i h_{J_i,E}\|_2^2$$

and we have

$$\|a \mathbb{1}_{E'} + \sum_{i=1,2,\dots} a_i' h_{J_i',E}\|_2^2 \leq \|a \mathbb{1}_E + \sum_{i=1,2,\dots} a_i h_{J_i,E}\|_2^2$$

which follows from $i \neq j \,:\, J_i \cap J_j = \varnothing$ for $a = 0$ and from Lemma A.13 if $a > 0$. Furthermore all restricted Haar functions w.r.t. most antiparallel intervals are translates and dilates of one another. Thus if we take $\tilde{a}$ in such a way that

$$\sum_{i=1,2,\dots} \|\tilde{a} h_{J_i',E'}\|_2^2 = \sum_{i=1,2,\dots} \|a_i' h_{J_i',E'}\|_2^2,$$

then

$$\|a \mathbb{1}_{E'} + \sum_{i=1,2,\dots} \tilde{a} h_{J_i',E'}\|_2^2 \leq \|a \mathbb{1}_{E'} + \sum_{i=1,2,\dots} a_i' h_{J_i',E'}\|_2^2$$

which follows from $i \neq j \,:\, J_i \cap J_j = \varnothing$ for $a = 0$, and from Lemma A.7 for $a > 0$ because for all $i = 1, 2, \dots$ we have $|J_i' \cap E'| = p|J_i'|$ and $\mathfrak{l}(J_i') \subset E'$. So we may take an $\tilde{E} \subset I$ and an interval $\tilde{J} \subset I$ with

$$\begin{aligned} |\tilde{E}| &= |E'|, \\ |\tilde{J}| &= |J_1'| + |J_2'| + \dots, \\ |\tilde{J} \cap \tilde{E}| &= p|\tilde{J}|, \\ \mathfrak{l}(\tilde{J}) &\subset \tilde{E} \end{aligned}$$

so that

$$\begin{aligned} |\tilde{J} \cap \tilde{E}| &= \sum_{i=1,2,\dots} |J_i' \cap E'| \leq \sum_{i=1,2,\dots} |J_i \cap E|, \\ |\tilde{J} \cap \tilde{E}^{\complement}| &= \sum_{i=1,2,\dots} |J_i' \cap E'^{\complement}| \leq \sum_{i=1,2,\dots} |J_i \cap E^{\complement}|, \end{aligned}$$

$$\|\tilde{a}h_{\tilde{J},\tilde{E}}\|_2^2 = \sum_{i=1,2,\dots} \|a_i' h_{J_i',E'}\|_2^2,$$

$$\|a\mathbb{1}_{\tilde{E}} + \tilde{a}h_{\tilde{J},\tilde{E}}\|_2^2 = \|a\mathbb{1}_{E'} + \sum_{i=1,2,\dots} a_i' h_{J_i',E}\|_2^2.$$

□

Proposition 4.9. Let $1 \geq p_1 \geq p_2 \geq 0$ and $p \coloneqq \frac{p_1+p_2}{2} \geq \frac{1}{2}$. Let $E = [0, \frac{p_1}{2}) \cup [\frac{1}{2}, \frac{1+p_2}{2})$ and let I be most p-antiparallel to $[0, \frac{1}{2})$. Then for $S = \{a_0[-1,1), a[0,1), a_I I\}$ we have

$$\frac{B_{[0,p)}\big([-1,1),[0,1)\big)}{A_{[0,p)}\big([-1,1),[0,1)\big)} \leq \frac{B_E(S)}{A_E(S)}.$$

Proof. For $a_0 = 0$ we might as well consider $\{a[-1,1), a_I I\}$ instead of S, and then the proposition follows from Lemma A.13. For $a_0 \neq 0$ it suffices to consider $a_0 = 1$ after rescaling. Then we split the proof into the cases $a \leq 0,\ a \geq 1,\ 0 \leq a \leq 1$.

Case $a \leq 0$ Here it suffices to consider $a_I \geq 0$ because $\mathbb{1}_E + ah_{[0,1),E}$ is positive on $I \cap E$. Set $\tilde{E} \coloneqq [0, \frac{p_1}{2}) \cup [\frac{1}{2}, \frac{1+p_1}{2})$. Then

$$\sphericalangle\big[h_{[-1,1),E}, h_{[0,1),E}\big] \leq 0 = \sphericalangle\big[h_{[-1,1),\tilde{E}}, h_{[0,1),\tilde{E}}\big]. \tag{63}$$

Now take $\tilde{a} \geq 0$ s.t.

$$\|h_{[-1,1),E} + ah_{[0,1),E}\|_2 = \|\tilde{a}h_{[-1,1),\tilde{E}} + \tilde{a}h_{[0,1),\tilde{E}}\|_2 = \|2\tilde{a}\mathbb{1}_{[0,\frac{p_1}{2})}\|_2^2. \tag{64}$$

Then by (63) and $a \geq 0$ we have

$$\|h_{[-1,1),E}\|_2^2 + \|ah_{[0,1),E}\|_2^2 \leq \|\tilde{a}h_{[-1,1),\tilde{E}}\|_2^2 + \|\tilde{a}h_{[0,1),\tilde{E}}\|_2^2 = \|2\tilde{a}\mathbb{1}_{[0,\frac{p_1}{2})}\|_2^2. \tag{65}$$

(64) reads $\frac{p_1}{2}(1+|a|)^2 + \frac{p_2}{2}(1-|a|)^2 = \frac{p_1}{2}(2\tilde{a})^2$ which implies

$$1 + |a| \leq 2\tilde{a}. \tag{66}$$

Furthermore note that on $[0, \frac{1}{2})$ the functions $h_{I,E}$ and $h_{I,\tilde{E}}$ are equal, and $h_{[-1,1),E} + ah_{[0,1),E}$ attains the value $1+|a|$. Therefore using $\langle a_I h_{I,E}, \mathbb{1}_{[0,\frac{p_1}{2})}\rangle \leq 0$ and (66) we have

$$\langle a_I h_{I,E}, h_{[-1,1),E} + ah_{[0,1),E}\rangle = (1+|a|)\langle a_I h_{I,E}, \mathbb{1}_{[0,\frac{p_1}{2})}\rangle \geq \langle a_I h_{I,\tilde{E}}, 2\tilde{a}\mathbb{1}_{[0,\frac{p_1}{2})}\rangle. \tag{67}$$

Now by (65) we get

$$A_E(S) = A_{[0,\frac{p_1}{2})}\big(2a[-1,1), a_I I\big) \tag{68}$$

and by (68), $\langle h_{[-1,1),E}, ah_{[0,1),E}\rangle \geq 0$ and (67) we get

$$B_E(S) \geq B_{[0,\frac{p_1}{2})}\big(2a[-1,1), a_I I\big).$$

Then the proposition for this case $a \leq 0$ follows by translation, dilation, scaling and Lemma A.5 and Lemma A.13.

Case $a \geq 1$ Consider $\tilde{S} = \{a[-1,1),[0,1),-a_I I\}$ instead. Then $A_E(S) = A_E(\tilde{S})$ since $\|h_{[-1,1),E}\|_2 = \|h_{[0,1),E}\|_2$. Also $B_E(S) = B_E(\tilde{S})$ because $h_{[-1,1),E} + ah_{[0,1),E}$ and $ah_{[-1,1),E} + h_{[0,1),E}$ are equal on $[\frac{1}{2},1)$ and equal up to a factor -1 on $[0,\frac{1}{2})$. Now $\frac{B}{A}$ does not change if we further pass to $\{[-1,1),\frac{1}{a}[0,1),-\frac{a_I}{a}I\}$. Hence it suffices to consider the case $0 \leq a \leq 1$.

Case $0 \leq a \leq 1$

$$\langle h_{[-1,1),E} + ah_{[0,1),E}, h_{I,E}\rangle = \langle (1-a)\mathbb{1}_{I\cap E}, h_{I,E}\rangle = (1-a)\langle \mathbb{1}_{I\cap E}, h_{I,E}\rangle \leq 0.$$

Hence we may always pass to the case $a_I \geq 0$. Then

$$\langle ah_{[0,1),E}, a_I h_{I,E}\rangle = aa_I\langle -\mathbb{1}_{E\cap I}, h_{I,E}\rangle \geq 0$$

and thus

$$\frac{B_E(S)}{A_E(S)} \geq \frac{B_E(S) - 2\langle ah_{[0,1),E}, a_I h_{I,E}\rangle}{A_E(S)} = \frac{A_E(S) + 2\langle h_{[-1,1),E}, ah_{[0,1),E} + a_I h_{I,E}\rangle}{A_E(S)} \tag{69}$$

Now for some $0 \leq r \leq p_1$ to be chosen soon, set

$$E_\| := [0,\frac{r}{2}) \cup [\frac{1}{2},\frac{1+r(2p-1)}{2}),$$
$$E_\perp := [\frac{r}{2},\frac{p_1}{2}) \cup [\frac{1+r(2p-1)}{2},\frac{1+p_2}{2}).$$

Note that $r(2p-1) = r(p_1+p_2-1) \leq rp_2 \leq p_2$. So $E_\| \cup E_\perp$ is a partition of E which means that

$$h_{[0,1),E} = h_{[0,1),E_\|} + h_{[0,1),E_\perp}$$

is an orthogonal decomposition and with

$$S' := S \setminus \{ah_{[0,1),E}\} \cup \{ah_{[0,1),E_\|}\}$$

we have $A_E(S) = A_E(S') + \|ah_{[0,1),E_\perp}\|_2^2$ by Lemma A.8. Hence the right hand side of (69) equals

$$\frac{A_E(S') + \|h_{[0,1),E_\|}\|_2^2 + 2\langle h_{[-1,1),E}, ah_{[0,1),E} + a_I h_{I,E}\rangle}{A_E(S') + \|h_{[0,1),E_\|}\|_2^2}. \tag{70}$$

Now since $p_1 \geq p_2$ and $a, a_I \geq 0$ we have $\langle h_{[-1,1),E}, ah_{[0,1),E} + a_I h_{I,E}\rangle \leq 0$ so that by Lemma A.25, (70) is

$$\geq \frac{A_E(S') + 2\langle h_{[-1,1),E}, ah_{[0,1),E} + a_I h_{I,E}\rangle}{A_E(S')}. \tag{71}$$

Now we want to take r s.t.

$$
\begin{aligned}
&& |E_\perp \cap [\tfrac{1}{2}, 1)| = |E_\perp \cap [0, \tfrac{1}{2})| && (72)\\
\iff && p_2 - r(2p-1) = p_1 - r &&\\
\iff && r := \frac{p_1 - p_2}{2(1-p)} = \frac{p_1 - p}{1-p} \geq 0, &&\\
\iff && p_1 - r = \frac{p_1(1-p) - (p_1 - p)}{1-p} = \frac{p(1-p_1)}{1-p} \geq 0 &&
\end{aligned}
$$

Note that for $p = 1$ we may take r arbitrarily in $[0, 1]$. (72) makes $h_{[0,1),E_\perp}$ orthogonal to $h_{[-1,1),E}$ so that (71) equals

$$\frac{A_E(S') + 2\langle h_{[-1,1),E}, ah_{[0,1),E_\parallel} + a_I h_{I,E}\rangle}{A_E(S')}. \tag{73}$$

Further note that

$$
\begin{aligned}
|E_\parallel| &= \frac{r}{2} + \frac{r(2p-1)}{2} = rp,\\
\frac{|E_\parallel \cap [\frac{1}{2}, 1)|}{|E_\parallel|} &= \frac{p - \frac{1}{2}}{p} = \frac{|[0,p) \cap [\frac{1}{2}, 1)|}{|[0,p)|}
\end{aligned}
$$

and since I is most antiparallel to $[0, \frac{1}{2})$ w.r.t. $[0, \frac{p_1}{2})$ we have

$$r + 2|I| = \frac{p_1 - p}{1-p} + 2\left(\frac{1}{2} - \frac{\frac{p_1}{2} - p\frac{1}{2}}{1-p}\right) = 1.$$

That means there are $E_1, E_2 \subset [0, 1)$ such that $E_\parallel, E_1, E_2$ are disjoint with

$$
\begin{gathered}
E_\parallel \cup E_1 \cup E_2 = [0, p),\\
|E_1| = |E_2| = p|I|,\\
\frac{|E_\parallel \cap [\frac{1}{2}, 1)|}{|E_\parallel|} = \frac{|E_1 \cap [\frac{1}{2}, 1)|}{|E_1|} = \frac{|E_2 \cap [\frac{1}{2}, 1)|}{|E_2|} = \frac{|[0,p) \cap [\frac{1}{2}, 1)|}{|[0,p)|}.
\end{gathered}
$$

Then we have

$$
\begin{aligned}
\|h_{[-1,1),[0,p)}\|_2^2 &= \|h_{[-1,1),E}\|_2^2,\\
\|h_{[0,1),E_1}\|_2^2 &= \|h_{I,E}\|_2^2,\\
\langle h_{[-1,1),[0,p)}, h_{[0,1),E_\parallel}\rangle &= \langle h_{[-1,1),E}, h_{[0,1),E_\parallel}\rangle,
\end{aligned}
$$

$$\langle h_{[-1,1),[0,p)}, h_{[0,1),E_1}\rangle = \langle h_{[-1,1),E}, h_{I,E}\rangle.$$

Therefore with

$$\tilde{S} \coloneqq \left\{h_{[-1,1),[0,p)}, ah_{[0,1),E_\parallel}, a_I h_{[0,1),E_1}\right\}$$

we can replace (73) summand for summand by

$$\frac{A_{[0,p)}(\tilde{S}) + 2\langle h_{[-1,1),[0,p)}, ah_{[0,1),E_\parallel} + a_I h_{[0,1),E_1}\rangle}{A_{[0,p)}(\tilde{S})}$$

and since $h_{[0,1),E_\parallel}$ and $h_{[0,1),E_1}$ are orthogonal this equals

$$\begin{aligned} &= \frac{B_{[0,p)}\big([-1,1), ah_{[0,1),E_\parallel}, a_I h_{[0,1),E_1}\big)}{A_{[0,p)}\big([-1,1), ah_{[0,1),E_\parallel}, a_I h_{[0,1),E_1}\big)} \\ &= \frac{B_{[0,p)}\big([-1,1), ah_{[0,1),E_\parallel}, a_I h_{[0,1),E_1}, 0h_{[0,1),E_2}\big)}{A_{[0,p)}\big([-1,1), ah_{[0,1),E_\parallel}, a_I h_{[0,1),E_1}, 0h_{[0,1),E_2}\big)}. \end{aligned} \tag{74}$$

Since by Lemma A.12 the functions $h_{[0,1),E_\parallel}, h_{[0,1),E_1}, h_{[0,1),E_2}$ can be seen as translates and dilates of one another, when we take $\tilde{a} \geq 0$ s.t.

$$\|ah_{[0,1),E_\parallel}\|_2^2 + \|a_I h_{[0,1),E_1}\|_2^2 + \|0h_{[0,1),E_2}\|_2^2 = \sum_{X\in\{E_\parallel,E_1,E_2\}} \|\tilde{a}h_{[0,1),X}\|_2^2$$

then by Lemma A.7, (74) is greater than

$$\frac{B_{[0,p)}\big([-1,1), \tilde{a}h_{[0,1),E_\parallel}, \tilde{a}h_{[0,1),E_1}, \tilde{a}h_{[0,1),E_2}\big)}{A_{[0,p)}\big([-1,1), \tilde{a}h_{[0,1),E_\parallel}, \tilde{a}h_{[0,1),E_1}, \tilde{a}h_{[0,1),E_2}\big)}.$$

By Lemma A.8 we have

$$= \frac{B_{[0,p)}\big([-1,1), \tilde{a}[0,1)\big)}{A_{[0,p)}\big([-1,1), \tilde{a}[0,1)\big)}$$

and by Lemma A.5 we have

$$\geq \frac{B_{[0,p)}\big([-1,1), [0,1)\big)}{A_{[0,p)}\big([-1,1), [0,1)\big)}.$$

□

Proposition 4.10. Let $p \geq \frac{2}{3}$. Let I be an interval, Let $1 \geq b_1 \geq b_2 \geq 0$, and with $b := \frac{b_1+b_2}{2}$ assume $b \geq p$. Let $E \subset I$ with $|\mathfrak{l}(I) \cap E| = b_1 |\mathfrak{l}(I)|$, $|\mathfrak{r}(I) \cap E| = b_2 |\mathfrak{r}(I)|$. Further assume that $\mathfrak{r}(I) \cap E$, $\mathfrak{l}(I) \cap E$ are intervals with the same left boundary as $\mathfrak{r}(I)$, $\mathfrak{r}(I)$ respectively and J_1, J_2 are the respective most antiparallel intervals of $\mathfrak{l}(I), \mathfrak{r}(I)$. Let $a, a_I, a_1, a_2 \in \mathbb{R}$. Denote

$$S := \{a\mathbb{1}_I, a_I I, a_1 J_1, a_2 J_2\}.$$

Then there is an $E' \subset I$ with $|E'| = |E|$, an interval partition

$$I = I'_{\|} \cup I'_{\perp},$$

an interval $J'_{\|} \subset \mathfrak{r}(I'_{\|})$ and $a_{\text{even}}, a_{\text{odd}} \in \mathbb{R}$ such that

$$\begin{aligned} S'_{\|,1} &:= \{a\mathbb{1}_{I'_{\|}}, a_I I'_{\|}, a_2 J'_{\|}\}, \\ S'_{\|,2} &:= \{a\mathbb{1}_{I'_{\perp}}, a_{\text{even}} I'_{\perp}\}, \\ S'_{\perp} &:= \{a_I \mathbb{1}_{I'_{\perp}}, a_{\text{odd}} I'_{\perp}\} \end{aligned}$$

are each compatible E'-dominant nested sequences and

$$A_E(S) = A_{E'}(S'_{\|,1} \cup S'_{\|,2}) + A_{E'}(S'_{\perp}), \tag{75}$$

$$B_E(S) = B_{E'}(S'_{\|,1} \cup S'_{\|,2}) + B_{E'}(S'_{\perp}). \tag{76}$$

Remark. (75) can be written as

$$\begin{aligned} A_E\big(S \setminus \{a\mathbb{1}_I\}\big) = {} & \|a_I h_{I'_{\|},E'}\|_2^2 + \|a_2 h_{J'_{\|},E'}\|_2^2 + \|a_{\text{even}} h_{I'_{\perp},E'}\|_2^2 \\ & + \|a_I \mathbb{1}_{I'_{\perp},E'}\|_2^2 + \|a_{\text{odd}} h_{I'_{\perp},E'}\|_2^2. \end{aligned}$$

Proof of Proposition 4.10. For $p = 1$ all functions in S are orthogonal which makes the statement uninteresting and obvious. Hence only consider $p < 1$ from now on. By translating and dilating it suffices to consider the case $I = [0, 1)$, and by scaling we may pass to $a = 1$.

Claim.

$$|J_1| \leq |J_2|.$$

Proof. By Lemma A.13 and $b_1 \geq p$ we have

$$|J_1| = \frac{1}{1-p} \frac{1-b_1}{2},$$

$$|J_2| = \begin{cases} \frac{1}{p}\frac{b_2}{2} & b_2 \le p \\ \frac{1}{1-p}\frac{1-b_2}{2} & b_2 \ge p \end{cases}.$$

Note that $|J_2|$ is linearly increasing in b_2 for $b_2 \le p$ and linearly decreasing for $b_2 \ge p$. Hence it is minimal for b_2 equal to one of the boundaries. For b_1 fixed, the lower bound for b_2 is given by $\frac{b_1+b_2}{2} \ge p$, i.e. $b_2 = 2p - b_1$ and the upper bound is $b_2 = b_1$. If $b_2 = b_1$ then $|J_2| = |J_1|\surd$. At the lower bound we have $b = p$ which implies $b_2 \le p$ since $b_1 \ge p$. Therefore

$$\begin{aligned}
|J_2| &= \frac{b_2}{2}\frac{1}{p} = \frac{1}{2}\frac{2p - b_1}{p}, \\
|J_2| - |J_1| &= \frac{(2p - b_1)(1-p) - (1-b_1)p}{2p(1-p)} = \frac{-b_1 + p + 2pb_1 - 2p^2}{2p(1-p)} \\
&= \frac{(2p-1)(b_1 - p)}{2p(1-p)} \ge 0,
\end{aligned}$$

which proves the claim. □

Now we may partition

$$J_2 = J_{2,\perp} \cup J_{2,\|}$$

in such a way that

$$|J_{2,\perp}| = |J_1|,$$

$$\frac{|\mathfrak{l}(J_2) \cap J_{2,\perp}|}{|J_{2,\perp}|} = \frac{|\mathfrak{l}(J_2) \cap J_{2,\|}|}{|J_{2,\|}|} = \frac{1}{2}, \tag{77}$$

$$J_{2,\perp} \cap \mathfrak{l}(J_2) \subset \mathfrak{l}(J_2) \subset E,$$

$$J_{2,\|} \cap \mathfrak{l}(J_2) \subset \mathfrak{l}(J_2) \subset E, \tag{78}$$

$$\frac{|J_{2,\|} \cap E|}{|J_{2,\|}|} = \frac{|J_{2,\perp} \cap E|}{|J_{2,\perp}|} = \frac{|J_2 \cap E|}{|J_2|} = p. \tag{79}$$

Of course it won't be an interval partition this time. Orthogonally decompose

$$h_{J_2,E} = h_{J_2,E\cap J_{2,\perp}} + h_{J_2,E\cap J_{2,\|}}$$

and set

$$\begin{aligned}
h_{\text{even}} &:= +h_{J_1,E} + h_{J_2,E\cap J_{2,\perp}}, \\
h_{\text{odd}} &:= -h_{J_1,E} + h_{J_2,E\cap J_{2,\perp}}.
\end{aligned}$$

Then h_{even} and h_{odd} are orthogonal. Set $a_{\mathrm{even}}, a_{\mathrm{odd}}$ such that

$$a_1 h_{J_1,E} + a_2 \mathbb{1}_{J_{2,\perp}} h_{J_2,E} = a_{\mathrm{even}} h_{\mathrm{even}} + a_{\mathrm{odd}} h_{\mathrm{odd}}.$$

Now define

$$\begin{aligned} I_\perp &:= J_1 \cup J_{2,\perp} \\ I_\parallel &:= [0,1) \setminus I_\perp \end{aligned}$$

and orthogonally decompose

$$h_{[0,1),E} = h_{[0,1),I_\perp \cap E} + h_{[0,1),I_\parallel \cap E}.$$

Now define

$$\begin{aligned} S_{\parallel,1} &:= \{ && \mathbb{1}_{I_\parallel \cap E}, && a_{[0,1)} h_{[0,1),E\cap I_\parallel}, && a_2 h_{J_2,E\cap J_{2,\parallel}} && \}, \\ S_{\parallel,2} &:= \{ && \mathbb{1}_{I_\perp \cap E}, && && a_{\mathrm{even}} h_{\mathrm{even}}, && \}, \\ S_\perp &:= \{ && && a_{[0,1)} h_{[0,1),E\cap I_\perp}, && a_{\mathrm{odd}} h_{\mathrm{odd}} && \} \end{aligned}$$

Since we only did orthogonal decompositions and recombinations and $S_{\parallel,1}, S_{\parallel,2}, S_\perp$ span orthogonal spaces, a few applications of Lemma A.8 show that

$$A_E(S) = A_E(S_{\parallel,1}) + A_E(S_{\parallel,2}) + A_E(S_\perp), \tag{80}$$

$$B_E(S) = B_E(S_{\parallel,1}) + B_E(S_{\parallel,2}) + B_E(S_\perp). \tag{81}$$

Now define

$$\begin{aligned} I'_\perp &:= \left[0, \frac{1-b_1}{1-p}\right), \\ I'_\parallel &:= \left[\frac{1-b_1}{1-p}, 1\right), \\ E' &:= \left[0, \frac{p(1-b_1)}{1-p}\right) \cup \left[\frac{1-b_1}{1-p}, 1 - \frac{b_1 - b_2}{2}\right) \end{aligned}$$

and the interval $J'_\parallel \subset \mathfrak{r}(I'_\parallel)$ with $|J'_\parallel| = |J_{2,\parallel}|$ and $|J'_\parallel \cap E'| = p|J'_\parallel|$. We will see soon that such $J'_\parallel$ exists. Note that

$$\begin{aligned} 1 - \frac{b_1 - b_2}{2} = 1 + b - b_1 &= \frac{1 + b - b_1 - p - pb + pb_1}{1-p} \\ &= \frac{1 - b_1 + (b-p) + p(b_1 - b)}{1-p} \geq \frac{1-b_1}{1-p}, \end{aligned}$$

$$|E'| = \frac{p(1-b_1) - (1-b_1) + (1-p)(1+b-b_1)}{1-p} = \frac{(1-p)b}{1-p} = b = |E|.$$

Now we would like to show that the three linear maps induced by

$$\mathscr{F} : S_{\|,1} \to S'_{\|,1} \qquad \left.\begin{array}{rl} \mathbb{1}_{I_\| \cap E} & \mapsto \mathbb{1}_{I'_\| \cap E'} \\ h_{[0,1),I_\| \cap E} & \mapsto h_{I'_\|,E'} \\ h_{J_2,J_{2,\|}\cap E} & \mapsto h_{J'_\|,E'} \end{array}\right\},$$

$$\mathscr{G} : S_{\|,2} \to S'_{\|,2} \qquad \left.\begin{array}{rl} \mathbb{1}_{I_\perp \cap E} & \mapsto \mathbb{1}_{I'_\perp \cap E'} \\ h_{\text{even}} & \mapsto h_{I'_\perp,E'} \end{array}\right\},$$

$$\mathscr{H} : S_\perp \to S'_\perp \qquad \left.\begin{array}{rl} h_{[0,1),I_\perp \cap E} & \mapsto \mathbb{1}_{I'_\perp \cap E'} \\ h_{\text{odd}} & \mapsto h_{I'_\perp,E'} \end{array}\right\}$$

are isometries.

Claim. $J'_\|$ exists, $I'_\|, I'_\perp, J'_\|$ are E'-dominant, and for

$$\begin{cases} T = S_{\|,1}, \\ \mathscr{I} = \mathscr{F} \end{cases} \qquad \text{and} \qquad \begin{cases} T = S_{\|,2}, \\ \mathscr{I} = \mathscr{G} \end{cases}$$

and $f \in T,\ i \in \{-1, 1\}$ we have

$$\left|\mathscr{I}(f)^{-1}(\{i\})\right| = \left|f^{-1}(\{i\})\right|.$$

Assume the claim holds. Then $S'_{\|,1}, S'_{\|,2}, S'_\perp$ are each nested sequences because for $T = I'_\|, I'_\perp, J'_\|$ we have $\mathfrak{l}(T) \subset E'$. Also note that

$$\begin{aligned} \operatorname{supp}(h_{[0,1),I_\|\cap E}) &\subset \mathbb{1}^{-1}_{I_\|\cap E}(\{1\}), \\ \operatorname{supp}(h_{J_2,J_{2,\|}\cap E}) &\subset h^{-1}_{[0,1),I_\|\cap E}(\{1\}), \\ \operatorname{supp}(h_{\text{even}}) &\subset \mathbb{1}^{-1}_{I_\perp\cap E}(\{1\}), \end{aligned}$$

and the same holds true for their images under $\mathscr{F}$ and $\mathscr{G}$ respectively. This means by the claim and Lemma A.12 that $\mathscr{F}, \mathscr{G}$ are isometries. Furthermore

$$\begin{aligned} h_{[0,1),I_\perp\cap E} &= h_{[0,1)} \cdot \mathbb{1}_{I_\perp,E}, \\ h_{\text{odd}} &= h_{[0,1)} \cdot h_{\text{even}} \end{aligned}$$

which implies that also $\mathscr{H}$ is an isometry since $\mathscr{G}$ is one. Now (75) and (76) follow from (80) and (81) because $S'_{\|,1}$ and $S'_{\|,2}$ span orthogonal spaces. Thus it suffices to prove the claim in order to finish the proof of the proposition.

Proof of claim.

$$|I'_{\|}| = 1 - \frac{1-b_1}{1-p} = 1 - 2|J_1| = |I_{\|}| \tag{82}$$

$$|I'_{\perp}| = 1 - |I'_{\|}| = 1 - |I_{\|}| = |I_{\perp}| \tag{83}$$

$$|I_{\perp} \cap E| = \left|(J_1 \cap E) \cup (J_{2,\perp} \cap E)\right| = p|J_1| + p|J_{2,\perp}| = p|I_{\perp}| = p|I'_{\perp}| = |I'_{\perp} \cap E'| \tag{84}$$

(84) proves the claim for $\mathbb{1}_{I_{\perp} \cap E}$. (84) also contains that $I'_{\perp}$ is E'-dominant and that

$$|I_{\|} \cap E| = |E| - |I_{\perp} \cap E| = |E'| - |I'_{\perp} \cap E'| = |I'_{\|} \cap E'|. \tag{85}$$

(85) proves the claim for $\mathbb{1}_{I_{\|} \cap E}$. And since $b \geq p$ we have

$$|I_{\|}| = 1 - |I_{\perp}| = 1 - 2|J_1|,$$
$$|I_{\|} \cap E| = |E| - |I_{\perp} \cap E| = b - 2p|J_1| \geq p|I_{\|}|$$

so that we get by (82) and (85) that $I'_{\|}$ is E'-dominant.

Recall that J_1 is most antiparallel to $[0, \frac{1}{2})$ w.r.t. $[0, \frac{b_1}{2})$ and that $b_1 \geq p$. Hence $|J_1 \setminus E| = (1-p)|J_1| = \frac{1}{2} - b_1 = |[0, \frac{1}{2}) \setminus E|$ and thus $[0, \frac{1}{2}) \setminus E \subset J_1$. Therefore

$$I_{\|} \cap [0, \frac{1}{2}) \cap E = \left([0, \frac{1}{2}) \setminus J_1\right) \cap E = [0, \frac{1}{2}) \setminus J_1 = I_{\|} \cap [0, \frac{1}{2}) = [0, \frac{1}{2}) \setminus J_1, \tag{86}$$

$$\implies |I_{\|} \cap [0, \frac{1}{2}) \cap E| = \frac{1}{2} - \frac{1}{2}\frac{1-b_1}{1-p} = \frac{1}{2}\frac{b_1 - p}{1-p}.$$

Now since $I'_{\|}$ and $I'_{\|} \cap E'$ are intervals with the same left boundary, $|I'_{\|} \cap E'|$ is E'-dominant and $p \geq \frac{1}{2}$, we have

$$|\mathfrak{l}(I'_{\|}) \cap E'| = |\mathfrak{l}(I'_{\|})| = \frac{1}{2}(1 - \frac{1-b_1}{1-p}) = |I_{\|} \cap [0, \frac{1}{2})| = |I_{\|} \cap [0, \frac{1}{2}) \cap E| \tag{87}$$

so that by (82) and (87) also

$$|\mathfrak{r}(I'_{\|})| = |I_{\|} \cap [\frac{1}{2}, 1)|. \tag{88}$$

Furthermore

$$|I_{\|} \cap [\frac{1}{2}, 1) \cap E| = |I_{\|} \cap E| - |I_{\|} \cap E \cap [0, \frac{1}{2})|$$

and by (82),(85) and (87) we have

$$= |I'_{\|} \cap E'| - |\mathfrak{l}(I'_{\|}) \cap E'| = |\mathfrak{r}(I'_{\|}) \cap E'| \tag{89}$$

(87) and (89) prove the claim for $h_{[0,1),I_{\|}\cap E}$.

Now to $h_{J_2,J_{2,\|}\cap E}$. Note that $\mathfrak{r}(I'_{\|})$ and $\mathfrak{r}(I'_{\|}) \cap E'$ are intervals with the same left boundary. Therefore when considering (77), (78) and (79), it suffices to find an interval $J'_{\|} \subset \mathfrak{r}(I'_{\|})$ with $|J'_{\|} \cap E'| = |J_{2,\|} \cap E|$ and $|J'_{\|}| = |J_{2,\|}|$. And since $J_{2,\|} \subset I_{\|} \cap [\frac{1}{2}, 1)$, the existence of such an interval $J'_{\|}$ follows from (88) and (89). So the claim also holds for $h_{J_2,J_{2,\|}\cap E}$.

For h_{even} we have by (84) that

$$\operatorname{supp}(h_{\text{even}}) = |J_1 \cap E| + |J_{2,\perp} \cap E| = |I_{\perp} \cap E| = |I'_{\perp} \cap E'| \tag{90}$$

and by (83) that

$$|h_{\text{even}}^{-1}(\{-1\})| = \frac{1}{2}|J_1| + \frac{1}{2}|J_{2,\perp}| = \frac{1}{2}|I_{\perp}| = \frac{1}{2}|I'_{\perp}| = h_{I'_{\perp},E'}^{-1}(\{-1\}). \tag{91}$$

(90) and (91) prove the claim for h_{even}. □

The claim is proven and thus so is the proposition. □

Proof of Theorem 4.7. By Proposition 4.8 it suffices to consider the case that for each i the sequence $K_{i1}, K_{i2}, \dots$ has length 2 and $K_{i1} \subset \mathfrak{l}(J_i)$, $K_{i2} \subset \mathfrak{r}(J_i)$. Furthermore abbreviate

$$S_0 := \{a_0[-1,1), a[0,1), a_1 I_1, a_2 I_2, \dots\}.$$

Then for each $i = 1, 2, \dots$ with $\|F_E(S_0)\|_{L^2(J_i)} = 0$ take $S_{J_i} := \{b_i J_i, c_{i1} K_{i1}, c_{i2} K_{i2}\}$, so that S_{J_i} is the subset of S that consists of all elements contained in J_i. Then $F_E(S \setminus S_{J_i})$ and $F_E(S_{J_i})$ are orthogonal. That means by Lemma A.9 that $S \setminus S_{J_i}$ or S_{J_i} has a smaller ratio $\frac{B}{A}$ than S. Therefore it suffices to prove Theorem 4.7 for each S_{J_i} instead of S, and for those S where for all $i = 1, 2, \dots$ we have

$$\|F_E(S_0)\|_{L^2(J_i)} > 0, \tag{92}$$

which by the way implies for $j = 1, 2$ that also

$$\|F_E(S_0)\|_{L^2(K_{ij})} > 0. \tag{93}$$

For the proof for S_{J_i}, note that

$$S_{J_i,1} := \{b_i \mathbb{1}_{\mathfrak{l}(J_i)\cap E}, -c_{i1} K_{i1}\},$$
$$S_{J_i,2} := \{b_i \mathbb{1}_{\mathfrak{r}(J_i)\cap E}, c_{i2} K_{i2}\}$$

are orthogonal and we have by Lemma A.8 that

$$A_E(S_{J_i}) = A_E(S_{J_i,1} \cup S_{J_i,2}),$$
$$B_E(S_{J_i}) = B_E(S_{J_i,1} \cup S_{J_i,2})$$

so that by Lemma A.9 it suffices to consider $S_{J_i,1}$ and $S_{J_i,2}$ separately. And from Lemma A.13 it follows that for $j = 1, 2$ we have

$$\frac{B_{[0,p)}\big([-1,1),[0,1)\big)}{A_{[0,p)}\big([-1,1),[0,1)\big)} \le \frac{B_E(S_{J_i,j})}{A_E(S_{J_i,j})},$$

which, after adding the remaining intervals of $\mathbb{I}^p$ with coefficient 0 to the left hand side, finishes the proof of Theorem 4.7 for S_{J_i} instead of S.

Thus it remains to consider such S which for $i = 1, 2, \ldots$ satisfy (92) from now on. By Proposition 4.8 applied to $[0, \frac{1}{2})$, there are $\tilde{E}$, $a_1' \in \mathbb{R}$, $I_1' \subset [0, \frac{1}{2})$ such that $\tilde{S}_0 := \{a_0[-1,1), a[0,1), a_1' I_1'\}$ is $\tilde{E}$-dominant and

$$A_{\tilde{E}}(\tilde{S}_0) = A_E(S_0),$$
$$B_{\tilde{E}}(\tilde{S}_0) \le B_E(S_0).$$

Then by Proposition 4.9 we have

$$\frac{B_{[0,p)}\big([-1,1),[0,1)\big)}{A_{[0,p)}\big([-1,1),[0,1)\big)} \le \frac{B_{\tilde{E}}(\tilde{S}_0)}{A_{\tilde{E}}(\tilde{S}_0)}.$$

Now rescale the coefficients in S in such a way that

$$B_E(S_0) = B_{[0,p)}\big([-1,1),[0,1)\big). \tag{94}$$

Then since $\frac{B_{\tilde{E}}(\tilde{S}_0)}{A_{\tilde{E}}(\tilde{S}_0)} \le \frac{B_E(S_0)}{A_E(S_0)}$ we have

$$A_E(S_0) \le A_{[0,p)}\big([-1,1),[0,1)\big). \tag{95}$$

(94) and (95) also hold if we replace $[0, p)$ by some E' with

$$|E'| = p, \qquad\qquad [0, \frac{1}{2}) \subset E'. \tag{96}$$

Due to (94) and since for $i = 1, 2, \ldots$ we have (92) and $\frac{|J_i \cap E|}{|J_i|} \ge p \ge 2p - 1 = \frac{|[\frac{1}{2},1) \cap E'|}{|[\frac{1}{2},1)|}$, we can take an E' with (96) and disjoint $J_1', J_2', \ldots \subset [\frac{1}{2}, 1)$ in such a way that for $i = 1, 2, \ldots$ we have

$$\frac{|\mathfrak{l}(J_i') \cap E'|}{|\mathfrak{l}(J_i')|} = \frac{|\mathfrak{l}(J_i) \cap E|}{|\mathfrak{l}(J_i)|}, \qquad \frac{|\mathfrak{r}(J_i') \cap E'|}{|\mathfrak{r}(J_i')|} = \frac{|\mathfrak{r}(J_i) \cap E|}{|\mathfrak{r}(J_i)|}, \tag{97}$$

$$\|F_{E'}\big([-1,1),[0,1)\big)\|^2_{L^2(J_i')} = \|F_E(S_0)\|^2_{L^2(J_i)}. \tag{98}$$

Due to (98) and since for $i = 1, 2, \dots$ we have (93) and (97), we can redistribute E' on $\mathfrak{l}(J_i')$ and $\mathfrak{r}(J_i')$ respectively and take $K_{i1}' \subset \mathfrak{l}(J_i')$ and $K_{i2}' \subset \mathfrak{r}(J_i')$ in such a way that for $j = 1, 2$ we have

$$\frac{|\mathfrak{l}(K_{ij}') \cap E'|}{|\mathfrak{l}(K_{ij}')|} = \frac{|\mathfrak{l}(K_{ij}) \cap E|}{|\mathfrak{l}(K_{ij})|}, \qquad \frac{|\mathfrak{r}(K_{ij}') \cap E'|}{|\mathfrak{r}(K_{ij}')|} = \frac{|\mathfrak{r}(K_{ij}) \cap E|}{|\mathfrak{r}(K_{ij})|}, \tag{99}$$

$$\|F_{E'}\big([-1,1),[0,1)\big)\|^2_{L^2(K_{ij}')} = \|F_E(S_0)\|^2_{L^2(K_{ij})}. \tag{100}$$

Now (98) and (100) further imply

$$\frac{|K_{ij}' \cap E'|}{|J_i' \cap E'|} = \frac{|K_{ij} \cap E|}{|J_i \cap E|}. \tag{101}$$

Now for $i = 1, 2, \dots$, (97) and (98) imply that when we take b_i' such that

$$\|b_i' h_{J_i',E'}\|_2^2 = \|b_i h_{J_i,E}\|_2^2, \tag{102}$$

and with the sign of b_i times the sign of $F_E(S_0)$ on J_i, then

$$\langle b_i' h_{J_i',E'}, F_{E'}\big([-1,1),[0,1)\big)\rangle = \langle b_i h_{J_i,E}, F_E(S_0)\rangle. \tag{103}$$

Similarly for $j = 1, 2$, (99) and (100) imply that when we take c_{ij}' such that

$$\|c_{ij}' h_{K_{ij}',E'}\|_2^2 = \|c_{ij} h_{K_{ij},E}\|_2^2, \tag{104}$$

and with the sign of c_{ij}' times the sign of $F_E(S_0)$ on J_i, then

$$\langle c_{ij}' h_{K_{ij}',E'}, F_{E'}\big([-1,1),[0,1)\big)\rangle = \langle c_{ij} h_{K_{ij},E}, F_E(S_0)\rangle. \tag{105}$$

Now by (97), (99), (101), (102), (104) and the choice of the signs we also have

$$\langle c_{ij}' h_{K_{ij}',E'}, b_i' h_{J_i',E'}\rangle = \langle c_{ij} h_{K_{ij},E}, b_i h_{J_i,E}\rangle. \tag{106}$$

Now using (94), (102), (103), (104), (105), (106) we get that

$$i = 1, 2, \dots,\ j = 1, 2 : \qquad \begin{aligned} F_E(S_0) &\mapsto F_{E'}\big([-1,1),[0,1)\big), \\ b_i h_{J_i,E} &\mapsto b_i' h_{J_i',E'}, \\ c_{ij} h_{K_{ij},E} &\mapsto c_{ij}' h_{K_{ij}',E'} \end{aligned}$$

induces an isometry. So with

$$S' \coloneqq \big\{[-1,1),[0,1)\big\} \cup \big\{b_1' J_1', b_2' J_2', \dots\big\} \cup \big\{c_{i1}' K_{i1}', c_{i2}' K_{i2}' \mid i = 1,2,\dots\big\}$$

and recalling (95) we have

$$\begin{aligned} A_{E'}(S') &= A_{E'}\big([-1,1),[0,1)\big) + A_{E'}\big(b_i' J_i', c_{i1}' K_{i1}', c_{i2}' K_{i2}' \mid i = 1,2,\dots\big) \\ &\geq A_E(S_0) + A_E\big(b_i J_i, c_{i1} K_{i1}, c_{i2} K_{i2} \mid i = 1,2,\dots\big) \\ &= A_E(S), \\ B_{E'}(S') &= B_{E'}\big(F_{E'}([-1,1),[0,1)), b_i' J_i', c_{i1}' K_{i1}', c_{i2}' K_{i2}' \mid i = 1,2,\dots\big) \\ &= B_E\big(F_E(S_0), b_i J_i, c_{i1} K_{i1}, c_{i2} K_{i2} \mid i = 1,2,\dots\big) \\ &= B_E(S). \end{aligned}$$

Now by an application of Lemma A.16 and Lemma A.9 it suffices to consider the case that $J_1', J_2', \dots$ consists of only one interval J'. So we also drop the index i at the K's. It also suffices to consider the case that K_1', K_2' are most antiparallel to $\mathfrak{l}(J')$ and $\mathfrak{r}(J')$ respectively. After possibly swapping $\mathfrak{l}(J')$ with $\mathfrak{r}(J')$ and flipping the sign of b', it suffices to consider the case that

$$\frac{|\mathfrak{l}(J') \cap E'|}{|\mathfrak{l}(J')|} \geq \frac{|\mathfrak{r}(J') \cap E'|}{|\mathfrak{r}(J')|}.$$

Now we apply Proposition 4.10 to J' and

$$S_{\mathrm{r}} \coloneqq \{2\mathbb{1}_{J' \cap E'}, b' h_{J',E'}, c_1' h_{K_1',E'}, c_2' h_{K_2',E'}\}.$$

Note that $2\mathbb{1}_{E' \cap J'} = \mathbb{1}_{E' \cap J'} + h_{[0,1),E' \cap J'}$. Proposition 4.10 states that we may rearrange E' into some $\tilde{E}$, divide J' into two intervals $I_\parallel'$ and $I_\perp'$ and replace S_{r} by three nested sequences $S_{\parallel,1}', S_{\parallel,2}', S_\perp'$ with

$$\begin{aligned} A_{E'}(S_{\mathrm{r}}) &= A_{\tilde{E}}(S_{\parallel,1}' \cup S_{\parallel,2}') + A_{\tilde{E}}(S_\perp'), \\ B_{E'}(S_{\mathrm{r}}) &= B_{\tilde{E}}(S_{\parallel,1}' \cup S_{\parallel,2}') + B_{\tilde{E}}(S_\perp'). \end{aligned}$$

Furthermore $S_{\parallel,1}'$ and $S_{\parallel,2}'$ are supported on $I_\parallel'$ and $I_\perp'$ respectively and we have $2\mathbb{1}_{\tilde{E} \cap I_\parallel'} \in S_{\parallel,1}'$, $2\mathbb{1}_{\tilde{E} \cap I_\perp'} \in S_{\parallel,1}'$. Note, that

$$2\mathbb{1}_{\tilde{E} \cap I_\parallel'} + 2\mathbb{1}_{\tilde{E} \cap I_\perp'} = \mathbb{1}_{\tilde{E} \cap J'} + h_{[0,1),\tilde{E} \cap J'}.$$

Now define

$$\tilde{S} \coloneqq \big\{[-1,1),[0,1)\big\} \cup (S_{\parallel,1}' \setminus \{2\mathbb{1}_{I_\parallel' \cap \tilde{E}}\}) \cup (S_{\parallel,2}' \setminus \{2\mathbb{1}_{I_\perp' \cap \tilde{E}}\})$$

$$= \{[-1,1),[0,1),b'I'_{\|},a_{\mathrm{even}}I'_{\perp},c'_2J'_{\|}\}.$$

Then using the remark after Proposition 4.10 we get also for the original set

$$S' = \{[-1,1),[0,1),b'J',c'_1K'_1,c'_2K'_2\}$$

that

$$A_{E'}(S') = A_{\tilde{E}}(\tilde{S}) + A_{\tilde{E}}(S'_{\perp}),$$
$$B_{E'}(S') = B_{\tilde{E}}(\tilde{S}) + B_{\tilde{E}}(S'_{\perp}).$$

By Lemma A.25 there is a $T \in \{\tilde{S}, S'_{\perp}\}$ for which we have

$$\frac{B_{E'}(T)}{A_{E'}(T)} \leq \frac{B_{E'}(S')}{A_{E'}(S')}.$$

If $T = S'_{\perp}$ then T is already of the form that we can apply Theorem 4.1 to it and are done. If $T = \tilde{S}$ we first apply Lemma A.16 and Lemma A.9 to replace T by a single nested sequence to which we thereafter can apply Theorem 4.1. □

5 On Question 1

For $E \subset [0,1)$ and a set of intervals $\mathbb{I}$ define

$$H_E(\mathbb{I}) := \{h_{I,E} \mid I \in \mathbb{I}\}.$$

Proposition 5.1. Let $[0,1) = E_0 \cup E_1$ be a partition. Further assume $n \geq 0$ and that E_0 and E_1 are unions of intervals of scale 2^n. Assume that $|E_1| > 0$. Then there is a partition $\mathscr{D} = \mathscr{D}_0 \cup \mathscr{D}_1$ such that $H(\mathscr{D}_0)_{E_0}$ and $H(\mathscr{D}_1)_{E_1}$ are Riesz basic sequences and

$$\mathbb{1}_{E_1} \notin \operatorname{span}(H_{E_1}(\mathscr{D}_1)).$$

However for any such partition we have

$$\mathbb{1}_{E_0} \in \operatorname{span}(H_{E_0}(\mathscr{D}_0)).$$

Remark. A few things about Proposition 5.1 should be stressed:

- Proposition 5.1 doesn't claim anything about the constants of the Riesz basic sequences.
- If for some i we have $|E_i| = 0$, then $E_{1-i} = [0,1)$ and we can answer Question 1 affirmatively by setting $\mathscr{D}_i := \varnothing$ and $\mathscr{D}_{1-i} := \mathscr{D}$. Furthermore if $|E_i| = 0$ then no matter how we choose $\mathscr{D}_i \subset \mathscr{D}$ we have $\mathbb{1}_{E_i} = 0 \in \operatorname{span}(H_{E_i}(\mathscr{D}_i))$.

- Hence $|E_0| > 0$ and $|E_1| > 0$ hold for any interesting partition $[0,1) = E_0 \cup E_1$. In that case Proposition 5.1 says that it is not possible to get a partition $\mathscr{D} = \mathscr{D}_0 \cup \mathscr{D}_1$ with

$$\mathbb{1}_{E_0} \notin \operatorname{span}(H_{E_0}(\mathscr{D}_0)),$$
$$\mathbb{1}_{E_1} \notin \operatorname{span}(H_{E_1}(\mathscr{D}_1)),$$

but we always have the two options

$$\begin{array}{c} \mathbb{1}_{E_0} \in \operatorname{span}(H_{E_0}(\mathscr{D}_0)), \\ \mathbb{1}_{E_1} \notin \operatorname{span}(H_{E_1}(\mathscr{D}_1)) \end{array} \quad \text{and} \quad \begin{array}{c} \mathbb{1}_{E_0} \notin \operatorname{span}(H_{E_0}(\mathscr{D}_0)), \\ \mathbb{1}_{E_1} \in \operatorname{span}(H_{E_1}(\mathscr{D}_1)) \end{array}.$$

The case

$$\mathbb{1}_{E_0} \in \operatorname{span}(H_{E_0}(\mathscr{D}_0)),$$
$$\mathbb{1}_{E_1} \in \operatorname{span}(H_{E_1}(\mathscr{D}_1))$$

is not very desirable, also because if we also want to add $[-1,1)$ to $\mathscr{D}_i$ for some i, then we need $\mathbb{1}_{E_i} \notin \operatorname{span}(H_{E_i}(D_i))$.

- The partitions are not unique. There are choices being made in the proof so that there actually are exponentially in n many such partitions.

Proof of Proposition 5.1. If for some i a dyadic interval I is contained in E_i then we have to put I into $\mathscr{D}_i$. Hence all the intervals I of size 2^{-n} and higher are already distributed into $\mathscr{D}_0, \mathscr{D}_1$ and for all such $I \in \mathscr{D}_i$ $h_{I,E_i} = h_I$ is orthogonal to all other restricted Haar functions. Hence it suffices to find a partition of the finite set of dyadic intervals of size at most 2^{-n}. That means being a Riesz basic sequence just means being a linearly independent set.

We proceed by induction on n. First consider the case $n = 0$. Then $E_1 = [0,1)$ and we must have $\mathscr{D}_1 = \mathscr{D}$. Then $\mathscr{D}_0, \mathscr{D}_1$ are Riesz basic sequences and $\mathbb{1}_{E_0} = 0 \in \operatorname{span} \varnothing = \operatorname{span}(H_{E_0}(\mathscr{D}_0))$.

Now assume the proposition holds for $n \geq 0$ and let E_0, E_1 be unions of intervals of size $2^{-(n+1)}$. If we have $|E_0| = 0$ then we are back to scale 2^0. Otherwise for $i = 0,1$ define

$$E_{0i} := [0, \frac{1}{2}) \cap E_i,$$
$$E_{1i} := [\frac{1}{2}, 1) \cap E_i.$$

Then for any $j, i = 1, 2$, E_{ji} translated and dilated to $[0,1)$ is a union of dyadic intervals of size 2^{-n}.

We first show the existence of a partition $\mathscr{D}_0 \cup \mathscr{D}_1$ of $\mathscr{D}$ into Riesz basic sequences with

$$\mathbb{1}_{E_1} \notin \operatorname{span} H_{E_1}(\mathscr{D}_1).$$

Since we have $|E_1| > 0$ and already handled the case $|E_0| = 0$, it suffices to consider the case

$$|E_{00}| > 0, \qquad\qquad |E_{11}| > 0$$

after possibly swapping $[0, \frac{1}{2})$ with $[\frac{1}{2}, 1)$. Then by inductive hypothesis there are partitions

$$\mathscr{D}_{00} \cup \mathscr{D}_{01} = \left\{ I \subset [0, \frac{1}{2}) \;\middle|\; I \in \mathscr{D} \right\},$$
$$\mathscr{D}_{10} \cup \mathscr{D}_{11} = \left\{ I \subset [\frac{1}{2}, 1) \;\middle|\; I \in \mathscr{D} \right\}$$

such that for all $j, i = 0, 1$ the set $H_{E_i}(\mathscr{D}_{ji})$ is linearly independent and

$$\mathbb{1}_{[0,\frac{1}{2})\cap E_0} \notin \operatorname{span}(H_{E_0}(\mathscr{D}_{00})), \tag{107}$$
$$\mathbb{1}_{[\frac{1}{2},1)\cap E_1} \notin \operatorname{span}(H_{E_1}(\mathscr{D}_{11})). \tag{108}$$

Define

$$\mathscr{D}_0 := \mathscr{D}_{00} \cup \mathscr{D}_{10} \cup \{[0, 1)\},$$
$$\mathscr{D}_1 := \mathscr{D}_{01} \cup \mathscr{D}_{11}.$$

Now for $i = 0, 1$ the set $H_{E_i}(\mathscr{D}_{0i} \cup \mathscr{D}_{1i})$ is linearly independent. Hence $H_{E_1}(\mathscr{D}_1)$ is linearly independent. Its span does not contain $\mathbb{1}_{E_1}$ because of (108). $H_{E_0}(\mathscr{D}_0)$ is linearly independent as well because of (107).

Now we show that for all partitions $\mathscr{D}_0 \cup \mathscr{D}_1$ which are Riesz basic sequences with

$$\mathbb{1}_{E_1} \notin \operatorname{span} H_{E_1}(\mathscr{D}_1) \tag{109}$$

we must have

$$\mathbb{1}_{E_0} \in \operatorname{span} H_{E_0}(\mathscr{D}_0). \tag{110}$$

For $i = 0, 1$ define

$$\mathscr{D}_{0i} := \left\{ I \in \mathscr{D}_i \;\middle|\; I \subset [0, \frac{1}{2}) \right\},$$
$$\mathscr{D}_{1i} := \left\{ I \in \mathscr{D}_i \;\middle|\; I \subset [\frac{1}{2}, 1) \right\}.$$

Then for $j, i = 0, 1$ the set $H_{E_i}(\mathscr{D}_{ji})$ is a Riesz basic sequence. Due to (109), one of

$$\mathbb{1}_{[0,\frac{1}{2})\cap E_1} \notin \operatorname{span} H_{E_1}(\mathscr{D}_{01}), \tag{111}$$

$$\mathbb{1}_{[\frac{1}{2},1)\cap E_1} \notin \operatorname{span} H_{E_1}(\mathscr{D}_{11}) \tag{112}$$

has to hold. After possibly swapping $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1)$ it suffices to consider (112). Then by inductive hypothesis we must have

$$\mathbb{1}_{[\frac{1}{2},1)\cap E_0} \in \operatorname{span} H_{E_0}(\mathscr{D}_{10}). \tag{113}$$

Now if $[0, 1) \in \mathscr{D}_0$ then (110) follows from (113). If $[0, 1) \in \mathscr{D}_1$ then (109) requires (111) to hold as well, which by inductive hypothesis implies

$$\mathbb{1}_{[0,\frac{1}{2})\cap E_0} \in \operatorname{span} H_{E_0}(\mathscr{D}_{00}). \tag{114}$$

Now (113) and (114) imply (110). □

A Elementary Lemmas

This appendix consists of a bunch of lemmas that are used in this thesis. They are put here instead of into the main text for the following reasons:

- Some lemmas are used many times and throughout large parts of the thesis.
- Some lemmas hold in a more general setting than in the setting of the thesis.
- Some lemmas and their proofs are not very interesting or surprising.

A.1 Lemmas on Inner Product Spaces

Some lemmas do not only hold in the setting of restricted Haar functions but also more generally in the setting of a vector space with a scalar product. We list these lemmas in this subsection.

Lemma A.1. Let $v_1, \dots, v_n$ be vectors such that for any $i \neq j$ we have

$$\langle v_i, v_j \rangle \leq 0$$

and $a_1, \dots, a_n \in \mathbb{R}$. Then

$$\| \sum_{i=1}^{n} |a_i| v_i \|^2 \leq \| \sum_{i=1}^{n} a_i v_i \|^2.$$

Proof.

$$\|\sum_{i=1}^{n} a_i v_i\|^2 = \sum_{i,j=1}^{n} a_i a_j \langle v_i, v_j \rangle = \sum_{i=1}^{n} a_i^2 \|v_i\|^2 + \sum_{i \neq j} a_i a_j \underbrace{\langle v_i, v_j \rangle}_{\leq 0}$$
$$\geq \sum_{i=1}^{n} |a_i|^2 \|v_i\|^2 + \sum_{i \neq j} |a_i a_j| \langle v_i, v_j \rangle = \|\sum_{i=1}^{n} |a_i| v_i\|^2.$$

□

Lemma A.2. Let F and G be Riesz basic sequences that are orthogonal to one another and with constants c_F and c_G. Then $F \cup G$ is a Riesz basic sequence with constant $\max(c_F, c_G)$.

Proof. Let $a \in l^2(F \cup G)$. Then

$$\begin{aligned}\sum_{v \in F \cup G} |a_v|^2 &= \sum_{v \in F} |a_v|^2 + \sum_{v \in G} |a_v|^2 \\ &\leq \max(c_F, c_G)(\|\sum_{v \in F} a_v v\|^2 + \|\sum_{v \in G} a_v v\|^2) \\ &= \max(c_F, c_G)\|\sum_{v \in F} a_v v + \sum_{v \in G} a_v v\|^2\end{aligned}$$

□

Lemma A.3. Let F and G be Bessel sequences that are orthogonal to one another and with constants c_F and c_G. Then $F \cup G$ is a Bessel sequence with constant $\max(c_F, c_G)$.

Proof. Let u be a vector in the surrounding space. Denote by π_F and π_G the projections onto span F and span G respectively. Then

$$\begin{aligned}\sum_{v \in F \cup G} \langle v, u \rangle^2 &= \sum_{v \in F} \langle v, u \rangle^2 + \sum_{v \in G} \langle v, u \rangle^2 \\ &= \sum_{v \in F} \langle \pi_F(u), v \rangle^2 + \sum_{v \in G} \langle \pi_G(u), v \rangle^2 \\ &\leq \max(c_F, c_G)(\|\pi_G(u)\|^2 + \|\pi_G(u)\|^2) \\ &\leq \max(c_F, c_G)\|u\|^2\end{aligned}$$

□

Lemma A.4. Let u, v be vectors and $v \neq 0$. Then the infimum over $a \in \mathbb{R}$ of

$$\|u + av\|^2$$

is attained at

$$a = -\frac{\langle u, v \rangle}{\|v\|^2}.$$

Proof.
$$\|u + av\|^2 = \|u\|^2 + 2a\langle u, v\rangle + a^2\|v\|^2$$
is a second degree polynomial with positive leading coefficient which attains its infimum at $a = -\frac{\langle u,v\rangle}{\|v\|^2}$. ☐

Lemma A.5. Let u, v be vectors with $\|u\| = \|v\|$. Then the infimum over $a \in \mathbb{R}$ of
$$\frac{\|u + av\|^2}{\|u\|^2 + \|av\|^2}$$
is attained at
$$a = -\operatorname{sign}\langle u, v\rangle.$$

Lemma A.5 applied to the setting $\left|[0, \frac{1}{2}) \cap E\right| > \left|[\frac{1}{2}, 1) \cap E\right|$ and $u := \mathbb{1}_{[0,\frac{1}{2})\cap E}$, $v := h_{[0,1),E}$ was the first idea to Proposition 4.9. It also already proves it for the case $a_I = 0$. The observation that $\operatorname{sign}\langle u, v\rangle = -1$ and $\mathbb{1}_E + h_{[0,1),E}$ is supported on $[\frac{1}{2}, 1)$ then led to the role of Proposition 4.9 in the proof of Theorem 4.7.

Proof. By scaling it suffices to consider $\|u\| = \|v\| = 1$. Then the fraction is equal to
$$\frac{1 + 2a\langle u, v\rangle + a^2}{1 + a^2} = 1 + 2\langle u, v\rangle \frac{a}{1 + a^2}.$$
If $\langle u, v\rangle = 0$ then any choice for a works. Now consider $\langle u, v\rangle \neq 0$. For $|a| \to \infty$ we have $\frac{a}{1+a^2} \to 0$ while for $a \gtrless 0$ we have $\frac{a}{1+a^2} \gtrless 0$. That means the optima will be attained at
$$0 = \frac{\partial}{\partial a}\frac{a}{1 + a^2} = \frac{1 + a^2 - 2a^2}{(1 + a^2)^2} = \frac{1 - a^2}{(1 + a^2)^2} \qquad \Longleftrightarrow \qquad a \in \{-1, 1\}.$$
Note that of the two the infimum is attained at $a = -\operatorname{sign}\langle u, v\rangle$. ☐

Lemma A.6 is only used to prove Lemma A.7.

Lemma A.6. Let u, v, w be vectors and $u \perp v$ and $v \perp w$. Then
$$\sup_{a,b\in\mathbb{R}} |\sphericalangle\big[u, av + bw\big]\rangle| = |\sphericalangle\big[u, w\big]|.$$

Proof. If $v = 0$ then argument on the left hand side is constantly equal to the right hand side except for $b = 0$. If $w = 0$ then both sides vanish. Hence it suffices to consider $\|v\|, \|w\| = 1$. Furthermore because the argument on the left hand side vanishes for $a = b = 0$, it suffices to consider $a^2 + b^2 = 1$. Then
$$|\sphericalangle\big[u, av + bw\big]| = |a\langle uv\rangle + b\langle uw\rangle| = |b\langle uw\rangle|$$
which is maximal for $a = 0, b = 1$. ☐

Lemma A.7. Let $E \subset \mathbb{R}$ and $\operatorname{supp}(f) \subset E$ and g be dilate and translate of f such that $\operatorname{supp}(f)$ and $\operatorname{supp}(g)$ are disjoint subsets of E. Then for all $v \in \operatorname{span}\{f, g\}$ we have

$$\sup_{a,b \in \mathbb{R}} |\sphericalangle[\mathbb{1}_E, af + bg]| = |\sphericalangle[\mathbb{1}_E, f + g]|.$$

Note that Lemma A.7 can be extended inductively to sets of more than two functions that are disjointly supported translates and dilates of one another.

Proof of Lemma A.7. First note that

$$f + g, \qquad\qquad \frac{f}{\|f\|_2^2} - \frac{g}{\|g\|_2^2}$$

are orthogonal and span $\operatorname{span}\{f, g\}$. $\frac{f}{\|f\|_2^2} - \frac{g}{\|g\|_2^2}$ is orthogonal to $\mathbb{1}_E$ since

$$\langle \mathbb{1}_E, g\rangle = \int g = \frac{\int g}{\int f}\langle \mathbb{1}_E, f\rangle = \frac{\operatorname{supp}(g)}{\operatorname{supp}(f)}\langle \mathbb{1}_E, f\rangle = \frac{\|g\|_2^2}{\|f\|_2^2}\langle \mathbb{1}_E, f\rangle.$$

Therefore by Lemma A.6, $f + g$ optimizes the supremum. □

Lemma A.8. Let $E \subset [0, 1)$, $S \subset L^2(E)$ and let $u, v \in L^2(E)$ be orthogonal to one another. Then

$$\begin{aligned} A_E\big(S \cup \{u + v\}\big) &= A_E\big(S \cup \{u, v\}\big), \\ F_E\big(S \cup \{u + v\}\big) &= F_E\big(S \cup \{u, v\}\big), \\ B_E\big(S \cup \{u + v\}\big) &= B_E\big(S \cup \{u, v\}\big). \end{aligned}$$

Proof. The equality for F is evident from the definition and implies the equality for B. For A it follows from $\|v + u\|_2^2 = \|v\|_2^2 + \|u\|_2^2$. □

Lemma A.9 is a corollary of Lemma A.25.

Lemma A.9. Let V_1 and V_2 be two sets of vectors where $\sum_{v \in V_1} v$ and $\sum_{v \in V_2} v$ are orthogonal. Then there is an $i \in \{1, 2\}$ with

$$\frac{\|\sum_{v \in V_i} v\|_2^2}{\sum_{v \in V_i} \|v\|_2^2} \le \frac{\|\sum_{v \in V_1 \cup V_2} v\|_2^2}{\sum_{v \in V_1 \cup V_2} \|v\|_2^2}.$$

Proof. Lemma A.9 follows from

$$\begin{aligned} \|\sum_{v \in V_1 \cup V_2} v\|_2^2 &= \|\sum_{v \in V_1} v\|_2^2 + \|\sum_{v \in V_2} v\|_2^2, \\ \sum_{v \in V_1 \cup V_2} \|v\|_2^2 &= \sum_{v \in V_1} \|v\|_2^2 + \sum_{v \in V_2} \|v\|_2^2, \end{aligned}$$

and Lemma A.25. □

Lemma A.10. Let V be a finite set of vectors to all of which u is orthogonal. Further assume
$$\|u + \sum_{v\in V} v\|^2 \leq \|u\|^2 + \sum_{v\in V} \|v\|^2.$$
Then
$$\frac{\| \sum_{v\in V} v\|^2}{\sum_{v\in V} \|v\|^2} \leq \frac{\|u + \sum_{v\in V} v\|^2}{\|u\|^2 + \sum_{v\in V} \|v\|^2}$$

Lemma A.8 and A.10 will often be used in conjunction: First a function $f \in S$ will be orthogonally split into $f = u + v$, of which u will be orthogonal also to all other functions in S. Then $\{f\}$ can be replaced by $\{u, v\}$ in S by Lemma A.8, after which u can again be discarded using Lemma A.10. That means f has been replaced by its part v that is more (anti-)parallel to the other functions in S.

Proof of Lemma A.10. By orthogonality we have
$$\|u + \sum_{v\in V} v\|^2 = \|u\|^2 + \| \sum_{v\in V} v\|^2$$
so that the assumption of the lemma implies
$$\frac{\| \sum_{v\in V} v\|^2}{\sum_{v\in V} \|v\|^2} \leq 1 = \frac{\|u\|^2}{\|u\|^2}.$$
Then Lemma A.10 follows from by Lemma A.9 applied to V and $\{u\}$. □

Lemma A.11. Let $\|u\| = \|v\|$, $-\|u\|\|v\| < \langle u, v\rangle \leq 0$ and $|a| \geq 1$. Then there is a $b \geq 0$ s.t.
$$\|u + av\|^2 = \|u + v\|^2 + \|bu + bv\|^2. \tag{115}$$
Furthermore for any such b we have
$$\|av\|^2 \leq \|v\|^2 + \|bu\|^2 + \|bv\|^2. \tag{116}$$

Proof. Since $-\|u\|\|v\| < \langle u, v\rangle$ implies $\|u\| > 0$, it suffices to consider $\|u\| = \|v\| = 1$ after rescaling. By Lemma A.5 and $|a| \geq 1$ we have
$$\|u + v\|^2 \leq \frac{\|u\|^2 + \|v\|^2}{\|u\|^2 + \|av\|^2}\|u + av\|^2 \leq \|u + av\|^2.$$
This means that a b that satisfies (115) exists, because by $\langle u, v\rangle > -1$ we have $\|u+v\|^2 > 0$.

Write $x := -\langle u, v\rangle$. Then (115) reads
$$1 - 2xa + a^2 = 1 - 2x + 1 + b^2(1 - 2x + 1) = 2(1 - x)(1 + b^2)$$

and we have to show

$$a^2 \leq 1 + 2b^2.$$

Now

$$\begin{aligned} 1 + 2b^2 - a^2 = 2(1+b^2) - 1 - a^2 &= \frac{1 - 2xa + a^2}{1-x} - 1 - a^2 \\ &= \frac{1 - 2xa + a^2 - 1 + x - a^2 + a^2 x}{1-x} \\ &= \frac{-2xa + x + a^2 x}{1-x} \\ &= \frac{x(a-1)^2}{1-x} \geq 0. \end{aligned}$$

□

A.2 Lemmas on Restricted Haar Functions

Lemma A.12. Let $f, f', g, g' : [0,1) \to \{-1, 0, 1\}$ with $\operatorname{supp} g \subset f^{-1}(\{1\})$ and $\operatorname{supp} g' \subset f'^{-1}(\{1\})$ and

$$\begin{aligned} |\operatorname{supp} f| &= |\operatorname{supp} f'|, \\ \left|g^{-1}(\{1\})\right| &= \left|g'^{-1}(\{1\})\right|, \\ \left|g^{-1}(\{-1\})\right| &= \left|g'^{-1}(\{-1\})\right|. \end{aligned}$$

Then the linear map induced by

$$\begin{aligned} f &\mapsto f', \\ f\mathbb{1}_{f=1} &\mapsto f'\mathbb{1}_{f'=1}, \\ f\mathbb{1}_{f=-1} &\mapsto f'\mathbb{1}_{f'=-1}, \\ g &\mapsto g' \\ g\mathbb{1}_{g=1} &\mapsto g'\mathbb{1}_{g'=1}, \\ g\mathbb{1}_{g=-1} &\mapsto g'\mathbb{1}_{g'=-1} \end{aligned}$$

is an isometry.

Proof. straightforward calculation. □

Lemma A.12 can clearly be extended to sets of more than two functions.

Lemma A.13. Let $p \in [\frac{1}{2}, 1)$ and I be an interval and $E \subset I$. Then there are unique $E_{\|}, I_{\|}$ with $E_{\|} \subset I$, $|E_{\|}| = |E|$ and $I_{\|}$ is the most p-antiparallel interval to I with respect to $E_{\|}$. Furthermore for all E, p-dominant $J \subset I$ we have

$$\sphericalangle\left[\mathbb{1}_{[0,p)}, h_{[0,1),[0,p)}\right] \leq \sphericalangle\left[\mathbb{1}_{E_{\|}}, h_{I_{\|},E_{\|}}\right] \leq \sphericalangle\left[\mathbb{1}_{E}, h_{J,E}\right]. \tag{117}$$

Remark. Here are some examples of most antiparallel intervals:

- If $I_{\|}$ is most antiparallel to some interval I w.r.t. E and $I_{\|} \subset J \subset I$ then $I_{\|}$ is also most antiparallel to J w.r.t. E.
- I is most antiparallel to I w.r.t. E if and only if I and $E \cap I$ are intervals with the same left boundary and $|I \cap E| = p$. In particular $[0,1)$ is most antiparallel to $[0,1)$ w.r.t. $[0,p)$.

Proof of Lemma A.13. We first prove the existence of a most antiparallel interval $I_{\|}$. For that it suffices to consider the case that $I = \big[0,|I|\big)$. Now set $E_{\|} := \big[0,|E|\big)$ and

$$I_{\|} := \begin{cases} \big[0, \frac{|E|}{p}\big) & |E| \le p|I| \\ \big[\frac{|E|-p|I|}{1-p}, |I|\big) & |E| \ge p|I| \end{cases}.$$

For $|E| \le p|I|$ it is evident from the definition that $I_{\|}$ is most p-antiparallel to I w.r.t $E_{\|}$. For $|E| \ge p|I|$ we have $\frac{|E|-p|I|}{1-p} \ge 0$ and $\frac{|E|-p|I|}{1-p} \le \frac{|I|-p|I|}{1-p} = |I|$. Also

$$\begin{aligned} |I_{\|}| &= \frac{|I|(1-p) - (|E| - p|I|)}{1-p} = \frac{|I| - |E|}{1-p}, \\ |E_{\|} \cap I_{\|}| &= \frac{|E|(1-p) - (|E| - p|I|)}{1-p} = \frac{p(|I| - |E|)}{1-p} = p|I_{\|}|. \end{aligned}$$

Hence $I_{\|}$ is most antiparallel.

$E_{\|}$ is clearly the unique interval contained in I with the same left boundary as I and $|E_{\|}| = |E|$. $I_{\|}$ is the unique most antiparallel interval to I w.r.t. $E_{\|}$, because $I = E_{\|} \cup (I \setminus E_{\|})$ is an interval partition and the definition of being most antiparallel fixes $|\mathfrak{l}(I_{\|}) \cap E_{\|}|$ and $|\mathfrak{r}(I_{\|}) \cap E|$.

Now we prove the first inequality in (117). First calculate

$$\sphericalangle\big[\mathbb{1}_{E_{\|}}, h_{I_{\|},E_{\|}}\big] = \frac{(p-\frac{1}{2})|I_{\|}| - \frac{1}{2}|I_{\|}|}{\sqrt{p|I_{\|}|}\sqrt{|E_{\|}|}} = \frac{p-1}{\sqrt{p}} \frac{\sqrt{|I_{\|}|}}{\sqrt{|E_{\|}|}}.$$

The same calculation holds for $E_{\|} = [0,p),\ I_{\|} = [0,1)$ so that

$$\sphericalangle\big[\mathbb{1}_{[0,p)}, h_{[0,1),[0,p)}\big] = \frac{p-1}{\sqrt{p}} \frac{1}{\sqrt{p}}.$$

Now

$$|E_{\|}| \le p|I|: \quad \frac{|I_{\|}|}{|E_{\|}|} = \frac{1}{p},$$

$$|E_{\|}| \geq p|I| : \quad \frac{|I_{\|}|}{|E_{\|}|} = \frac{\frac{|I|(1-p)-(|E_{\|}|-p|I|)}{1-p}}{|E_{\|}|} = \frac{|I| - |E_{\|}|}{(1-p)|E_{\|}|} = \frac{\frac{|I|}{|E_{\|}|} - 1}{1-p} \leq \frac{\frac{1}{p} - 1}{1-p} = \frac{1}{p}.$$

This proves the first inequality.

Now for the second inequality in (117) let $E_{\|} \subset I$, $|E_{\|}| = |E|$ and $I_{\|}$ be most antiparallel to I w.r.t. $E_{\|}$. Since J is E-dominant we have

$$\frac{|\mathfrak{r}(J) \cap E|}{|\mathfrak{l}(J) \cap E|} \geq 2p - 1,$$

and it suffices to consider the case

$$\frac{|\mathfrak{r}(J) \cap E|}{|\mathfrak{l}(J) \cap E|} < 1.$$

since otherwise $\sphericalangle\left[\mathbb{1}_E, h_{I,E}\right] \geq 0$. That means there is a partition

$$J = J_{\|} \cup J_{\perp}$$

such that

$$\frac{|\mathfrak{r}(J) \cap J_{\|}|}{|\mathfrak{l}(J) \cap J_{\|}|} = \frac{|\mathfrak{r}(J) \cap J_{\perp}|}{|\mathfrak{l}(J) \cap J_{\perp}|} = \frac{|\mathfrak{r}(J) \cap J_{\perp} \cap E|}{|\mathfrak{l}(J) \cap J_{\perp} \cap E|} = 1,$$

$$\frac{|\mathfrak{r}(J) \cap J_{\|} \cap E|}{|\mathfrak{l}(J) \cap J_{\|} \cap E|} = 2p - 1,$$

$$|J_{\|} \cap E| = p|J_{\|}|,$$

and $|J_{\|}| > 0$. Now we have an orthogonal decomposition

$$h_{J,E} = h_{J,E\cap J_{\|}} + h_{J,E\cap J_{\perp}}$$

where $h_{J,E\cap J_{\perp}}$ is orthogonal to $\mathbb{1}_E$. Thus because $\sphericalangle\left[\mathbb{1}_E, h_{J,E\cap J_{\|}}\right] < 0$ we have

$$\sphericalangle\left[\mathbb{1}_E, h_{J,E\cap J_{\|}}\right] \leq \sphericalangle\left[\mathbb{1}_E, h_{J,E}\right] \tag{118}$$

Claim.

$$|J_{\|}| \leq |I_{\|}|.$$

Proof.

$$|E| \leq p|I| : \qquad |J_{\|}| = \frac{|J_{\|} \cap E|}{p} \leq \frac{|E|}{p} = |I_{\|}|,$$

$$|E| \geq p|I| : \qquad |J_{\|}| = \frac{|J_{\|} \setminus E|}{1-p} \leq \frac{|I| - |E|}{1-p} = |I_{\|}|.$$

□

By the claim we may take $D \subset E \setminus J_{\parallel}$ such that

$$\frac{|J_{\parallel}|}{|E \setminus D|} = \frac{|I_{\parallel}|}{|E_{\parallel}|}.$$

Note that the definition of most antiparallel implies that $\mathfrak{l}(I_{\parallel}) \subset E_{\parallel}$. So

$$\frac{|\mathfrak{r}(J) \cap J_{\parallel} \cap E|}{|\mathfrak{l}(J) \cap J_{\parallel} \cap E|} = (2p-1) = \frac{|\mathfrak{r}(I_{\parallel}) \cap E_{\parallel}|}{|\mathfrak{l}(I_{\parallel}) \cap E_{\parallel}|}$$

and we have by Lemma A.12 that

$$\sphericalangle\left[\mathbb{1}_{E\setminus D}, h_{J,E\cap J_{\parallel}}\right] = \sphericalangle\left[\mathbb{1}_{E_{\parallel}}, h_{I_{\parallel},E_{\parallel}}\right]. \tag{119}$$

Now because (119) ≤ 0 and $\mathbb{1}_E = \mathbb{1}_{E\setminus D} + \mathbb{1}_D$ is an orthogonal decomposition and $\mathbb{1}_D \perp h_{J,E\cap J_{\parallel}}$ we get

$$\sphericalangle\left[\mathbb{1}_E, h_{J,E\cap J_{\parallel}}\right] \geq \sphericalangle\left[\mathbb{1}_{E\setminus D}, h_{J,E\cap J_{\parallel}}\right]. \tag{120}$$

(118), (119) and (120) prove the second inequality in (117). □

Lemma A.14. Let $E \subset [0,1)$ and S be a compatible E-dominant sequence with $[-1,1) \in \mathbb{I}(S)$. Then there is an $E' \subset [0,1)$ and a compatible E'-dominant sequence S' with $[-1,1) \notin \mathbb{I}(S)$ and

$$\frac{B_{E'}(S')}{A_{E'}(S')} = \frac{B_E(S)}{A_E(S)}.$$

Proof. By rescaling it suffices to consider the case that the coefficient in front of $[-1,1)$ is 1. Then let E_1 be E translated and dilated to $[0,\frac{1}{2})$ and $E_2 = E_1 + \frac{1}{2}$. Define S_1, S_2 similarly, applied to $S \setminus \{[-1,1)\}$. Further define $E' = E_1 \cup E_2$ and $S' = \{[0,1)\} \cup (-S_1) \cup S_2$. Then

$$\begin{aligned}
A_E(S) &= \frac{1}{2}A_E(S) + \frac{1}{2}A_E(S) \\
&= \|\mathbb{1}_{E_1}\|_2^2 + A_{E_1}(S_1) + \|\mathbb{1}_{E_1}\|_2^2 + A_{E_2}(S_2) \\
&= \|-\mathbb{1}_{E_1} + \mathbb{1}_{E_2}\|_2^2 + A_{E_1}(-S_1) + A_{E_2}(S_2) \\
&= A_{E'}(S') \\
B_E(S) &= \frac{1}{2}B_E(S) + \frac{1}{2}B_E(S) \\
&= \|\mathbb{1}_{E_1} + F_{E_1}(S_1)\|_2^2 + \|\mathbb{1}_{E_2} + F_{E_2}(S_2)\|_2^2 \\
&= \|-\mathbb{1}_{E_1} - F_{E_1}(S_1)\|_2^2 + \|\mathbb{1}_{E_2} + F_{E_2}(S_2)\|_2^2 \\
&= \|-\mathbb{1}_{E_1} - F_{E_1}(S_1) + \mathbb{1}_{E_2} + F_{E_2}(S_2)\|_2^2 \\
&= B_{E'}(S')
\end{aligned}$$

Also, $[0,1), \mathbb{I}(S_1), \mathbb{I}(S_2)$ are all E'-dominant. □

Remark. The proof shows that if $\mathbb{I}(S) \subset \mathscr{D}$ then we can also take $\mathbb{I}(S') \subset \mathscr{D}$.

Lemma A.15. Let $p \in [0,1]$. For $i = 1,2,\dots$ let $E_i \subset [0,1)$ and let $\mathbb{I}_i$ consist of compatible E_i, p-dominant intervals so that $\{\frac{h_{I,E_i}}{\|h_{I,E_i}\|_2} \mid I \in \mathbb{I}_i\}$ is a Riesz basic sequence with maximal constant at most $\frac{1}{i}$. Then there is an $E \subset [0,1)$ and a compatible E, p-dominant $\mathbb{I} \subset \mathscr{D}$ such that $\{\frac{h_{I,E}}{\|h_{I,E}\|_2} \mid I \in \mathbb{I}\}$ is not a Riesz basic sequence.

Proof. By Lemma A.14 it suffices to consider the case $[-1,1) \notin \mathbb{I}_i$ so that all intervals in $\mathbb{I}_i$ are contained in $[0,1)$. Then let E and $\mathbb{I}$ be a disjoint union of translates and dilates of $E_1, E_2, \dots$ and $\mathbb{I}_1, \mathbb{I}_2, \dots$ respectively. For example work with dyadic numbers and define

$$E := \bigcup_{i=1,2,\dots} 0.\underbrace{1\dots1}_{i-1 \text{ times}} + 2^{-i} E_i,$$
$$\mathbb{I} := \bigcup_{i=1,2,\dots} 0.\underbrace{1\dots1}_{i-1 \text{ times}} + 2^{-i} \mathbb{I}_i.$$

Then $\mathbb{I}$ is compatible and E, p-dominant, and for all $i = 1,2,\dots$ the maximal constant of $\left\{\frac{h_{I,E}}{\|h_{I,E}\|_2} \mid I \in \mathbb{I}\right\}$ can be at most $\frac{1}{i}$, because it contains a translate and dilate of $\left\{\frac{h_{I,E_i}}{\|h_{I,E_i}\|_2} \mid I \in \mathbb{I}_i\right\}$. Hence the maximal constant has to be zero which means that it is no Riesz basic sequence. $\square$

Lemma A.16. Let $\frac{1}{2} \leq p \leq 1$ and $E \subset [0,1)$ with $[0,\frac{1}{2}) \subset E$ and $|E| \geq p$. Let $J_0, J_1, \dots \subset [\frac{1}{2}, 1)$ be disjoint E-dominant intervals and for $i = 0,1,\dots$ let T_i be a finite subset of $L^2(J_i)$. Assume

$$S = \{h_{[-1,1),E}, ah_{[0,1),E}\} \cup T_0 \cup T_1 \cup \dots. \tag{121}$$

Then there is a partition into intervals $[0,1) = I_0' \cup I_1' \cup \dots$ and for each $i = 0,1,\dots$ there is a translate J_i', T_i' of J_i, T_i with $J_i \subset \mathfrak{r}(I_i')$. Take $E' \subset [0,1)$ with $|E'| = |E|$, $\mathfrak{l}(I_i') \subset E'$ and such that $E' \cap J_i'$ is the image of $E \cap J_i$ under the translation $J_i \to J_i'$. Then

$$A_E(S) = \sum_i A_{E'}(\{\mathbb{1}_{I_i' \cap E}, ah_{I_i',E}\} \cup T_i')$$
$$B_E(S) = \sum_i B_{E'}(\{\mathbb{1}_{I_i' \cap E}, ah_{I_i',E}\} \cup T_i')$$

This Lemma A.16 may be used in conjunction with Lemma A.9 in order to pass from an S of the form (121) to one where the sequence $T_0, T_1, \dots$ consists of only one element.

Proof of Lemma A.16. We would first like to redistribute $E \cap [\frac{1}{2}, 1)$ and translate $T_0, T_1, \dots$ isometrically so that there is an interval partition

$$[\frac{1}{2}, 1) = I_{01} \cup I_{11} \cup I_{21} \cup \dots \tag{122}$$

such that for all $i = 0, 1, \dots$ we have $J_i \subset I_{i1}$ and

$$\frac{|E \cap I_{i1}|}{|I_{i1}|} \geq 2p - 1. \tag{123}$$

This can be done as follows: For $i = 0, 1, \dots$ define $I_{i1} \coloneqq J_i$. Then (123) holds. If (122) holds or

$$\frac{\left|([\frac{1}{2}, 1) \setminus \bigcup_{i=0,1,\dots} I_{i1}) \cap E\right|}{\left|[\frac{1}{2}, 1) \setminus \bigcup_{i=0,1,\dots} I_{i1}\right|} \geq 2p - 1 \tag{124}$$

add $[\frac{1}{2}, 1) \setminus \bigcup_{i=0,1,\dots} I_{i1}$ to I_{01}. Otherwise, since

$$\frac{|E \cap [\frac{1}{2}, 1)|}{|[\frac{1}{2}, 1)|} \geq 2p - 1$$

there must be a j s.t. $\frac{|I_{j1} \cap E|}{|I_{j1}|} > 2p - 1$. Then add as much of $[\frac{1}{2}, 1) \setminus \bigcup_{i=0,1,\dots} I_{i1} \setminus E$ to I_{j1} so that (124) holds or $\frac{|I_{j1} \cap E|}{|I_{j1}|} = 2p - 1$. If (124) still doesn't hold, then there are $j', j'', \dots$ for which we can repeat the procedure until (124) holds. After that, add $[\frac{1}{2}, 1) \setminus \bigcup_{i=0,1,\dots} I_{i1}$ to I_{01}. Now redistribute and/or translate $E \cap [\frac{1}{2}, 1)$ and $I_{01}, I_{11}, \dots$ and $T_0, T_1, \dots$ isometrically so that $I_{01}, I_{11}, \dots$ are intervals. Then (122) holds as an interval partition and (123) holds.

Now for $i = 0, 1, \dots$ set

$$\begin{aligned} I_{i0} &\coloneqq I_{i1} - \frac{1}{2}, \\ I_i &\coloneqq I_{i0} \cup I_{i1}, \\ E_i &\coloneqq E \cap I_i. \end{aligned}$$

Then

$$\begin{aligned} h_{[-1,1),E} &= \sum_{i=0,1,\dots} h_{[-1,1),E_i}, \\ h_{[0,1),E} &= \sum_{i=0,1,\dots} h_{[0,1),E_i} \end{aligned}$$

are orthogonal decompositions. Now for $i = 0, 1, \ldots$ define

$$S_i' := \{h_{[-1,1),E_i}, ah_{[0,1),E_i}\} \cup T_i.$$

Then $S_0', S_1', \ldots$ are pairwise orthogonal and by Lemma A.8 we have

$$A_E(S) = A_E(S_0' \cup S_1' \cup \ldots),$$
$$B_E(S) = B_E(S_0' \cup S_1' \cup \ldots).$$

Now for each i redistribute translate I_{i0} and I_{i1} next to each other such that they become the left and right halves of I_i which is now an interval. Redistribute the domains in S_i' and E_i accordingly, i.e. such that that S_i' is of the form

$$\{\mathbb{1}_{I_i \cap E_i}, ah_{I_i,E_i}\} \cup T_i'.$$

Set $E' := E_0 \cup E_1 \cup \ldots$ and the lemma follows. □

Lemma A.17. Let I be an interval, $0 \leq b_1, b_2 \leq 1$, $b := \frac{b_1+b_2}{2}$ and $E \subset I$ with $|\mathfrak{l}(I) \cap E| = b_1|\mathfrak{l}(I)|$, $|\mathfrak{r}(I) \cap E| = b_2|\mathfrak{r}(I)|$. Then the infimum over $a \in \mathbb{R}$ of

$$\|\mathbb{1}_E + ah_{I,E}\|_2^2$$

is attained at

$$a = a_{\min} := \frac{b_1 - b_2}{2b}.$$

Furthermore if $b \geq \frac{2}{3}$ then for all $|a| \leq 1$ and

$$\begin{cases} I_1 = \mathfrak{l}(I), \\ I_2 = \mathfrak{r}(I) \end{cases} \qquad \text{and} \qquad \begin{cases} I_1 = \mathfrak{r}(I), \\ I_2 = \mathfrak{l}(I) \end{cases}$$

there is an a' with the same sign as a and $|a| \leq |a'| \leq 1$ and

$$\|\mathbb{1}_E + ah_{I,E}\|_{L^2(I_1)}^2 \leq \|\mathbb{1}_E + a'h_{I,E}\|_{L^2(I_2)}^2, \tag{125}$$
$$\|\mathbb{1}_E + ah_{I,E}\|_2^2 = \|\mathbb{1}_E + a'h_{I,E}\|_2^2. \tag{126}$$

Proof. By translation and dilation it suffices to consider $I = [0, 1)$. Then the value for a where the infimum is attained follows Lemma A.4 and

$$\|h_{[0,1),E}\|_2^2 = |E| = b,$$

$$\langle \mathbb{1}_E, h_{[0,1),E}\rangle = \frac{b_2 - b_1}{2}.$$

For the second part, the changes $I_1 \leftrightarrow I_2$, $a \mapsto -a$ and $b_1 \leftrightarrow b_2$ all have the same effect. Hence it suffices to consider the case $I_1 = \mathfrak{l}(I)$, $I_2 = \mathfrak{r}(I)$ and $a \geq 0$.

If $b_1 \leq b_2$ then (125) and (126) hold for $a' = a$. It remains to consider $b_1 \geq b_2$. Rewrite $a = a_{\min} + \varepsilon$ and $a' = a_{\min} + \delta$. Then

$$\begin{aligned}
&\|\mathbb{1}_E + a' h_{[0,1),E}\|^2_{L^2([\frac{1}{2},1))} - \|\mathbb{1}_E + a h_{[0,1),E}\|^2_{L^2([0,\frac{1}{2}))} \\
&= \frac{b_2}{2}[1 + a_{\min} + \delta]^2 - \frac{b_1}{2}[1 - a_{\min} - \varepsilon]^2 \\
&= \frac{b_2}{2}\frac{(2b + b_1 - b_2 + 2b\delta)^2}{(2b)^2} - \frac{b_1}{2}\frac{(2b - b_1 + b_2 - 2b\varepsilon)^2}{(2b)^2} \\
&= \frac{b_2}{2}\frac{(2b_1 + 2b\delta)^2}{(2b)^2} - \frac{b_1}{2}\frac{(2b_2 - 2b\varepsilon)^2}{(2b)^2} \\
&= \frac{(b_1 - b_2)b_1 b_2 + 2b_1 b_2 b(\delta + \varepsilon) + b^2(b_2\delta^2 - b_1\varepsilon^2)}{2b^2} \qquad (127)
\end{aligned}$$

We set $\delta := |\varepsilon|$. Then (126) holds because $x \mapsto \|\mathbb{1}_E + x h_{I,E}\|_2^2$ is a quadratic polynomial with minimum $x = a_{\min}$. Also $a' \geq a_{\min} \geq 0$. For $\varepsilon \geq 0$ we have $a' = a \leq 1$. For $\varepsilon \leq 0$ we still have $a \geq 0$ i.e. $\varepsilon \geq -a_{\min}$ so that $\delta \leq a_{\min}$ and

$$\begin{aligned}
a' &\leq 2a_{\min} = 2\frac{b_1 - b_2}{2b} = 2\frac{b_1 - b_2}{b_1 + b_2} = 2\frac{1 - \frac{b_2}{b_1}}{1 + \frac{b_2}{b_1}} \\
&\leq 2\frac{1 - (2b - 1)}{1 + (2b - 1)} = 2\frac{2 - 2b}{2b} = 2\frac{1 - b}{b} \leq 2\frac{\frac{1}{3}}{\frac{2}{3}} = 1.
\end{aligned}$$

Because $b_1 \geq b_2$ the choice $\delta = |\varepsilon|$ makes (127) a quadratic polynomial in ε with negative leading coefficient, in the domains $\varepsilon \geq 0$ and $\varepsilon \leq 0$ each. Thus in order to ensure its positivity we only need to check it at the boundaries of the domains, which are at $\varepsilon = -a_{\min}$, $\varepsilon = 0$ and $\varepsilon = a_{\min}$. Of the three summands in (127) the first does not depend on ε, δ. The second one vanishes for $\delta = \varepsilon = 0$ and $\delta = -\varepsilon = a_{\min}$ and is positive for $\delta = \varepsilon = a_{\min}$. The last one vanishes for $\delta = \varepsilon = 0$ and is equal and negative for $\delta = \varepsilon = a_{\min}$ and $\delta = -\varepsilon = a_{\min}$. Thus it suffices to consider $\delta = -\varepsilon = a_{\min}$ because (127) is minimal there. If $b_1 = b_2$ then (127) $= 0$. Otherwise

$$\begin{aligned}
\frac{2b^2}{b_1 - b_2}(127) &= b_1 b_2 - b^2 a_{\min}^2 = b_1 b_2 - \frac{(b_1 - b_2)^2}{4} = \frac{3}{2}b_1 b_2 - \frac{b_1^2}{4} - \frac{b_2^2}{4} \\
&= b_1 b_2 \Big[\frac{3}{2} - \frac{1}{4}(\frac{b_1}{b_2} + \frac{b_2}{b_1})\Big]
\end{aligned}$$

and since $0 = \frac{\mathrm{d}}{\mathrm{d}x}(\frac{1}{x} + x) = -\frac{1}{x^2} + 1 \leq 0$ for $x \leq 1$ we maximize $\frac{b_1}{b_2} + \frac{b_2}{b_1}$ for $\frac{b_2}{b_1}$ minimal. So since $\frac{b_2}{b_1} \geq 2b - 1 > \frac{4}{3} - 1 = \frac{1}{3}$ we get

$$\geq b_1 b_2 \big[\frac{3}{2} - \frac{1}{4}(3 + \frac{1}{3})\big] = b_1 b_2 \big[\frac{3}{2} - \frac{1}{4}\frac{10}{3}\big] = b_1 b_2 \frac{9-5}{6} > 0.$$

□

Lemma A.18. Let $p \geq \frac{1}{2}$, $E \subset I$ be intervals with the same left boundary, $|E| = p|I|$ and $J \subset I$ with $|J \cap E| = p|J|$ and $a, a_I, a_J \geq 0$. Now if

$$\|a_J h_{J,E}\|_2^2 = \|a_I h_{I,E}\|_2^2$$

then

$$\|a \mathbb{1}_E + a_J h_{J,E}\|_{L^2(\mathfrak{r}(J))}^2 \leq \|a \mathbb{1}_E + a_I h_{I,E}\|_{L^2(\mathfrak{r}(I))}^2$$

Proof. The assumption $|J \cap E| a_J^2 = |I \cap E| a_I^2$ implies

$$|J| a_J^2 = |I| a_I^2$$

which by $|J| \leq |I|$ also implies $|J|^2 a_J^2 \leq |I|^2 a_I^2$ and thus

$$|J| a_J \leq |I| a_I.$$

Therefore

$$\begin{aligned}
\|a \mathbb{1}_E + a_J h_{J,E}\|_{L^2(\mathfrak{r}(J))}^2 &= |J|(p - \frac{1}{2})(a + a_J)^2 \\
&= (p - \frac{1}{2})\big(|J|a^2 + 2|J|a a_J + |J| a_J^2\big) \\
&\leq (p - \frac{1}{2})\big(|I|a^2 + 2|I|a a_I + |I| a_I^2\big) \\
&= \|a \mathbb{1}_E + a_I h_{I,E}\|_{L^2(\mathfrak{r}(I))}
\end{aligned}$$

□

Lemma A.19. Let $E \subset J \subset I$ be three intervals with the same left boundary and $|E| > \frac{1}{2}|I|$. Let $0 \leq a \leq a_0$. Then there is an $(a_{\max}, J_{\max})$ where $a \leq a_{\max} \leq a_0$, $J \subset J_{\max} \subset I$, and $J_{\max}$ also has the same left boundary as I and with

$$\|a_0 \mathbb{1}_E + a_{\max} h_{J_{\max},E}\|_{L^2(\mathfrak{r}(I))}^2 = \|a_0 \mathbb{1}_E + a h_{J,E}\|_{L^2(\mathfrak{r}(I))}^2$$

and

$$\|a_0 \mathbb{1}_E + a_{\max} h_{J_{\max},E}\|_2^2 \leq \|a_0 \mathbb{1}_E + a h_{J,E}\|_2^2$$

and where $a_{\max} = a_0$ or $J_{\max} = I$.

Proof. If $a_0 = 0$ we may just keep $a_{\max} := a = 0,\ J_{\max} := J$. Otherwise if $a_0 > 0$ by rescaling it suffices to consider $a_0 = 1$. Then note that

$$\frac{1}{|E|}\|\mathbb{1}_E + ah_{J,E}\|^2_{L^2(\mathfrak{r}(I))} = (1 - \frac{|J|}{2|E|})(1+a)^2,$$
$$\frac{1}{|E|}\|\mathbb{1}_E + ah_{J,E}\|^2_2 = \frac{|J|}{2|E|}(1-a)^2 + (1 - \frac{|J|}{2|E|})(1+a)^2.$$

Now Lemma A.26 allows us to increase a and extend J to the right in such a way that $\|\mathbb{1}_E + ah_{J,E}\|^2_{L^2(\mathfrak{r}(J))}$ stays constant and $\|\mathbb{1}_E + ah_{J,E}\|^2_2$ decreases. We do this until one of the bounds $a = 1,\ J = I$ is reached. □

A.3 Calculations

Some statements in this thesis can be seen as results of simple calculations. We list most of these calculations in this subsubsection.

Lemma A.20. Let $f : [0,1] \to \mathbb{R}$ be continuous and *midpoint convex*, i.e. for all $x, y \in [0,1)$ we have

$$\frac{f(x)+f(y)}{2} \geq f\Big(\frac{x+y}{2}\Big).$$

Then f is also *convex*, i.e. for all $x, y, t \in [0,1]$ we have

$$tf(x) + (1-t)f(y) \geq f(tx + (1-t)y).$$

Proof. Assume f is not convex. Then there are $x, y, t \in [0,1]$ with

$$tf(x) + (1-t)f(y) < f(tx + (1-t)y).$$

Since both sides are continuous in t, there is an interval $I \ni t$ such that for all $s \in I$ the above strict inequality holds for s instead of t. Let J be the union of all such intervals I. J is also an interval. At the endpoints of J the strict inequality cannot hold anymore, for then we could extend it even further by continuity. Hence we may write $J = (t_0, t_1)$. Since for $s = 0, 1$ the inequality does not hold, we have $\{t_0, t_1\} \subset [0,1]$. By continuity we must have equality for $s = t_0, t_1$. Therefore

$$\begin{aligned}
&\frac{f(t_0x + (1-t_0)y) + f(t_1x + (1-t_1)y)}{2} \\
&= \frac{t_0}{2}f(x) + \frac{1-t_0}{2}f(y) + \frac{t_1}{2}f(x) + \frac{1-t_1}{2}f(y) \\
&= \frac{t_0+t_1}{2}f(x) + (1 - \frac{t_0+t_1}{2})f(y)
\end{aligned}$$

and since $\frac{t_0+t_1}{2} \in \boldsymbol{J}$ we have

$$\begin{aligned} &< f\Big(\frac{t_0+t_1}{2}x + (1-\frac{t_0+t_1}{2})y\Big) \\ &= f\Big(\frac{t_0x+(1-t_0)y+t_1x+(1-t_1)y}{2}\Big) \end{aligned}$$

which means that f is not midpoint convex. □

Lemma A.21. For $c_- + c_+ > 2$ and $c \in \mathbb{R}$ we have

$$\begin{aligned} \inf_{x\in\mathbb{R}} \frac{(1-x)^2}{2}c_- + \frac{(1+x)^2}{2}c_+ - c - x^2 &= \frac{c_- + c_+}{2} - \frac{1}{2}\frac{(c_+ - c_-)^2}{c_+ + c_- - 2} - c \\ &= \frac{2c_+c_- - (1+c)(c_+ + c_-) + 2c}{c_+ + c_- - 2}. \end{aligned}$$

Proof.

$$\begin{aligned} &\frac{(1-x)^2}{2}c_- + \frac{(1+x)^2}{2}c_+ - c - x^2 \\ &= (\frac{c_+ + c_-}{2} - 1)x^2 + (c_+ - c_-)x + (\frac{c_+ + c_-}{2} - c) \\ &= (\frac{c_+ + c_-}{2} - 1)\big(x + \frac{c_+ - c_-}{c_- + c_+ - 2}\big)^2 - \frac{1}{2}\frac{(c_+ - c_-)^2}{c_+ + c_- - 2} + \frac{c_- + c_+}{2} - c \end{aligned}$$

whose minimum is

$$\frac{c_- + c_+}{2} - \frac{1}{2}\frac{(c_+ - c_-)^2}{c_+ + c_- - 2} - c.$$

Now multiplying this by $c_+ + c_- - 2$ yields

$$\begin{aligned} &\frac{1}{2}(c_- + c_+)(c_+ + c_- - 2) - \frac{1}{2}(c_+ - c_-)^2 - c(c_+ + c_- - 2) \\ &= 2c_+c_- - (1+c)(c_+ + c_-) + 2c. \end{aligned}$$

□

Lemma A.22. Let $1 < b < a$. Then for all $x \in \mathbb{R}$ and $q_1, q_2 \in [0,1]$ and $q \coloneqq \frac{q_1+q_2}{2}$ we have

$$\frac{(1-x)^2}{2}\frac{a-q_1}{b-q_1} + \frac{(1+x)^2}{2}\frac{a-q_2}{b-q_2} - \frac{a-q}{b-q} \geq x^2$$

Proof. Abbreviate

$$a_1 = a - q_1 \qquad\qquad a_2 = a - q_2$$

$$b_1 = b - q_1 \qquad\qquad b_2 = b - q_2$$

Then we have to check that

$$\frac{(1-x)^2}{2}\frac{a_1}{b_1} + \frac{(1+x)^2}{2}\frac{a_2}{b_2} - \frac{a_1+a_2}{b_1+b_2} - x^2 \geq 0$$

Since $a_1 > b_1 > 0$ and $a_2 > b_2 > 0$ we may invoke Lemma A.21 for that, which implies that it suffices to confirm the positivity of

$$\begin{aligned}
&2\frac{a_1a_2}{b_1b_2} - (1 + \frac{a_1+a_2}{b_1+b_2})(\frac{a_1}{b_1} + \frac{a_2}{b_2}) + 2\frac{a_1+a_2}{b_1+b_2}\\
&= \frac{2a_1a_2(b_1+b_2) - (b_1+b_2+a_1+a_2)(a_1b_2+a_2b_1) + 2b_1b_2(a_1+a_2)}{b_1b_2(b_1+b_2)}\\
&= \frac{a_1a_2(b_1+b_2) + b_1b_2(a_1+a_2) - a_1b_2(a_1+b_2) - a_2b_1(a_2+b_1)}{b_1b_2(b_1+b_2)}\\
&= \frac{a_1a_2b_1 + a_1a_2b_2 - a_1a_1b_2 - a_2a_2b_1 + a_1b_1b_2 + a_2b_1b_2 - a_1b_2b_2 - a_2b_1b_1}{b_1b_2(b_1+b_2)}\\
&= \frac{(a_1-a_2)a_2b_1 + a_1(a_2-a_1)b_2 + a_1(b_1-b_2)b_2 + a_2b_1(b_2-b_1)}{b_1b_2(b_1+b_2)}
\end{aligned}$$

which since $a_1 - a_2 = q_1 - q_2 = b_1 - b_2$ is

$$= 0$$

□

Lemma A.23. Let $\varepsilon > 0$ and $p = \frac{2}{3} + \varepsilon$. Then the infimum of

$$\frac{a-1}{b-1}$$

under the conditions

$$a \geq b > 1, \tag{128}$$

$$\frac{2p-1}{p}\frac{a-p}{b-p} \geq \frac{a-(2p-1)}{b-(2p-1)} \tag{129}$$

is

$$\frac{8}{81}\frac{1}{\varepsilon^2} + \mathcal{O}(\frac{1}{\varepsilon}).$$

Furthermore

$$a = \frac{4}{27}\frac{1}{\varepsilon} + \frac{10}{9} + \frac{4}{3}\varepsilon,$$

$$b = 1 + \frac{3}{2}\varepsilon$$

satisfy (128), (129) and $\frac{a-1}{b-1} = \frac{8}{81}\frac{1}{\varepsilon^2} + \mathcal{O}(\frac{1}{\varepsilon})$.

Proof. First we check that it suffices to consider the case of equality in (129): Increasing b decreases the right hand side of (129) less than the left hand side because $p > 2p-1$. At the same time it decreases $\frac{a-1}{b-1}$. Hence if the inequality in (129) is strict then we can just increase b until it is not strict anymore. This will happen for some $b \leq a$, since for $a = b$ the inequality is reversed as $2p - 1 < p$.

Now (129) holds with equality if and only if

$$\begin{aligned}
&0 = \frac{(p-(1-p))(a-p)}{p(b-p)} - \frac{a-p+1-p}{b-p+1-p} \\
\iff\ &0 = (p-(1-p))(a-p)(b-p+1-p) - (a-p+1-p)p(b-p) \\
&= p(a-p)(b-p) + p(a-p)(1-p) - (1-p)(a-p)(b-p) - (1-p)^2(a-p) \\
&\quad - p(b-p)(a-p) - p(b-p)(1-p) \\
&= p(1-p)(a-p) - (1-p)(a-p)(b-p) - (1-p)^2(a-p) - p(1-p)(b-p)
\end{aligned}$$

dividing by $(1-p)$ implies

$$\begin{aligned}
\iff\ &0 = p(a-p) - (a-p)(b-p) - (1-p)(a-p) - p(b-p) \\
&= p(a-b) - (a-p)(1-p+b-p) \\
&= -ab + (2p-(1-p))a + p((1-p)-p) \\
\iff\ &a = \frac{p(p-(1-p))}{2p-(1-p)-b} = \frac{\frac{2}{9}+\frac{5}{3}\varepsilon+2\varepsilon^2}{1+3\varepsilon-b}
\end{aligned}$$

Thus having $a \geq 0$ requires $b \leq 1 + 3\varepsilon$. Plugging the result into $\frac{a-1}{b-1}$ yields

$$\frac{a-1}{b-1} = \frac{\frac{2}{9}+\frac{5}{3}\varepsilon+2\varepsilon^2-1-3\varepsilon+b}{(b-1)(1+3\varepsilon-b)} = \frac{b-\frac{7}{9}-\frac{4}{3}\varepsilon+2\varepsilon^2}{(b-1)(1+3\varepsilon-b)}.$$

If ε is small, and since we already know $1 < b < 1+3\varepsilon$, we cannot influence the nominator a lot by our choice of b. Hence we minimize the fraction about where we maximize the denominator, i.e. for

$$b = 1 + \frac{3}{2}\varepsilon + \mathcal{O}(\varepsilon^2),$$

That means

$$\begin{aligned}
\frac{a-1}{b-1} &= \frac{\frac{2}{9}+\mathcal{O}(\varepsilon)}{\frac{9}{4}\varepsilon^2+\mathcal{O}(\varepsilon^3)} = \frac{8}{81}\frac{1}{\varepsilon^2} + \mathcal{O}(\frac{1}{\varepsilon}), \\
a &= \frac{\frac{2}{9}+\frac{5}{3}\varepsilon+2\varepsilon^2}{\frac{3}{2}\varepsilon+\mathcal{O}(\varepsilon^2)} = \frac{4}{27}\frac{1}{\varepsilon} + \mathcal{O}(1).
\end{aligned}$$

If we instead take

$$b = 1 + \frac{3}{2}\varepsilon$$

we get the same result for $\frac{a-1}{b-1}$ up to another $\mathcal{O}(\frac{1}{\varepsilon})$ and

$$a = \frac{4}{27}\frac{1}{\varepsilon} + \frac{10}{9} + \frac{4}{3}\varepsilon$$

and

$$a - b = \frac{4}{27}\frac{1}{\varepsilon} + \frac{1}{9} - \frac{1}{6}\varepsilon \geq \frac{4}{27} \cdot 3 + \frac{1}{9} - \frac{1}{6}\frac{1}{3} = \frac{8+2-1}{18} = \frac{1}{2} \geq 0$$

□

Lemma A.24. For all $x \in [\sqrt{2}, 2)$ we have

$$\frac{2-x}{x-1} \leq \frac{2+x}{x+1}.$$

Proof.

$$\frac{2+x}{x+1} - \frac{2-x}{x-1} = \frac{(2+x)(x-1) - (2-x)(x+1)}{x^2-1} = \frac{2x^2-4}{x^2-1} \geq 0.$$

□

Lemma A.25 is used here to prove Lemma A.9 and in similar situations.

Lemma A.25. Let $a_1, a_2 \geq 0,\ b_1, b_2 > 0$ and $\frac{a_1}{b_1} \leq \frac{a_2}{b_2}$. Then

$$\frac{a_1}{b_1} \leq \frac{a_1 + a_2}{b_1 + b_2} \leq \frac{a_2}{b_2}.$$

Proof. We have

$$(a_1 + a_2)b_1 - (b_1 + b_2)a_1 = a_2 b_1 - b_2 a_1 \geq 0$$

and thus

$$\frac{a_1 + a_2}{b_1 + b_2} \geq \frac{a_1}{b_1}.$$

Similarly

$$(a_1 + a_2)b_2 - (b_1 + b_2)a_2 = a_1 b_2 - b_1 a_2 \leq 0$$

and thus

$$\frac{a_1 + a_2}{b_1 + b_2} \leq \frac{a_2}{b_2}.$$

□

Lemma A.26 is the calculation that leads to Lemma A.19.

Lemma A.26. For $R \geq 0$ fixed, define

$$c(a) \coloneqq 1 - \frac{R}{(1+a)^2},$$

i.e. s.t. for all $a > -1$ we have

$$R = (1 - c(a))(1+a)^2.$$

Then for all $0 \leq a \leq 1$ with $\frac{1}{2} \leq c(a) \leq 1$ we have

$$\frac{\mathrm{d}}{\mathrm{d}a}[c(a)(1-a)^2 + (1-c(a))(1+a)^2] \leq 0.$$

Proof. $c(a) \geq \frac{1}{2}$ is equivalent to

$$\begin{aligned} \frac{R}{(1+a)^2} &\leq \frac{1}{2}, \\ 2R &\leq (1+a)^2. \end{aligned}$$

Now

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}a}\Big[c(a)(1-a)^2 + (1-c(a))(1+a)^2\Big] &= \frac{\mathrm{d}}{\mathrm{d}a}\Big[c(a)(1-a)^2 + R\Big] \\ &= \frac{\mathrm{d}}{\mathrm{d}a}c(a)(1-a)^2 \\ &= c'(a)(1-a)^2 - 2c(a)(1-a) \\ &= (1-a)\Big[2\frac{R}{(1+a)^3}(1-a) - 2 + 2\frac{R}{(1+a)^2}\Big] \\ &= 2\frac{1-a}{(1+a)^3}\Big[R(1-a) - (1+a)^3 + R(1+a)\Big] \\ &= 2\frac{1-a}{(1+a)^3}\Big[2R - (1+a)^3\Big] \\ &\leq 2\frac{1-a}{(1+a)^3}\Big[(1+a)^2(1-(1+a))\Big] \\ &\leq 0. \end{aligned}$$

□

B Notation

- For $a \in \mathbb{N}$ and $b \in \mathbb{N} \cup \{\infty\}$ a tuple $(F_i)_{i=a}^{b}$ we denote

$$\{F_a, F_{a+1}, \dots\} = \{F_a, \dots, F_b\} = \left\{F_i \mid a \leq i \leq b\right\}.$$

That means that even if $\{F_i \mid a \leq i \leq b\}$ is finite we sometimes write $\{F_a, F_{a+1}, \dots\}$. In fact, most of our tuples will be finite but we usually don't care about how many members they have exactly, only that they are finite. For some elements $x, y, z, \dots$ we write

$$\{x, y, z, \dots, F_a, F_{a+1}, \dots\} = \{x, y, z, \dots\} \cup \{F_a, F_{a+1}, \dots\}.$$

Similarly we write

$$F_a + F_{a+1} + \dots = \sum_{i=a}^{b} F_i,$$
$$F_a \cup F_{a+1} \cup \dots = \bigcup_{i=a}^{b} F_i,$$
$$\dots$$

We do that so that we don't have to introduce a variable for the length of a sequence even though we don't actually care about its value.

- We write 'positive' when we mean 'nonnegative' and 'strictly positive' when we mean 'positive'. Similarly for 'smaller, greater,...'.

- '$\subset$' means the same as '$\subseteq$'

- Whenever we assume a set E to be a subset of $\mathbb{R}$ we implicitly assume it to be Lebesgue measurable. Since usually we only care about how E effects $\langle h_{I,E}, h_{J,E}\rangle$ for I, J belonging to a finite set of intervals $\mathbb{I}$, by Lemma A.12 we could even assume E to be a finite union of intervals without reducing the strength of statements.

- For a subset $E \subset \mathbb{R}$ we denotes its Lebesgue measure by $|E|$.

- Since we only care about subsets of real numbers in terms of integrals, we do not care about measure zero sets. That means whenever we write words like 'subset' or 'disjoint' we mean 'subset/disjoint up to measure zero'.

- $f(x) = \mathcal{O}(g(x))$ means

$$\limsup_{x \to 0} |\frac{f(x)}{g(x)}| < \infty.$$

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References

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